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SEQUENTIAL CONVERGENCES IN LATTICES

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Summary. The notion of sequential convergence on a lattice is defined in a natural way. In the present paper we investigate the system Conv L of all sequential convergences on a lattice L.

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In this paper the notion of sequential convergence in a lattice L will be introduced. It is defined to be a FLUSH convergence on the set L (cf., e.g., [10], [11]) such that the lattice operations are continuous and a certain convexity condition is fulfilled; for a thorough definition cf. Section 1 below. The system Conv L of all sequential convergences in L will be investigated (this system being partially ordered by inclusion). The main results deal with the case when L is a distributive lattice.

The analogous notions of sequential convergence in a lattice ordered group or in a Boolean algebra were studied in [2]–[9].

1. Preliminaries

Throughout the present paper, L denotes a lattice. Let $\mathbf{N}$ be the set of all positive integers. The direct product $\Pi_{n\in\mathbf{N}}L_n$, where $L_n = L$ for each $n \in \mathbf{N}$, will be denoted by $L^{\mathbf{N}}$. The elements of $L^{\mathbf{N}}$ are called sequences in L and they will be written as (x_n) (instead of n, sometimes other notation for indices will be applied). The notion of a subsequence has the usual meaning. If $x \in L$, $(x_n) \in L^{\mathbf{N}}$ and $x_n = x$ for each $n \in \mathbf{N}$, then we denote $(x_n) = \text{const}\, x$.

Let $\alpha \subseteq L^{\mathbf{N}} \times L$. A relation of the form $((x_n), x) \in \alpha$ will be recorded also by writing $x_n \to_\alpha x$.

Definition 1.1. A subset α of $L^{\mathbb{N}} \times L$ will be called a convergence in L, if the following conditions are satisfied:

(i) If $x_n \to_\alpha x$ and (y_n) is a subsequence of (x_n), then $y_n \to_\alpha x$.

(ii) If $(x_n) \in L^{\mathbb{N}}$, $x \in L$ and if for each subsequence (y_n) of (x_n) there is a subsequence (z_n) of (y_n) such that $z_n \to_\alpha x$, then $x_n \to_\alpha x$.

(iii) If $(x_n) \in L^{\mathbb{N}}$, $x \in L$, $(x_n) = \text{const}\, x$, then $x_n \to_\alpha x$.

(iv) If $x_n \to_\alpha x$ and $x_n \to_\alpha y$, then $x = y$.

(v) If $x_n \to_\alpha x$ and $y_n \to_\alpha y$, then $x_n \wedge y_n \to_\alpha x \wedge y$ and $x_n \wedge y_n \to_\alpha x \vee y$.

(vi) If $x_n \leqslant y_n \leqslant z_n$ is valid for each $n \in \mathbb{N}$ and if $x_n \to_\alpha x$, $z_n \to_\alpha x$, then $y_n \to_\alpha x$.

If all the above conditions except (iv) are assumed to be valid then α is called a multivalued convergence (shorter: m-convergence) in L.

The conditions (i) – (iv) say that L is a FLUSH convergence space (cf., e.g., [10] or [11]); the condition (v) means that α is a sublattice of the lattice $L^{\mathbb{N}} \times L$.

The system of all convergences (or m-convergences) in L will be denoted by Conv L (or $\text{Conv}_m L$, respectively); both these systems are partially ordered by inclusion.

Let $d \subset L^{\mathbb{N}} \times L$ be defined as follows: $x_n \to_\alpha x$ if there exists $m \in \mathbb{N}$ such that $x_n = x$ for each $n \geqslant m$.

The following assertion is obvious.

Lemma 1.2. *d is the least element in both* Conv L *and* $\text{Conv}_m L$ *If $\{\alpha_i\}_{i \in I}$ is a nonempty subset of* $\text{Conv}_m L$, *then $\bigcap_{i \in I} \alpha_i$ is the greatest lower bound of the set $\{\alpha_i\}_{i \in I}$ in* $\text{Conv}_m L$. *An analogous result holds for* Conv L.

From 1.2 we obtain as a corollary:

Lemma 1.3. $\text{Conv}_m L$ *is a $\wedge$-similattice. If $\alpha \in$* $\text{Conv}_m L$, *then the interval $[d, \alpha]$ of* $\text{Conv}_m L$ *is a complete lattice. Analogous results hold for* Conv L.

The set $L^{\mathbb{N}} \times L$ belongs to $\text{Conv}_m L$. Hence from 1.3 we infer:

Corollary 1.4. $\text{Conv}_m L$ *is a complete lattice. The following conditions are equivalent:*

(i) Conv L *is a complete lattice.*

(ii) Conv L *possesses a greatest element.*

(iii) *Each nonempty subset of* Conv L *is upper-bounded.*

Remark 1.5. In [9] the notion of convergence in a Boolean algebra B was introduced; it differs from that of 1.1 only by adding to the condition (v) in 1.1 the assumption that the implication

$$x_n \to_\alpha x \Rightarrow x'_n \to_\alpha x'$$

is valid (x'_n or x' is the complement of x_n or x, respectively).

Remark 1.6. The partially ordered set Conv L need not have, in general, a greatest element. To verify this it suffices to consider the same example which was applied in [9] (for proving that the system of all convergences on a Boolean algebra need not have a greatest element).

2. Constructive description of the join in $\mathrm{Conv}_m L$

The existence of the join of any subset of $\mathrm{Conv}_m L$ is guaranteed by 1.4. In this section we want to search for a constructive description of this operation. As consequences we obtain some results concerning Conv L.

The system of all lattice polynomials will be denoted by F. If $f \in F$, then $n(f)$ denotes the arity of f.

Let A be a nonempty subset of $L^{\mathbf{N}} \times L$. We denote by $[A]$ the system of all $((x_n), x)$ in $L^{\mathbf{N}} \times L$ which have the following property: there are f_1, $f_2 \in F$ with $n(f_1) = k(1) \geqslant 1$, $n(f_2) = k(2) \geqslant 1$ and elements

$$((y_n^1), y^1),\ ((y_n^2), y^2),\ \ldots,\ ((y_n^{k(1)}), y^{k(1)}),$$
$$((z_n^1), z^1),\ ((z_n^2), z^2),\ \ldots,\ ((z_n^{k(2)}), z^{k(2)})$$

in A such that

$$f_1(y^1, y^2, \ldots, y^{k(1)}) = f_2(z^1, z^2, \ldots, z^{k(2)}) = x$$

and for each $n \in \mathbf{N}$,

$$f_1(y_n^1, y_n^2,\ \ldots,\ y_n^{k(1)}) \leqslant x_n \leqslant f(z_n^1, z_n^2,\ \ldots,\ z_n^{k(2)}).$$

Next, let A^* be the set of all $((v_n), v)$ in $L^{\mathbf{N}} \times L$ such that for each subsequence $(v_{n(1)})$ of (v_n) there exists a subsequence $(v_{n(2)})$ of $(v_{n(1)})$ with the property that $((v_{n(2)}), v)$ belongs to A. Finally, let A^1 be the set of all $((x_n), x) \in L^{\mathbf{N}} \times L$ such that either

(i) there exists $((y_n), y) \in A$ such that $x = y$ and (y_n) is a subsequence of (x_n), or

(ii) there is $m \in \mathbf{N}$ such that $x_n = x$ for each $n \geqslant m$. The following lemma is obvious.

2.1. *Let* $\emptyset \neq A \subseteq L^{\mathbf{N}} \times L$. *Then* $[[A]] = [A] \supseteq A$, $A^{**} = A^* \supseteq A$ *and* $[A^1]^1 = [A^1]$.

Lemma 2.2. *Let A be as in 2.1. Then* $[[A^1]^*] = [A^1]^*$.

Proof. Let $((x_n), x) \in [[A^1]^*]$. We have to verify that $((x_n), x) \in [A^1]^*$. There exist $((y_n^1), y^1)$, …, $((y_n^{k(1)}), y^{k(1)})$, $((z_n^1), z^1)$, …, $((z_n^{k(1)}), z^{k(1)})$ having the properties as above with the distinction that we now have $[A^1]^*$ instead of A. Thus

$$\tag{1} \begin{gathered} ((y_n^j), y^j), ((z_n^t), z^t) \in [A^1]^* \\ \text{for each } j \in \{1, 2, \ldots, k(1)\} \text{ and each } t \in \{1, 2, \ldots, k(2)\}. \end{gathered}$$

Let $(x_{n(1)})$ be a subsequence of (x_n). In view of 2.1 and (1) there exists a subsequence $(x_{n(2)})$ of $(x_{n(1)})$ such that

$$\tag{2} ((y_{n(2)}^j), y^j), ((z_{n(2)}^t), z^t) \in [A^1] \quad \text{for each } t \in \{1, 2, \ldots, k\}.$$

By virtue of (2) and in view of the above relation we infer that

$$((x_{n(2)}), x) \in [[A^1]] = [A^1].$$

Therefore $((x_n), x) \in [A^1]^*$. □

Lemma 2.3. *Let A be as in 2.1. Then $[A^1]^* \in \operatorname{Conv}_m L$.*

Proof. The validity of the conditions (i), (ii) and (iii) follows immediately from the definition of $[A^1]^*$. By virtue of 2.2, the conditions (v) and (vi) are satisfied as well. □

Lemma 2.4. *Let A be as in 2.1 and let $\alpha \in \operatorname{Conv}_m L$, $A \subseteq \alpha$. Then $[A^1]^* \subseteq \alpha$.*

Proof. In view of (i), (ii) and (iii) from 1.1 we obtain $A^1 \subseteq \alpha$. The conditions (v) and (vi) of 1.1 imply $[A] \subseteq \alpha$. Hence in view of the condition (ii) of 1.1 we infer that $[A^1]^* \subseteq \alpha$. □

The m-convergence $[A^1]^*$ will be said to be generated by the set A.

The set A will be said to be regular (with respect to L) if there exists $\alpha \in \operatorname{Conv} L$ such that $A \subseteq \alpha$.

The following assertions 2.5, 2.6 and 2.7 are immediate consequences of 2.3 and 2.4.

Theorem 2.5. *Let $\{\alpha_i\}_{i \in I}$ be a nonempty system of m-convergences in L. Then in the complete lattice $\operatorname{Conv}_m L$ we have*

$$\tag{3} \bigvee_{i \in I} \alpha_i = \Big[\bigcup_{i \in I} \alpha_i\Big].$$

Theorem 2.6. *Let A be a nonempty subset of $L^{\mathbb{N}} \times L$. Then the following conditions are equivalent:*

(i) *A is regular.*

(ii) *The system $[A^1]^*$ satisfies ~~the condition (iv) from~~* 1.1.

Theorem 2.7. *Let $\{\alpha_i\}_{i \in I}$ be a nonempty system of convergences in L. Then the following conditions are equivalent:*

(i) *The system $\{\alpha_i\}_{i \in I}$ is upper bounded in* Conv L.

(ii) *The set $[\bigcup_{i \in I} \alpha_i]$ satisfies the condition* (iv) *from* 1.1.

If (ii) *holds, then the relation* (3) *is valid in the partially ordered set* Conv L.

3. Positive and negative m-convergences

An m-convergence α in L will be called positive (or negative, respectively) if, whenever $x_n \rightarrow_\alpha x$, then there is $m \in \mathbb{N}$ such that $x_n \geqslant x$ $(x_n \leqslant x)$ for each $n \geqslant m$. Let $\mathrm{Conv}_m L^+ (\mathrm{Conv}_m L^-)$ be the set of all positive (or negative, respectively) convergences in L. Next, let $\mathrm{Conv}\, L^+$ and $\mathrm{Conv}\, L^-$ be defined analogously. Then $\mathrm{Conv}_m L^+ \cap \mathrm{Conv}_m L^- = \{d\}$. For $\alpha \in \mathrm{Conv}_m L$ let α^+ be the set of all $((x_n), x) \in \alpha$ with the property that there is $m \in \mathbb{N}$ such that $x_n \geqslant x$ for each $n \geqslant m$. The set α^- is defined analogously.

In view of 1.1 we obviously have

Lemma 3.1. *If $\alpha \in \mathrm{Conv}_m L$ ($\alpha \in \mathrm{Conv}\, L$), then both α^+ and α^- belong to $\mathrm{Conv}_m L$ (or to $\mathrm{Conv}\, L$, respectively).*

Lemma 3.2. *Let $\alpha \in \mathrm{Conv}_m L$, $((x_n), x) \in L^{\mathbb{N}} \times L$. Then the following conditions are equivalent:*

(a) $x_n \rightarrow_\alpha x$.

(b) *$x_n \wedge y \rightarrow_\alpha x \wedge y$ and $x_n \vee y \rightarrow_\alpha x \vee y$ for every $y \in L$.*

Proof. From the conditions (iii) and (v) in 1.1 we obtain that (a) $\Rightarrow$ (b). Next, the condition (vi) in 1.1 yields that (b) $\Rightarrow$ (a) is valid. □

From 3.1 and 3.2 we infer:

Lemma 3.3. *Let $\alpha \in \mathrm{Conv}_m L$. Then in the partially ordered set $\mathrm{Conv}_m L$ we have $\alpha = \alpha^+ \vee \alpha^-$. An analogous assertion holds for* Conv L.

Proposition 3.4. *Let $\alpha \in \mathrm{Conv}_m L^+$, $\beta \in \mathrm{Conv}_m L^-$. We denote by γ the set of all elements $((x_n), x)$ of $L^{\mathbb{N}} \times L$ such that $x_n \vee x \rightarrow_\alpha x$ and $x_n \wedge x \rightarrow_\beta x$. Then*

(i) $\gamma \in \mathrm{Conv}_m L$;
(ii) $\gamma = \alpha \vee \beta$ *in* $\mathrm{Conv}_m L$;
(iii) $\gamma^+ = \alpha$ *and* $\gamma^- = \beta$.

Proof. (i) The conditions (i), (ii), (iii) and (vi) from 1.1 are obviously valid for γ. Let us verify that the condition (v) from 1.1 holds for γ.

Assume that $x_n \to_\gamma y$. Hence

$$x_n \vee x \to_\alpha x, y_n \vee y \to_\alpha y, \tag{1}$$

$$x_n \wedge x \to_\beta x, y_n \wedge y \to_\beta y. \tag{2}$$

In view of (1) we have

$$(x_n \vee y_n) \vee (x \vee y) \to_\alpha x \vee y. \tag{3}$$

In each lattice the following relation is valid:

$$(x_n \wedge x) \vee (y_n \wedge y) \leqslant (x_n \vee y_n) \wedge (x \vee y) \leqslant x \vee y. \tag{4}$$

The relation (2) yields that

$$(x_n \wedge x) \vee (x_n \wedge y) \to_\beta x \vee y,$$

hence according to (4) we obtain

$$(x_n \vee y_n) \wedge (x \vee y) \to_\beta x \vee y. \tag{5}$$

From (3) and (5) we infer that

$$x_n \vee y_n \to_\gamma x \vee y$$

is valid. In a similar way we can prove that $x_n \wedge y_n \to_\gamma x \wedge y$ holds. We have proved that (i) holds.

The assertion (ii) is an easy consequence of (i). The verification of (iii) is routine and it is omitted. □

Proposition 3.5. *The mapping* $f(\alpha) = (\alpha^+, \alpha^-)$ *where (*α *runs over* $\mathrm{Conv}_m L$*) is an isomorphism of the partially ordered set* $\mathrm{Conv}_m L$ *onto the direct product* $\mathrm{Conv}_m L^+ \times \mathrm{Conv}_m L^-$.

Proof. If $\alpha, \beta \in \mathrm{Conv}_m L$ and $\alpha \leqslant \beta$, then clearly $\alpha^+ \leqslant \beta^+$ and $\alpha^- \leqslant \beta^-$. Next, from 3.4 (iii) we infer that for each $\alpha_1 \in \mathrm{Conv}_m L^+$ and $\alpha_2 \in \mathrm{Conv}\, L^-$ there exists $\alpha \in \mathrm{Conv}_m L$ with $f(\alpha) = (\alpha_1, \alpha_2)$. In view of 3.4 (ii) we have

$$f(\alpha) \leqslant f(\beta) \Rightarrow \alpha \leqslant \beta.$$

Thus f is an isomorphism. □

Proposition 3.6. *Let $\alpha \in \operatorname{Conv} L^+$, $\beta \in \operatorname{Conv} L^-$ and assume that the set $\{\alpha, \beta\}$ is upper-bounded in* Conv L. *Let γ be as in 3.4. Then $\gamma = \alpha \vee \beta$ in* Conv L.

Proof. Since $\{\alpha, \beta\}$ is upper-bounded in Conv L, then in view of 1.3 the element $\alpha \vee \beta$ exists in Conv L. According to 2.5 and 2.7 this element coincides with the least upper bound of the set $\{\alpha, \beta\}$ in $\operatorname{Conv}_m L$. Therefore 3.4 (ii) yields that $\alpha \vee \beta = \gamma$ in Conv L.

By applying 3.6 and the same method as in the proof of 3.5 we obtain: □

Proposition 3.7. *The mapping $g(\alpha) = (\alpha^+, \alpha^-)$ (where α runs over* Conv L*) is an isomorphism of the partially ordered set* Conv L *onto the direct product* $\operatorname{Conv} L^+ \times \operatorname{Conv} L^-$.

Acknowledgement. The author is indebted to the referee for pointing out that the assumption of the distributivity of L (which was applied in the original version of the proof of 3.4) can be omitted.

4. Convergences on linearly ordered sets

In this section we assume that L is a linearly ordered set.

Let $\alpha(o)$ be the set of all elements $((x_n), x)$ of $L^{\mathbb{N}} \times L$ such that (x_n) o-converges to x (cf., e.g., Birkhoff [1]). In view of 1.1 we immediately obtain:

Lemma 4.1. *$\alpha(o)$ belongs to* Conv L.

Lemma 4.2. *Let $\alpha \in \operatorname{Conv} L^+$, $((x_n), x) \in \alpha$. Then $((x_n), x) \in \alpha(o)$.*

Proof. Let $m \in \mathbb{N}$. In view of 1.1 (ii), (iii) and (v) the set $\{k \in \mathbb{N} : k \geqslant m \text{ and } x_k \geqslant x_m\}$ is finite, hence there exists an element

$$y_m = \max\{x_k : k \geqslant m\}.$$

We have $y_1 \geqslant y_2 \geqslant \ldots$ and $\bigwedge_{n=1}^{\infty} y_n = x$. Because of $y_n \geqslant x_n \geqslant x$ for each $n \in \mathbb{N}$ we infer that (x_n) o-converges to x. □

An analogous result holds for $\alpha \in \operatorname{Conv} L^-$, whence in view of 3.7 we infer:

Proposition 4.3. *$\alpha(o)$ is the greatest element of* Conv L.

Lemma 4.4. *Let $\alpha \in \operatorname{Conv} L^+$, $((x_n), x) \in \alpha \setminus d$, $((z_n), x) \in \alpha(o)^+$. Then $((z_n), x) \in \alpha$.*

Proof. Let $(z_{n(1)})$ be a subsequence of (z_n). Next, let (y_n) be as in the proof 4.2. Evidently we have $((y_n), x) \in \alpha$. For each $n \in \mathbb{N}$ there exists $n(2) \in \{n(1)\}$ with $n(2) \geqslant n$ such that $z_{n(2)} \leqslant y(n)$. Therefore $((z_{n(2)}), x) \in \alpha$. By virtue of 1.1 (ii) we have $((z_n), x) \in \alpha$. □

An analogous result holds for $\alpha \in \operatorname{Conv} L^-$.

Let L_1^+ be the set of all $x \in L$ with the property that there exists a strictly decreasing sequence (x_n) in L with $\bigwedge_n x_n = x$.

Next, let L_1^- be defined analogously.

For each $x \in L_1^+$ let $C(x)^+$ be the set of all $((x_n), x) \in L^{\mathbb{N}} \times L$ such that $((x_n), x) \in \alpha(o)^+$. For $x \in L_1^-$ let $C(x)^-$ have an analogous meaning. From 4.3 and 4.4 we have:

Proposition 4.5. (i) *Let M be a subset of L_1^+. Put*

$$\alpha(M)^+ = \{d\} \cup \Big(\bigcup_{x \in M} C(x)^+ \Big).$$

Then $\alpha(M)^+ \in \operatorname{Conv} L^+$.

(ii) *Let $\alpha \in \operatorname{Conv} L^+$. Let M be the set of all $x \in L$ such that there exists $(x_n) \in L^{\mathbb{N}}$ with $((x_n), x) \in \alpha \setminus d$. Then $\alpha = \alpha(M)^+$.*

An analogous result holds for negative convergences in L. From 4.5, 4.3 and 3.7 we obtain:

Theorem 4.6. *Let L be a linearly ordered set. Then the lattice* Conv L *is isomorphic to the direct product*

$$\{\alpha(M_1)^+\} \times \{\alpha(M_2)^-\},$$

where M_1 runs over the set of all subsets of L_1^+ and M_2 runs over the set of all subsets of L_1^- (the systems $\{\alpha(M_1)^+\}$ and $\{\alpha(M_2)^-\}$ being partially ordered by inclusion).

Corollary 4.7. *If $L_1^+ \neq \emptyset$ or $L_1^- \neq \emptyset$, then* Conv L *is a completely distributive Boolean algebra.*

For a related result concerning convergences in linearly ordered groups cf. [10].

In this section we assume that L is a distributive lattice. It will be proved that each interval of Conv L is a Brouwerian lattice.

Lemma 5.1. *Let $\alpha_i (i \in I)$ be elements of* Conv L^+ *and assume that $((x_n), x) \in \left[\bigcup_{i\in I} \alpha_i\right]$. Then there exist $i(1), \ldots, i(k(1)) \in I$ and $((t_n^1), x) \in \alpha_{i(1)}, \ldots, ((t_n^{k(1)}), x) \in \alpha_{i(k(1))}$ such that $x_n = t_n^1 \vee t_n^2 \vee \ldots \vee t_n^{k(1)}$ for each $n \in \mathbb{N}$.*

Proof. By the definition of [A] (cf. Section 2), for $A = \bigcup_{i\in I} \alpha_i$ there are $((z_n^1), z^1)$, $((z_n^2), z^2), \ldots, ((z_n^k), z^k) \in A$ and $f \in F$ such that

$$f(z^1, z^2, \ldots, z^k) = x, \tag{1}$$

$$x \leqslant x_n \leqslant f(z_n^1, \ldots, z_n^k) \text{ for each } n \in \mathbb{N}. \tag{2}$$

Put $s^j = z^j \vee x$, $s_n^j = z_n^j \vee x_n$ for each $j \in \{1, 2, \ldots, k\}$ and each $n \in \mathbb{N}$. Hence $((s_n^j), s^j) \in A$ for $j = 1, 2, \ldots, k$ and (in view of the distributivity of L)

$$f(s^1, s^2, \ldots, s^k) = x, \tag{3}$$

$$x \leqslant x_n \leqslant f(s_n^1, \ldots, s_n^k). \tag{4}$$

By applying the distributivity of L again we infer that $f(s^1, s^2, \ldots, s^n)$ is the join of a finite number (say $k(1)$) of meets of some subsets of $\{s^1, s^2, \ldots, s^n\}$, and that $f(s_n^1, \ldots, s_n^k)$ can also be expressed in the corresponding way. Let these meets be denoted by $t^1, t^2, \ldots, t^{k(1)}$ or $t_{on}^1, t_{on}^2, \ldots, t_{on}^{k(1)}$, respectively. Because of $s^1 \geqslant x, \ldots, s^k \geqslant x$ we obtain that

$$t^1 = t^2 = \ldots = t^k = x. \tag{5}$$

Also, $((t_{on}^j), t^j) \in \alpha_{i(j)}$ for $j = 1, 2, \ldots, k(1)$, whence

$$((t_{on}^j), x) \in A \text{ for each } j \in \{1, 2, \ldots, k(1)\}.$$

In view of (4) we have

$$x_n \leqslant t_{on}^1 \vee t_{on}^2 \vee \ldots \vee t_{on}^{k(1)}) \text{ for each } n \in \mathbb{N},$$

hence

$$x_n = (x \wedge t_{on}^1) \vee (x \wedge t_{on}^2) \vee \ldots \vee (x \wedge t_{on}^{k(1)}).$$

Put $x \wedge t_{on}^j = t_n^j$ for each $j \in \{1, 2, \ldots, k(1)\}$ and each $n \in \mathbb{N}$. We have $((t_n^j), x) \in A$ for each $J \in \{1, 2, \ldots, k(1)\}$ and $x_n = t_n^1 \vee t_n^2 \vee \ldots \vee t_n^{k(1)}$ for each $n \in \mathbb{N}$. □

From 5.1, the assertion dual to 5.1 and from 3.2 we obtain:

Lemma 5.2. *Let $\alpha_i (i \in I)$ be elements of* Conv L *and assume that* $((x_n), x) \in \left[\bigcup_{i\in I} \alpha_i\right]$. *Then there exist* $i(1), i(2), \ldots, i(k(1)), j(1), j(2), \ldots, j(k(2)) \in I$ *and*

$$((t_n^1), x) \in \alpha^+_{i(1)}, \ldots, ((t_n^{k(1)}), x) \in \alpha^+_{i(k(1))},$$
$$((q_n^1), x) \in \alpha^-_{j(1)}, \ldots, ((q_n^{k(2)}), x) \in \alpha^-_{j(k(2))}$$

such that

$$x_n \vee x = t_n^1 \vee t_n^2 \vee \ldots \vee T_n^{k(1)} \text{ for each } n \in \mathbf{N},$$
$$x_n \wedge x = q_n^1 \wedge q_n^2 \wedge \ldots \wedge q_n^{k(2)} \text{ for each } n \in \mathbf{N}.$$

Lemma 5.3. *Let $\alpha, \beta \in$ Conv L and let $\{\alpha_i\}_{i\in I}$ be a nonempty subset of $[d, \beta]$. Then* $\alpha \wedge \left(\bigvee_{i\in I} \alpha_i\right) = \bigvee_{i\in I} (\alpha \wedge \alpha_i)$.

Proof. In view of 1.3, both $\bigvee_{i\in I} \alpha_i$ and $\bigvee_{i\in I} (\alpha \wedge \alpha_i)$ do exist in $[d, \beta]$. The relation $\bigwedge_{i\in I}(\alpha \wedge \alpha_i) \leqslant \alpha \wedge \left(\bigvee_{i\in I} \alpha_i\right)$ being obvious, we have to verify that

$$\alpha \wedge \left(\bigvee_{i\in I} \alpha_i\right) \leqslant \bigvee_{i\in I} (\alpha \wedge \alpha_i)$$

is valid. Thus in view of 2.7 and 1.2 we have to verify that

$$\alpha \cap \left[\bigcup_{i\in I} \alpha_i\right]^* \subseteq \left[\bigcup_{i\in I} (\alpha \cap \alpha_i)\right]^*$$

holds.

Let $((x_n), x) \in \alpha \cap \left[\bigcup_{i\in I} \alpha_i\right]^*$. Let $(x_{n(1)})$ be a subsequence of (x_n). There exists a subsequence $(x_{n(2)})$ of $(x_{n(1)})$ such that

(1) $$((x_{n(2)}), x) \in \left[\bigcup_{i\in I} \alpha_i\right].$$

Clearly $((x_{n(2)}), x) \in \alpha$.

According to 5.2 there exist $(t_n^1), \ldots, (t_n^{k(1)})$ and $(q_n^1), \ldots, (q_n^{k(2)}) \in L^{\mathbf{N}}$ with the properties as in 5.2 with the distinction that we now have $(x_{n(2)})$ instead of (x_n).

Then

$$((t_n^1), x), \ldots, ((t_n^{k(1)}), x), ((q_n^1), x), \ldots, ((q_n^{k(2)}), x) \in \alpha,$$

whence

$$((t_n^1), x) \in \alpha \cap \alpha_{i(1)}, \ \ldots, \ ((q_n^{k(2)}), x) \in \alpha \cap \alpha_{j(k(2))}.$$

Therefore

$$((x_{n(2)} \vee x), x), \ ((x_{n(2)} \wedge x), x) \in \Big[\bigcup_{i \in I} (\alpha \cap_{\alpha_i})\Big]$$

and thus

$$((x_{n(2)}), x) \in \Big[\bigcup_{i \in I} (\alpha \cap \alpha_i)\Big].$$

Hence

$$((x_n), x) \in \Big[\bigcup_{i \in I} (\alpha \cap \alpha_i)\Big]^*,$$

completing the proof. □

Now, 5.3 and 1.3 yield:

Theorem 5.4. *Let L be a distributive lattice. Then each interval of* Conv L *is a Brouwerian lattice.*

The question whether the assumption of the distributivity in 5.4 can be omitted remains open.

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