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Jaromír Duda

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CONGRUENCE RESTRICTIONS ON AXES

JAROMÍR DUDA, Brno

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Summary. We give Mal'cev conditions for varieties V whose congruences on the product $A \times B$, $A, B \in V$, are determined by their restrictions on the axes in $A \times B$.

Keywords: Congruence, axis in the product, variety of algebras, Mal'cev conditions

AMS classification: 08B05

The present paper is a contribution to the problem of restoration of a congruence from its given subset. Recall that any regular congruence is uniquely determined by any one of its blocks, see [1], [8], [11]. For weakly regular congruences it suffices to give up the congruence blocks at the nullary operations $c_1, \ldots, c_n$, see [6] and [8]. Subregular congruences are determined by their blocks on an arbitrary subalgebra, see [2] and [4]. The recent paper [5] investigates congruences on the square $A \times A$ which uniquely correspond to their blocks over the diagonal Δ_A. Here we study congruences on the product $A \times B$ which are uniquely determined by their restrictions on axes in $A \times B$.

Definition 1. Let A, B be similar algebras, $a \in A$, $b \in B$ arbitrary elements. The subsets $A \times \{b\}$, $\{a\} \times B$ of $A \times B$ are called *axes* in the product $A \times B$.

1. CONGRUENCES DETERMINED BY TRACES ON AXES

Let ψ be a congruence on an algebra A, S a subset of A. Then the trace $\psi \cap S \times S$ of ψ on S is denoted by $\psi \restriction S$. The symbol $\theta(S_1, \ldots, S_n)$ denotes the least congruence on A containing subsets $S_1, \ldots, S_n$ of $A \times A$.

Definition 2. A congruence ψ on the product $A \times B$ of similar algebras A, B is said to be *determined by its traces* $\psi \restriction A \times \{b\}$, $\psi \restriction \{a\} \times B$ on the axes $A \times \{b\}$, $\{a\} \times B$, respectively, whenever $\psi = \theta(\psi \restriction A \times \{b\}, \psi \restriction \{a\} \times B)$.

We say that a variety V has *congruences determined by traces* on axes whenever each congruence on the product $A \times B$ of any $A, B \in V$ has this property.

Before stating the main theorem of this section we prove an auxiliary result.

Lemma 1. *Let a, b, c be elements of an algebra A. Then the traces $\Phi \restriction A \times \{c\}$, $\Phi \restriction \{c\} \times A$ of the principal congruence $\Phi = \theta(\langle a, a\rangle, \langle b, b\rangle) \in \mathrm{Con}\, A \times A$ on the axes $A \times \{c\}$, $\{c\} \times A$, respectively, satisfy the symmetry law $\langle\langle f, c\rangle, \langle g, c\rangle\rangle \in \Phi \restriction A \times \{c\}$ iff $\langle\langle c, f\rangle, \langle c, g\rangle\rangle \in \Phi \restriction \{c\} \times A$.*

Proof. Let $\langle\langle f, c\rangle, \langle g, c\rangle\rangle \in \Phi \restriction A \times \{c\}$. From $\langle\langle f, c\rangle, \langle g, c\rangle\rangle \in \Phi = \theta(\langle a, a\rangle, \langle b, b\rangle)$ we get that

$$\begin{aligned} f &= \varphi_1(a, b), \\ c &= \varphi_1(a, b), \\ \varphi_i(b, a) &= \varphi_{i+1}(a, b), \\ \varphi_i(b, a) &= \varphi_{i+1}(a, b), \quad 1 \leqslant i < n, \\ g &= \varphi_n(b, a), \\ c &= \varphi_n(b, a) \end{aligned}$$

for an integer $n \geqslant 1$ and suitable algebraic functions $\varphi_1, \ldots, \varphi_n$ over $A \times A$, see e.g. [3; Thm. 1]. Apparently the above equalities yield that also $\langle\langle c, f\rangle, \langle c, g\rangle\rangle \in \Phi$. Altogether, $\langle\langle c, f\rangle, \langle c, g\rangle\rangle \in \Phi \restriction \{c\} \times A$ and the proof is complete. □

The symbol c stands for a finite sequence $c_1, \ldots, c_k$.

Theorem 1. *For a variety V the following conditions are equivalent:*

(1) V has congruences determined by traces on axes;

(2) there exist ternary terms $p_1, \ldots, p_m$, $q_1, \ldots, q_m$, $c_1, \ldots, c_k$, $d_1, \ldots, d_k$, $f_1, \ldots, f_l$, $g_1, \ldots, g_l$, $(4+k)$-ary terms $t_1, \ldots, t_m$, and $(2+1)$-ary terms $s_1^i, \ldots, s_n^i$, $1 \leqslant i \leqslant m$, such that

$$\begin{aligned} x &= t_1(p_1(x, y, z), q_1(x, y, z), z, z, c(x, y, z)), \\ x &= t_1(z, z, p_1(x, y, z), q_1(x, y, z), d(x, y, z)), \end{aligned}$$

$$
\begin{aligned}
&t_i\big(q_i(x,y,z),p_i(x,y,z),z,z,c(x,y,z)\big) = \\
&\quad = t_{i+1}\big(p_{i+1}(x,y,z),q_{i+1}(x,y,z),z,z,c(x,y,z)\big),\\
\text{(i)}\qquad &t_i\big(z,z,q_i(x,y,z),p_i(x,y,z),d(x,y,z)\big) = \\
&\quad = t_{i+1}\big(z,z,p_{i+1}(x,y,z),q_{i+1}(x,y,z),d(x,y,z)\big), \quad 1 \leqslant i < m,\\
&y = t_m\big(q_m(x,y,z),p_m(x,y,z),z,z,c(x,y,z)\big),\\
&y = t_m\big(z,z,q_m(x,y,z),p_m(x,y,z),d(x,y,z)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
p_i(x,y,z) &= s_1^i\big(x,y,f(x,y,z)\big),\\
z &= s_1^i\big(x,y,g(x,y,z)\big),\\
\text{(ii)}\qquad s_j^i\big(y,x,f(x,y,z)\big) &= s_{j+1}^i\big(x,y,f(x,y,z)\big),\\
s_j^i\big(y,x,g(x,y,z)\big) &= s_{j+1}^i\big(x,y,g(x,y,z)\big), \quad 1 \leqslant j < n,\\
q_i(x,y,z) &= s_n^i\big(y,x,f(x,y,z)\big),\\
z &= s_n^i\big(y,x,g(x,y,z)\big), \quad 1 \leqslant i \leqslant m,
\end{aligned}
$$

are identities in V.

Proof. (1) $\Rightarrow$ (2): Denote by Φ the principal congruence $\theta(\langle x,x\rangle, \langle y,y\rangle)$ on the square $A \times A = F_V(x,y,z) \times F_V(x,y,z)$. Consider the axes $A \times \{z\}$ and $\{z\} \times A$ in $A \times A$. By hypothesis is determined by its traces on the axes, so $\Phi = \theta(\Phi \restriction A \times \{z\}, \Phi \restriction \{z\} \times A)$. Since Φ is finitely generated we infer that the above equality holds true for some finite subsets of $\Phi \restriction A \times \{z\}$, $\Phi \restriction \{z\} \times A$. Furthermore, using Lemma 1 we can state that

$$\Phi = \bigvee_{1 \leqslant i \leqslant h} \theta\big(\langle\langle p_i,z\rangle, \langle q_i,z\rangle\rangle, \langle\langle z,p_i\rangle, \langle z,q_i\rangle\rangle\big)$$

for some $p_1, \ldots, p_h, q_1, \ldots, q_h \in A$. Now the relation

$$\langle\langle x,x\rangle, \langle y,y\rangle\rangle \in \bigvee_{1 \leqslant i \leqslant h} \theta\big(\langle\langle p_i,z\rangle, \langle q_i,z\rangle\rangle, \langle\langle z,p_i\rangle, \langle z,q_i\rangle\rangle\big)$$

yields the identities (2) (i) where $\{p_1,\ldots,p_m\} = \{p_1,\ldots,p_h\}$ and $\{q_1,\ldots,q_m\} = \{q_1,\ldots,q_h\}$, see [2; Thm. 1]. Similarly from $\langle\langle p_i,z\rangle, \langle q_i,z\rangle\rangle \in \theta(\langle x,x\rangle, \langle y,y\rangle)$, $1 \leqslant i \leqslant m$, we obtain the other identities (2) (ii).

Notice that the identities (2) (ii) ensure also the relations $\langle\langle z,p_i\rangle, \langle z,q_i\rangle\rangle \in \theta(\langle x,x\rangle, \langle y,y\rangle)$, $1 \leqslant i \leqslant m$.

(2) $\Rightarrow$ (1): Let ψ be a congruence on the product $A \times B$ of algebras $A, B \in V$. Choose arbitrary elements $a \in A$, $b \in B$. We have to verify the equality $\psi = \theta(\psi \restriction A \times$

$\{b\}, \psi \restriction \{a\} \times \mathsf{B})$. To this end take a pair $\langle\langle x,y\rangle, \langle u,v\rangle\rangle \in \psi$. Setting $z := a$, $y := u$ in the odd identities from (2) (ii) and $z := b$, $y := v$, $x := y$ in the even identities from (2) (ii) we get that $\langle\langle p_i(x,u,a),b\rangle, \langle q_i(x,u,a),b\rangle\rangle \in \psi$ for $1 \leqslant i \leqslant m$. Similarly, setting $z := a$, $y := u$ in the even identities from (2) (ii) and $z := b$, $y := v$, $x := y$ in the odd identities from (2) (ii) we obtain that $\langle\langle a, p_i(y,v,b)\rangle, \langle a, q_i(y,v,b)\rangle\rangle \in \psi$ for $1 \leqslant i \leqslant m$. Further, setting $z := a$, $y := u$ in the odd identities from (2) (i) and $z := b$, $y := v$, $x := y$ in the even identities from (2) (i) we find that $\langle\langle x,y\rangle, \langle u,v\rangle\rangle \in \bigvee_{1\leqslant i\leqslant m} \theta\big(\langle\langle p_i(x,u,a),b\rangle, \langle q_i(x,u,a),b\rangle\rangle, \langle\langle a,p_i(y,v,b)\rangle, \langle a,q_i(y,v,b)\rangle\rangle\big)$ and thus $\langle\langle x,y\rangle, \langle u,v\rangle\rangle \in \theta(\psi \restriction \mathsf{A} \times \{b\}, \psi \restriction \{a\} \times \mathsf{B})$. Since $\langle\langle x,y\rangle, \langle u,v\rangle\rangle$ is an arbitrary element from ψ we conclude that $\psi \subseteq \theta(\psi \restriction \mathsf{A} \times \{b\}, \psi \restriction \{a\} \times \mathsf{B})$, which was to be proved. □

E x a m p l e 1. Let V be a variety of rings with 1. We propose the terms from Theorem 1 (2) as follows:

$$\begin{aligned}
&p_1(x,y,z) = x,\\
&q_1(x,y,z) = y,\\
&t_1(a,b,u,v,c_1,c_2) = a \cdot c_1 + u \cdot c_2,\\
&c_1(x,y,z) = 1, \quad c_2(x,y,z) = 0,\\
&d_1(x,y,z) = 0, \quad d_2(x,y,z) = 1,\\
&s_1^1(a,b,f_1,f_2) = a \cdot f_1 + f_2,\\
&f_1(x,y,z) = 1, \quad f_2(x,y,z) = 0,\\
&g_1(x,y,z) = 0, \quad g_2(x,y,z) = z.
\end{aligned}$$

Then

$$\begin{aligned}
t_1\big(p_1(x,y,z), q_1(x,y,z), z, z, c_1(x,y,z), c_2(x,y,z)\big) &= p_1(x,y,z) \approx x,\\
t_1\big(z, z, p_1(x,y,z), q_1(x,y,z), d_1(x,y,z), d_2(x,y,z)\big) &= p_1(x,y,z) \approx x,\\
t_1\big(q_1(x,y,z), p_1(x,y,z), z, z, c_1(x,y,z), c_2(x,y,z)\big) &= q_1(x,y,z) \approx y,\\
t_1\big(z, z, q_1(x,y,z), p_1(x,y,z), d_1(x,y,z), d_2(x,y,z)\big) &= q_1(x,y,z) \approx y,
\end{aligned}$$

and

$$\begin{aligned}
s_1^1\big(x, y, f_1(x,y,z), f_2(x,y,z)\big) &= x \cdot 1 + 0 = x = p_1(x,y,z),\\
s_1^1\big(x, y, g_1(x,y,z), g_2(x,y,z)\big) &= x \cdot 0 + z = z,\\
s_1^1\big(y, x, f_1(x,y,z), f_2(x,y,z)\big) &= y \cdot 1 + 0 = y = q_1(x,y,z),\\
s_1^1\big(y, x, g_1(x,y,z), g_2(x,y,z)\big) &= y \cdot 0 + z = z.
\end{aligned}$$

Other examples follow from our next observation.

Corollary 1. *Any variety satisfying the Fraser-Horn property, see* [7], *has congruences determined by traces on the axes.*

Proof. Immediate. □

2. Congruences determined by pairs on different axes

Definition 3. A congruence ψ on the product $\mathsf{A} \times \mathsf{B}$ of similar algebras A, B is said to be *determined by pairs on different axes* whenever $\psi = \theta(\psi \cap \mathsf{A} \times \{b\} \times \{a\} \times \mathsf{B})$ for any elements $a \in \mathsf{A}$, $b \in \mathsf{B}$.

We say that a variety V has *congruences determined by pairs on different axes* whenever each congruence on the product $\mathsf{A} \times \mathsf{B}$ of any algebras $\mathsf{A}, \mathsf{B} \in V$ has this property.

Theorem 2. *For a variety V the following conditions are equivalent:*

(1) *V has congruences determined by pairs on different axes;*

(2) *there exist ternary terms $p_1, \ldots, p_m$, $q_1, \ldots, q_m$, $c_1, \ldots, c_k$, $d_1, \ldots, d_k$, $f_1, \ldots, f_l$, $g_1, \ldots, g_l$, $(1+k)$-ary terms $t_1, \ldots, t_m$ and $(1+l)$-ary terms $s_1^i, \ldots, s_n^i$, $1 \leqslant i \leqslant m$, such that*

$$
\text{(i)} \qquad
\begin{aligned}
&x = t_1\big(p_1(x,y,z), c(x,y,z)\big),\\
&x = t_1\big(z, d(x,y,z)\big),\\
&t_i\big(z, c(x,y,z)\big) = t_{i+1}\big(p_{i+1}(x,y,z), c(x,y,z)\big),\\
&t_i\big(q_i(x,y,z), d(x,y,z)\big) = t_{i+1}\big(z, d(x,y,z)\big), \quad 1 \leqslant i < m,\\
&y = t_m\big(z, c(x,y,z)\big),\\
&y = t_m\big(q_m(x,y,z), d(x,y,z)\big),
\end{aligned}
$$

and

$$
\text{(ii)} \qquad
\begin{aligned}
p_i(x,y,z) &= s_1^i\big(x, f(x,y,z)\big),\\
z &= s_1^i\big(x, g(x,y,z)\big),\\
s_j^i\big(y, f(x,y,z)\big) &= s_{j+1}^i\big(x, f(x,y,z)\big),\\
s_j^i\big(y, g(x,y,z)\big) &= s_{j+1}^i\big(x, g(x,y,z)\big), \quad 1 \leqslant j < n,\\
z &= s_n^i\big(y, f(x,y,z)\big),\\
q_i(x,y,z) &= s_n^i\big(y, g(x,y,z)\big), \quad 1 \leqslant i \leqslant m,
\end{aligned}
$$

are *identities in* V.

Proof. (1) $\Rightarrow$ (2): Choose algebras $A = B = F_V(x,y,z) \in V$, the principal congruence $\theta(\langle x,x\rangle, \langle y,y\rangle)$ on $A \times B$ and the axes $A \times \{z\}$, $\{z\} \times A$ in $A \times B$. By hypothesis $\theta(\langle x,x\rangle, \langle y,y\rangle)$ is uniquely determined by pairs on the axes $A \times \{z\}$ and $\{z\} \times A$, i.e. we have $\theta(\langle x,x\rangle, \langle y,y\rangle) = \bigvee_{1\leqslant i\leqslant h} \theta(\langle p_i, z\rangle, \langle z, q_i\rangle)$ for some $p_1, \ldots, p_h, q_1, \ldots, q_h \in A$. The relation $\langle\langle x,x\rangle, \langle y,y\rangle\rangle \in \bigvee_{1\leqslant i\leqslant h} \theta(\langle p_i, z\rangle, \langle z, q_i\rangle)$ yields the identities

$$
\begin{aligned}
&x = t_1(p_1(x,y,z), z, c(x,y,z)),\\
&x = t_1(z, q_1(x,y,z), d(x,y,z)),\\
\text{(I1)}\quad &t_i(z, p_i(x,y,z), c(x,y,z)) = t_{i+1}(p_{i+1}(x,y,z), z, c(x,y,z)),\\
&t_i(q_i(x,y,z), z, d(x,y,z)) = t_{i+1}(z, q_{i+1}(x,y,z), d(x,y,z)), \quad 1 \leqslant i < m,\\
&y = t_m(z, p_m(x,y,z), c(x,y,z)),\\
&y = t_m(q_m(x,y,z), z, d(x,y,z)),
\end{aligned}
$$

where $\{p_1,\ldots,p_m\} = \{p_1,\ldots,p_h\}$, $\{q_1,\ldots,q_m\} = \{q_1,\ldots,q_h\}$, see [3] again, and similarly form $\langle\langle p_i, z\rangle, \langle z, q_i\rangle\rangle \in \theta(\langle x,x\rangle, \langle y,y\rangle)$, $1 \leqslant i \leqslant m$, we obtain the identities

$$
\begin{aligned}
p_i(x,y,z) &= s_1^i(x,y,f(x,y,z)),\\
z &= s_1^i(x,y,g(x,y,z)),\\
\text{(I2)}\qquad s_j^i(y,x,f(x,y,z)) &= s_{j+1}^i(x,y,f(x,y,z)),\\
s_j^i(y,x,g(x,y,z)) &= s_{j+1}^i(x,y,g(x,y,z)), \quad 1 \leqslant j < n,\\
z &= s_n^i(y,x,f(x,y,z)),\\
q_i(x,y,z) &= s_n^i(y,x,g(x,y,z)) \quad \text{for } 1 \leqslant i \leqslant m.
\end{aligned}
$$

Now the implication $(p(x,y,z) = z,\ 1 \leqslant i \leqslant m) \Rightarrow x = y$ is a consequence of the identities (I1), and $p(x,x,z) = z$, $1 \leqslant i \leqslant m$, follow from the identities (I2). Altogether we find that $p_1, \ldots, p_m$ are Csákány terms ensuring the congruence regularity of V, see [1]. Hence by [9] V has n-permutable congruences for some $n > 1$, and we can state that the terms $t_1, \ldots, t_m$ as well as the terms $s_1^i, \ldots, s_n^i$, $1 \leqslant i \leqslant m$, do not depend on the second variable, see [3]. The identities (2) (i) and (2) (ii) follow.

(2) $\Rightarrow$ (1): Let ψ be a congruence on $A \times B$, $A, B \in V$, $a \in A$, $b \in B$. Consider the axes $A \times \{b\}$ and $\{a\} \times B$ in $A \times B$. Take an element $\langle\langle x,y\rangle, \langle u,v\rangle\rangle \in \psi$. Setting $z := a$, $y := u$ in the odd identities from (2) (ii) and $z := b$, $y := v$, $x := y$ in the even identities from (2) (ii) we obtain that also $\langle\langle p_i(x,u,a), b\rangle, \langle a, q_i(y,v,b)\rangle\rangle \in \psi$

for $1 \leqslant i \leqslant m$. Applying the same substitutions in the identities (2) (i) we find that $\langle\langle x,y\rangle,\langle u,v\rangle\rangle \in \bigvee_{1\leqslant i\leqslant m} \theta(\langle p_i(x,u,a),b\rangle,\langle a,q_i(y,v,b)\rangle) \subseteq \theta(\psi\cap \mathsf{A}\times\{b\}\times\{a\}\times\mathsf{B})$, which proves that ψ is determined by pairs on the different axes $\mathsf{A}\times\{b\}$ and $\{a\}\times\mathsf{B}$. The proof is complete. □

Example 2. Let V be a variety of Abelian groups. We propose the terms from Theorem 2(2) as follows:

$$\begin{aligned}
&p_1(x,y,z) = x - y + z,\\
&q_1(x,y,z) = y - x + z,\\
&t_1(a,c_1,c_2) = a + c_1 - c_2,\\
&c_1(x,y,z) = y, \quad c_2(x,y,z) = z,\\
&d_1(x,y,z) = x, \quad d_2(x,y,z) = z,\\
&s_1^1(a,f_1,f_2) = a - f_1 + f_2,\\
&f_1(x,y,z) = y, \quad f_2(x,y,z) = z,\\
&g_1(x,y,z) = x, \quad g_2(x,y,z) = z.
\end{aligned}$$

Then

$$\begin{aligned}
&t_1\big(p_1(x,y,z),c_1(x,y,z),c_2(x,y,z)\big) = (x-y+z)+y-z = x,\\
&t_1\big(z,d_1(x,y,z),d_2(x,y,z)\big) = z+x-z = x,\\
&t_1\big(z,c_1(x,y,z),c_2(x,y,z)\big) = z+y-z = y,\\
&t_1\big(q_1(x,y,z),d_1(x,y,z),d_2(x,y,z)\big) = (y-x+z)+x-z = y
\end{aligned}$$

and

$$\begin{aligned}
&s_1^1\big(x,f_1(x,y,z),f_2(x,y,z)\big) = x-y+z = p_1(x,y,z),\\
&s_1^1\big(x,g_1(x,y,z),g_2(x,y,z)\big) = x-x+z = z,\\
&s_1^1\big(y,f_1(x,y,z),f_2(x,y,z)\big) = y-y+z = z,\\
&s_1^1\big(y,g_1(x,y,z),g_2(x,y,z)\big) = y-x+z = q_1(x,y,z).
\end{aligned}$$

Corollary 2. *Any variety whose congruences are determined by pairs on different axes is congruence regular and hence congruence modular and n-permutable for an integer $n > 1$.*

Proof. Congruence regularity was already verified in the proof of Theorem 2. The remaining conclusions are due to J. Hagemann [9]. □

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Souhrn

STOPY KONGRUENCÍ NA OSÁCH

JAROMÍR DUDA

Jsou ukázány Mal'cevovské podmínky pro variety V jejichž kongruence na součinu $\mathsf{A} \times \mathsf{B}$, $\mathsf{A}, \mathsf{B} \in V$, jsou určeny již stopami na osách v $\mathsf{A} \times \mathsf{B}$.

Author's address: Kroftova 21, 616 00 Brno, Czechoslovakia.