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*Mathematica Bohemica*, Vol. 117 (1992), No. 3, 225–238

Persistent URL: http://dml.cz/dmlcz/126284

117 (1992) MATHEMATICA BOHEMICA No. 3, 225–238

# PERIODIC AND ALMOST PERIODIC FLOWS OF PERIODIC ITO EQUATIONS

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(Received January 30, 1989)

*Summary.* Under the uniform asymptotic stability of a finite dimensional Ito equation with periodic coefficients, the asymptotically almost periodicity of the $L^p$-bounded solution and the existence of a trajectory of an almost periodic flow defined on the space of all probability measures are established.

*Keywords*: Ito equations, periodic and almost periodic flows, asymptotically almost periodic solution

*AMS classification*: 60H20

## 1. Introduction

There are many interesting results for deterministic differential equations concerning the existence of periodic, almost periodic and asymptotically almost periodic solutions under some stability assumptions.

In this paper we consider finite dimensional Ito equations with periodic coefficients. We introduce three stability concepts in $L^r$ ($L^r$-uniform stability, $L^r$-asymptotic uniform stability and $L^r$-uniform asymptotic stability in the large) which are natural extensions of the corresponding ones from the deterministic situation. Assuming the existence of an $L^p$-bounded solution we prove that it is asymptotically almost periodic in distribution under the uniform stability hypothesis.

Moreover, the existence of a trajectory of an almost periodic (periodic) flow defined on the space of all probability measures is also proved under the uniform asymptotic stability (uniform asymptotic stability in the large) assumption. Related problems for affine Ito equations have been considered in [1], [2], [7] and for nonlinear Ito equations with asymptotic almost periodic coefficients in [12].

## 2. Almost periodic functions with values in the space of probability measures

Let $(X, |.|)$ be a real separable Hilbert space and let $\mathscr{B}_X$ be its Borel field. We shall denote by $\Pr(X)$ the class of all probability measures on $\mathscr{B}_X$. $\Pr(X)$ is endowed with the weak topology; recall that $F_n$ converges weakly to $F$ ($F_n \Rightarrow F$ for short) if $\int f\,\mathrm{d}F_n \to \int f\,\mathrm{d}F$ for all $f \in C_b(X) := \{f : X \to \mathbf{R}; f$ continuous and $\|f\|_\infty := \sup_x |f(x)| < \infty\}$.

For $f \in C_b(X)$ we define

$$\|f\|_L = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|};\ x, y \in X,\ x \neq y\right\},$$
$$\|f\|_{BL} = \max(\|f\|_\infty, \|f\|_L),$$

and for $F, G \in \Pr(X)$ we define

$$d_{BL}(F, G) = \sup\left\{\left|\int f\,\mathrm{d}(F - G)\right|;\ \|f\|_{BL} \leqslant 1\right\}.$$

It is known that $d_{BL}$ is a complete metric on $\Pr(X)$ which generates the weak topology (*Kantorovitch metric*).

If $I = \mathbf{R}$ or $[s, \infty)$ then we shall denote by $C(I, \Pr(X))$ the set of all continuous functions from $I$ into $\Pr(X)$ endowed with the uniform convergence. If $F \in \Pr(X)$ we denote by $E(F)$ the expectation of $F$ and by $\operatorname{cov}(F)$ the covariance of $F$.

For an $\mathbf{R}^d$-valued random variable $f$ defined on a probability space $(\Omega, \mathscr{F}, P)$ we denote by $P \circ f^{-1}$ the distribution of $f$, by $E(f)$ the expectation of $f$ and by $\operatorname{cov}(f)$ the covariance of $f$. Finally $J$ is the interval $[0, 1]$, $\lambda$ is the Lebesgue measure and for $r \geqslant 1$, $L^r(J, \mathscr{B}_J, \lambda)$ is the standard space of all real valued functions $g$ defined on $J$ which are $\mathscr{B}_J$-measurable and $|g|^r$ is $\lambda$-integrable.

**Definition.** 1) A continuous mapping $\mu : [s, \infty) \to \Pr(X)$ is called *asymptotically almost periodic* (a.a.p. for short) if for every sequence $\{t_{n'}\}$, $t_{n'} \geqslant s$, $t_{n'} \to \infty$, there exists a subsequence $\{t_n\}$ such that $\int f\,\mathrm{d}\mu(t + t_n)$ convergences uniformly for $t \geqslant s$, for every $f \in C_b(X)$.

2) A continuous mapping $\mu : \mathbf{R} \to \Pr(X)$ is *almost periodic* (a.p. for short) if for every sequence $\{t_{n'}\} \subset \mathbf{R}$ there exists a subsequence $\{t_n\}$ such that $\int f\,\mathrm{d}\mu(t + t_n)$ converges uniformly for $t \in \mathbf{R}$, for every $f \in C_b(X)$.

**Lemma 2.1.** *Let $\mu : [s, \infty) \to \Pr(X)$ ($\mu : \mathbf{R} \to \Pr(X)$) be a continuous function. Then the following assertions are equivalent*

(a) *$\mu$ is a.a.p. (a.p.);*

(b) *for every sequence* $\{t_{n'}\} \subset [s, \infty)$, $t_{n'} \to \infty$ $(\{t_{n'}\} \subset \mathbf{R})$ *there exist a subsequence* $\{t_n\} \subset \{t_{n'}\}$ *and a continuous function* $\hat{\mu}: [s, \infty) \to \Pr(X)$ $(\hat{\mu}: \mathbf{R} \to \Pr(X))$ *such that*

$$\sup_{t \geqslant s} \left| \int f \, \mathrm{d}\mu(t + t_n) - \int f \, \mathrm{d}\hat{\mu}(t) \right| \to 0 \tag{2.1}$$

$$\left( \sup_{t \in \mathbf{R}} \left| \int f \, \mathrm{d}\mu(t + t_n) - \int f \, \mathrm{d}\hat{\mu}(t) \right| \to 0, \text{ respectively} \right) \tag{2.2}$$

*for all* $f \in C_b(X)$.

Proof. (a) $\Rightarrow$ (b). Choose a subsequence $\{t_n\} \subset \{t_{n'}\}$ such that for all $f \in C_b(X)$, $\int f \, \mathrm{d}\mu(t+t_n)$ converges uniformly for $t \geqslant s$ $(t \in \mathbf{R})$. Since, for fixed $t_0$, the sequence $\{\mu(t_0 + t_n)\}_n$ is relatively compact, we choose a subsequence $\{t_{n''}\} \subset \{t_n\}$ such that $\mu(t_0 + t_{n''})$ convergences weakly to $\hat{\mu}(t_0) \in \Pr(X)$.

In particular, we have $\mu(t_0 + t_n) \Rightarrow \hat{\mu}(t_0)$ and then

$$\left| \int f \, \mathrm{d}\mu(t + t_n) - \int f \, \mathrm{d}\hat{\mu}(t) \right| \to 0 \quad \text{uniformly for } t \geqslant s \quad (t \in \mathbf{R}),$$

for all $f \in C_b(X)$ and $\hat{\mu}$ is continuous.

The implication (b) $\Rightarrow$ (a) is immediate. □

**Theorem 2.2.** *A continuous function* $\mu: \mathbf{R} \to \Pr(X)$ *is a.p. if and only if the family* $\{\mu(t + .)\}_{t \in \mathbf{R}}$ *is relatively compact in* $C(\mathbf{R}, \Pr(X))$.

Proof. Assume that $\mu$ is a.p. and let $\{t_{n'}\} \subset \mathbf{R}$. Choose $\{t_n\} \subset \{t_{n'}\}$ and $\hat{\mu}: \mathbf{R} \to \Pr(X)$ continuous such that (2.2) holds.

Assume that $\overline{\lim}_{n \to \infty} \sup_{t \in \mathbf{R}} d_{BL}(\mu(t + t_n), \hat{\mu}(t)) > \varepsilon$ for some $\varepsilon > 0$. Then, for every $n$, there exist $f_n \in C_b(X)$ with $\|f_n\|_{BL} \leqslant 1$ and $s_n \in \mathbf{R}$ such that

$$\left| \int f_n \, \mathrm{d}\mu(s_n + t_n) - \int f_n \, \mathrm{d}\hat{\mu}(s_n) \right| > \varepsilon \quad \text{for every } n$$

(we pass to a subsequence if necessary).

Let $K \subset X$ be a compact set such that

$$\mu(s_n + t_n)(X \setminus K) \leqslant \frac{\varepsilon}{8}, \quad \hat{\mu}(s_n)(X \setminus K) \leqslant \frac{\varepsilon}{8} \quad \text{for all } n.$$

Without loss of generality we may assume that $f_n \to f$ uniformly on every compact, for some $f \in C_b(X)$. Then we have

$$
\begin{aligned}
0 < \varepsilon &\leqslant \left| \int f_n \, \mathrm{d}\mu(s_n + t_n) - \int f_n \, \mathrm{d}\hat{\mu}(s_n) \right| \\
&\leqslant \int |f_n - f| \, \mathrm{d}\mu(s_n + t_n) + \int |f_n - f| \, \mathrm{d}\hat{\mu}(s_n) \\
&\quad + \left| \int f \, \mathrm{d}\mu(s_n + t_n) - \int f \, \mathrm{d}\hat{\mu}(s_n) \right| \\
&\leqslant 2 \sup_{x \in K} |f_n(x) - f(x)| + \frac{\varepsilon}{2} + \sup_t \left| \int f \, \mathrm{d}\mu(t + t_n) - \int f \, \mathrm{d}\hat{\mu}(t) \right|,
\end{aligned}
$$

so that

$$
0 < \varepsilon \leqslant \varlimsup_{n \to \infty} \left| \int f_n \, \mathrm{d}\mu(s_n + t_n) - \int f_n \, \mathrm{d}\hat{\mu}(s_n) \right| \leqslant \frac{\varepsilon}{2},
$$

which is a contradiction.

Now suppose that $\{\mu(t + .)\}_{t \in \mathbf{R}}$ is relatively compact in $C(\mathbf{R}, \mathrm{Pr}(X))$. If $\{t_{n'}\} \subset \mathbf{R}$ than choose $\{t_n\} \subset \{t_{n'}\}$ and $\hat{\mu} \in C(\mathbf{R}, \mathrm{Pr}(X))$ such that

$$
\sup_{t \in \mathbf{R}} d_{BL}\big(\mu(t + t_n), \hat{\mu}(t)\big) \to o.
$$

In particular, $d_{BL}\big(\mu(t_n), \hat{\mu}(0)\big) \to 0$ and hence $\mu(t_n) \to \hat{\mu}(0)$. Now let $f \in C_b(X)$ be locally Lipschitz, i.e., for every $r > 0$ we have $|f(x) - f(y)| \leqslant L_r |x - y|$ if $|x| \leqslant r$, $|y| \leqslant r$, for some $L_r > 0$. Let $K \subset X$ be a compact set such that

$$
\sup_{t \in \mathbf{R}} \mu(t)(X \setminus K) \leqslant \varepsilon, \quad \sup_{t \in \mathbf{R}} \hat{\mu}(t)(X \setminus K) \leqslant \varepsilon,
$$

and let $r > 0$ be such that $K \subset \{x; |x| \leqslant r\}$.

From [8; Lemma 1] we have

$$
\begin{aligned}
\left| \int f \, \mathrm{d}\mu(t + t_n) - \int f \, \mathrm{d}\hat{\mu}(t) \right| &\leqslant C_1 d_{BL}\big(\mu(t + t_n), \hat{\mu}(t)\big) + C_2 \hat{\mu}(t)(\{x; |x| > r\}) \\
&\leqslant C_1 d_{BL}\big(\mu(t + t_n), \hat{\mu}(t)\big) + C_2 \hat{\mu}(t)(X \setminus K),
\end{aligned}
$$

so that

$$
\varlimsup_{n \to \infty} \sup_{t \in \mathbf{R}} \left| \int f \, \mathrm{d}\mu(t + t_n) - \int f \, \mathrm{d}\hat{\mu}(t) \right| \leqslant C_2 \varepsilon.
$$

If $f \in C_b(X)$ then choose $f_k$ locally Lipschitz, $\|f_k\|_\infty \leqslant \|f\|_\infty$ and $f_k \to f$ uniformly on every compact.

Then we have

$$
\begin{aligned}
&\sup_{t \in \mathbf{R}} \left| \int f \, \mathrm{d}\mu(t + t_n) - \int f \, \mathrm{d}\hat{\mu}(t) \right| \\
&\qquad \leqslant \sup_{x \in K} |f_k(x) - f(x)| + \sup_{t \in \mathbf{R}} \left| \int f_k \, \mathrm{d}\mu(t + t_n) - \int f_k \, \mathrm{d}\hat{\mu}(t) \right| + C_3 \varepsilon.
\end{aligned}
$$

Now take $\varlimsup\limits_{n \to \infty}$ and then $\lim\limits_{k \to \infty}$ to complete the proof. □

**Definition.** A set $E \subset \mathbf{R}$ is *relatively dense* if there exists $r > 0$ such that for every $a \in \mathbf{R}$ we have $[a, a+r] \cap E \neq \emptyset$.

**Corollary 2.3.** *Let $\mu \in C(\mathbf{R}, \Pr(X))$ be such that for some relatively dense set $E \subset \mathbf{R}$, the family $\{\mu(t+.)\}_{t \in E}$ is relatively compact in $C(\mathbf{R}, \Pr(X))$. Then $\mu$ is a.p.*

Proof. It follows from Theorem 2.2 and [3; Theorem 1]. □

**Corollary 2.4.** *For $\theta > 0$ let $E_\theta = \{m\theta;\ m = 0, \pm 1, \dots\}$ and $E_\theta^+ = \{m\theta;\ m = 0, 1, \dots\}$. Then $\mu \in C(\mathbf{R}, \Pr(X))$ ($\mu \in C(\mathbf{R}_+, \Pr(X))$) is a.p. (a.a.p.) if the family $\{\mu(t+.)\}_{t \in E_\theta}$ ($\{\mu(t+.)\}_{t \in E_\theta^+}$) is relatively compact in $C(\mathbf{R}, \Pr(X))$ ($C(\mathbf{R}_+, \Pr(X))$).*

Proof. The a.p. case follows from Corollary 2.3. We consider the a.a.p. case. First we note that $\mu \colon \mathbf{R} \to (\Pr(X), d_{BL})$ is uniformly continuous (since $\{\mu(t+.)\}_{t \in E_\theta^+}$ is relatively compact in $C([\theta, 2\theta], \Pr(X))$ and if $x_1 \leqslant x_2 \leqslant x_1 + \theta$ then we have $x_1, x_2 \in [m\theta, (m+2)\theta]$ for some $m = 0, 1, \dots$).

Let $\{t_k\}_k$ be a sequence, $0 \leqslant t_k \to \infty$, and for every $k$ choose $m_k = 0, 1, \dots$ such that $m_k\theta \leqslant t_k \leqslant (m_k + 1)\theta$. Then $t_k = m_k\theta + \sigma_k$ with $0 \leqslant \sigma_k < \theta$. Without loss of generality we may assume that $\sigma_k$ converges to $\sigma$ and $\sup_{t \geqslant 0} d_{BL}(\mu(m_k\theta + t), \hat{\mu}(t)) \to 0$ for some $\hat{\mu} \in C(\mathbf{R}_+, \Pr(X))$.

Therefore for $\varepsilon > 0$ we find $k_\varepsilon$ such that if $k \geqslant k_\varepsilon$ then we have

$$\begin{aligned} d_{BL}(\mu(t + t_k), \hat{\mu}(t + \sigma)) &\leqslant d_{BL}(\mu(t + t_k), \mu(t + m_k\theta + \sigma)) \\ &\quad + d_{BL}(\mu(t + m_k\theta + \sigma), \hat{\mu}(t + \sigma)) \\ &\leqslant \varepsilon + d_{BL}(\mu(t + m_k\theta + \sigma), \hat{\mu}(t + \sigma)) \\ &\leqslant \varepsilon + \sup_{t \geqslant 0} d_{BL}(\mu(m_k\theta + t), \hat{\mu}(t)). \end{aligned}$$

Now it is easy to complete the proof. □

Remark 2.5. From Corollary 2.4 we see that every $\theta$-periodic and continuous function $\mu \colon \mathbf{R} \to \Pr(X)$ is a.p.

**Definition.** A continuous mapping $\mu \colon [s, \infty) \to \Pr(X)$ is *a.a.p. in Bohr's sense* if for every $\varepsilon > 0$ there exists $k(\varepsilon)$ and $T(\varepsilon) > 0$ such that any interval of length $k(\varepsilon)$ contains a $\tau$ such that $d_{BL}(\mu(t + \tau), \mu(t)) \leqslant \varepsilon$ for $t, t + \tau \geqslant T(\varepsilon)$.

**Definition.** A continuous mapping $\mu \colon \mathbf{R} \to \Pr(X)$ is *a.p. in Bohr's sense* if for every $\varepsilon > 0$ there exist $k(\varepsilon)$ such that any interval of length $k(\varepsilon)$ contains a $\tau$ such that for all $t \in \mathbf{R}$ we have $d_{BL}(\mu(t + \tau), \mu(t)) \leqslant \varepsilon$.

As in the case of a.a.p. (a.p.) mappings with values in complete metric spaces we have (with the same proof) the following results.

**Theorem 2.6.** *A continuous function* $\mu\colon [s,\infty) \to \mathrm{Pr}(X)$ $(\mu\colon \mathbf{R} \to \mathrm{Pr}(X))$ *is* a.a.p. (a.p.) *if and only if* $\mu$ *is Bohr* a.a.p. (*Bohr* a.p.).

**Lemma 2.7.** *Assume that* $\mu\colon \mathbf{R} \to \mathrm{Pr}(X)$ *is* a.p. *and there is* $\alpha > 0$ *such that* $\lim_{t\to\infty} d_{BL}(\mu(t+\alpha), \mu(t)) = 0$. *Then* $\mu(t) = \mu(t+\alpha)$ *for all* $t$.

Proof. For $f \in C_b(X)$ with $\|f\|_{BL} \leqslant 1$ we have that the function $t \to \nu_t^f = \int f \,\mathrm{d}\mu(t)$ is a.p. with values in $\mathbf{R}$ and $\nu_{t+\alpha}^f - \nu_t^f \to 0$ as $t \to \infty$. Therefore we have $\nu_t^f + \nu_{t+\alpha}^f$ and hence $\mu(t+\alpha) = \mu(t)$ for all $t$. □

## 3. Periodic Ito equations

Let $F\colon \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$, $G\colon \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ be two measurable functions. We shall assume that

(j) $F(.,0)$, $G(.,0)$ are bounded;

(jj) there exists a concave increasing function $\varrho\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that $\int_{0+}^1 \frac{\mathrm{d}u}{\varrho(u)} = \infty$ and

$$|F(t,x) - F(t,y)|^2 + |G(t,x) - G(t,y)|^2 \leqslant \varrho(|x-y|^2) \tag{3.1}$$

for all $t \in \mathbf{R}$, $x, y \in \mathbf{R}^d$.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathbf{R}})$ be a filtered probability space on which a $d$-dimensional $\mathcal{F}_t$-adapted brownian motion $(w_t)_{t\in\mathbf{R}}$ is defined. Since for a process $(x_t)_t$ we are interested only in the one-dimensional distribution $t \to P \circ x_t^{-1}$, we choose a new (extended) filtered probability space adequate for our purpose. This probability space is $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P, (\tilde{\mathcal{F}}_t)_{t\in\mathbf{R}}) = (\Omega \times J, \mathcal{F} \otimes \mathcal{B}_J, P \otimes \lambda, (\mathcal{F}_t \otimes \mathcal{B}_J)_{t\in\mathbf{R}})$. We extend $w$ onto $\tilde\Omega$ in an obvious manner. On this probability space we consider the Ito equation

$$\mathrm{d}x_t = F(t,x_t)\,\mathrm{d}t + G(t,x_t)\,\mathrm{d}w_t, \quad t \in \mathbf{R}. \tag{3.2}$$

If $\tilde x_0$ is a $\tilde{\mathcal{F}}_0$-measurable $\mathbf{R}^d$-valued random variable and $s \in \mathbf{R}$ then we denote by $x_t(s, \tilde x_0)$ the unique strong solution of the Ito equation (see [11])

$$\begin{cases} \mathrm{d}x_t = F(t,x_t)\,\mathrm{d}t + G(t,x_t)\,\mathrm{d}w_t, & t \geqslant s,\\ x_s = \tilde x_0. \end{cases} \tag{3.3}$$

**Remark 3.1.** Since the pathwise uniqueness holds for (3.3) hence by a result of Yamada-Watanabe ([4], [10]) the distribution of $(x_t)_{t\geqslant s}$ depends only on $\tilde P \circ \tilde x_0^{-1}$ and $F$, $G$ and not the support of the probability space.

We shall denote by $\mu_t(s,\mu_0)$ the distribution of $x_t(s,\tilde{x}_0)$, where $\tilde{P}\circ\tilde{x}_0^{-1}=\mu_0$. The following stability concepts are useful.

**Definition.** Let $M\subset L^2(J,\mathscr{B}_J,\lambda)$ and $r\geqslant 1$.

a) We say that (3.2) is *$L^r$-uniformly stable with respect to $M$* ($L^r$-u.s. $M$ for short) if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that for all $s\in\mathbf{R}$, $x,y\in M$ with $E(|x-y|^r)\leqslant\delta(\varepsilon)$ we have

$$\sup_{t\geqslant s}E\big(|x_t(s,x)-x_t(s,y)|^r\big)\leqslant\varepsilon.$$

b) We say that (3.2) is *$L^r$-uniformly asymptotically stable with respect to $M$* ($L^r$-u.a.s. $M$ for short) if it is $L^r$-u.s. $M$ and there exists $\delta_0>0$ such that if $s\in\mathbf{R}$, $x,y\in M$ and $E(|x-y|^r)\leqslant\delta_0$ then

$$\lim_{t\to\infty}E\big(|x_t(s,x)-x_t(s,y)|^r\big)=0.$$

c) We say that (3.2) if $L^r$-uniformly asymptotically stable in the large with respect to $M$ ($L^r$-u.a.s.l. $M$) if it is $L^r$-u.s. $M$ and for all $s\in\mathbf{R}$, $x,y\in M$ we have

$$\lim_{t\to\infty}E\big(|x_t(s,x)-x_t(s,y)|^r\big)=0.$$

As in the deterministic case we can use Lyapunov functions in order to prove the stability of (3.2). In this sense we have the following result.

**Proposition 3.2.** *Let $V(t,x,y)\in C^{1,2}(\mathbf{R}\times\mathbf{R}^{2d})$ be such that $V\geqslant 0$,*

$$V(t,0,0)=0,\quad\left|\frac{\partial V}{\partial x}(t,x,y)\right|\leqslant\delta|x|^p,\quad\left|\frac{\partial V}{\partial y}(t,x,y)\right|\leqslant\delta|x|^p$$

*for some $\delta>0$, $0<p\leqslant 1$ and for all $t$, $x$, $y$.*

a) *Assume that*

$$\alpha|x-y|^r\leqslant V(t,x,y)\leqslant\beta|x-y|^r\tag{3.4}$$

*for some $\alpha,\beta>0$, $r\geqslant 1$ and for all $t$, $x$, $y$;*

$$\begin{aligned}LV(t,x,y)&=\frac{\partial V}{\partial x}(t,x,y)+\Big\langle F(t,x),\frac{\partial V}{\partial x}(t,x,y)\Big\rangle+\Big\langle F(t,y),\frac{\partial V}{\partial y}(t,x,y)\Big\rangle\\&\quad+\frac{1}{2}Tr\Big\{G^*(t,x)\frac{\partial^2 V}{\partial x^2}(t,x,y)G(t,x)\Big\}\\&\quad+Tr\Big\{G^*(t,x)\frac{\partial^2 V}{\partial x\partial y}(t,x,y)G(t,x)\Big\}+\frac{1}{2}Tr\Big\{G^*(t,y)\frac{\partial^2 V}{\partial y^2}(t,x,y)G(t,y)\Big\}\\&\leqslant 0\quad\text{for all } t,\ x,\ y.\end{aligned}$$

Then (3.2) *is* $L^r$-u.s. $L^r(J,\mathcal{B}_J,\lambda)$.

b) *Assume that for all* $t$, $x$, $y$

$$V(t,x,y) \geqslant \alpha|x-y|^r \quad \text{for some } \alpha > 0, \tag{3.6}$$

$$LV(t,x,y) \leqslant -\beta|x-y|^r \quad \text{for some } \beta > 0, \tag{3.7}$$

*Then* (3.2) *is* $L^r$-u.a.s.l. $L^r(J,\mathcal{B}_J,\lambda)$.

Proof. a) Let $s \in \mathbf{R}$ and $x, y \in L^r(J,\mathcal{B}_J,\lambda)$. By Ito's formula we have

$$\begin{aligned} E\big(|x_t(s,x)-x_t(s,y)|^r\big) &\leqslant \frac{1}{\alpha}E\big[V\big(t,x_t(s,x)-x_t(s,y)\big)\big] \\ &\leqslant \frac{1}{\alpha}E[V(s,x,y)] \leqslant \frac{\beta}{\alpha}E(|x-y|^r). \end{aligned}$$

b) As in a) we obtain

$$E\big(|x_t(s,x)-x_t(s,y)|^r\big) \leqslant \alpha_1 \exp\{-\beta_1(t-s)\}E(|x-y|^r) \quad \text{for } \alpha_1, \beta_1 > 0.$$

□

Example. We consider the semi-linear Ito equation

$$dx_t = [A(t)x_t + F_1(t,x_t)]\,dt + \sum_{j=1}^{d}[B_j(t)x_t + G_j(t,x_t)]\,dw_t^j, \quad t \in \mathbf{R}, \tag{3.8}$$

where $A\colon \mathbf{R} \to \mathbf{R}^d$, $B_j\colon \mathbf{R} \to \mathbf{R}^d \otimes \mathbf{R}^d$ are bounded and measurable and $F_1$, $G_j$ satisfy

($i_1$) $F_1(.,0)$, $G_j(.,0)$ are bounded,

($i_2$) $|F_1(t,x)-F_1(t,y)| + \sum_{j=1}^{d}|G_j(t,x)-G_j(t,y)| \leqslant \gamma|x-y|$ for some $\gamma > 0$ and for all $t$, $x$, $y$.

**Proposition 3.3.** *If the linear part in* (3.8) *is exponentially stable in mean square and* $\gamma$ *is small enough then* (3.8) *is* $L^2$-u.a.s.l. $L^2(J,\mathcal{B}_J,\lambda)$.

Proof. Choose a quadratic form $W(t,x) = \langle W(t)x, x\rangle$ such that $\alpha|x\|^2 \leqslant W(t,x) \leqslant \beta|x|^2$ for some $\alpha, \beta > 0$ and for all $t$, $x$, $LW(t,x) = -|x|^2$, where $L$ is the parabolic operator associated with the linear part of (3.8) (see [5; Theorem 32, pp. 248]). Then $V(t,x,y) = W(t,x-y)$ satisfies the hypotheses of Proposition 3.2 for $\gamma$ small enough. □

The following two theorems represent the main results.

**Theorem 3.4.** *Assume that $F(t,x)$, $G(t,x)$ are $\theta$-periodic in $t$ ($\theta > 0$) and satisfy* (j), (jj). *Suppose that the Ito equation*

$$x_t = x_0 + \int_0^t F(s,x_s)\,\mathrm{d}s + \int_0^t G(s,x_s)\,\mathrm{d}w_s; \quad t \geqslant 0 \tag{3.9}$$

*has an $L^p$-bounded solution ($p > 2$), i.e., $\sup_{t\geqslant 0} E(|x_t|^p) = K < \infty$. If (3.2) is $L^2$-u.s. $M$, where*

$$M = \{\tilde{x}_0 \in L^2(J, \mathscr{B}_J, \lambda);\ E(|\tilde{x}_0|^p) \leqslant K\},$$

*then the mapping $t \to \tilde{P} \circ x_t^{-1}$ is a.a.p. Moreover, there is a unique a.p. function $\hat{\mu}\colon \mathbf{R} \to \mathrm{Pr}(\mathbf{R}^d)$ such that*

$$\sup_t \int |x|^p \,\mathrm{d}\hat{\mu}_t(x) \leqslant K, \tag{3.10}$$

$$\mu_t(s, \hat{\mu}_s) = \hat{\mu}_t \quad \text{for } t \geqslant s \tag{3.11}$$

$$\lim_{t\to\infty} d_{BL}(\tilde{P} \circ x_t^{-1}, \hat{\mu}_t) = 0. \tag{3.12}$$

**Theorem 3.5.** *Assume the hypotheses of Theorem 3.4 are satisfied. If (3.2) is $L^2$-u.a.s. $M$ then there exists $\hat{\mu}\colon \mathbf{R} \to \mathrm{Pr}(\mathbf{R}^d)$ which satisfies (3.10)–(3.12) and is $m\theta$-periodic for an integer $m > 0$. Moreover, if (3.2) is $L^2$-u.a.s.l. $M$ then $\hat{\mu}$ can be taken $\theta$-periodic.*

Proof of Theorem 3.4. The $L^p$-boundedness implies that the family $(\tilde{P} \circ x_t^{-1})_{t\geqslant 0}$ is relatively compact. Let $0 \leqslant r_{k'} = m_{k'}\theta \to \infty$ and choose a subsequence $r_k = m_k\theta$ and $\tilde{x}^k, \tilde{x} \in M$ such that

a) $\tilde{P} \circ x_{r_k}^{-1} \Rightarrow \mu$ and $\tilde{x}^k \to \tilde{x}$ $\tilde{P}$-a.s.,

b) $\tilde{P} \circ [\tilde{x}^k]^{-1} = \tilde{P} \circ x_{r_k}^{-1}$ for all $k$ (by Skorokhod's theorem; [4; Theorem 2.7] or [9; pp. 10]).

In particular, $\tilde{x}^k$ and $\mathscr{B}(w_t; t \in \mathbf{R})$ are independent ($\tilde{x}^k$, $\tilde{x}$ are extended onto $\tilde{\Omega}$ in an obvious manner) and $E(|\tilde{x}^k - \tilde{x}|^2) \to 0$. Let $(\tilde{x}_t^k)_{t\geqslant 0}$, $(\tilde{x}_t)_{t\geqslant 0}$ be solutions on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ of the equations

$$\tilde{x}_t^k = \tilde{x}^k + \int_0^t F(s,\tilde{x}_s^k)\,\mathrm{d}s + \int_0^t G(s,\tilde{x}_s^k)\,\mathrm{d}w_s; \quad t \geqslant 0, \tag{3.13}$$

$$\tilde{x}_t = \tilde{x} + \int_0^t F(s,\tilde{x}_s)\,\mathrm{d}s + \int_0^t G(s,\tilde{x}_s)\,\mathrm{d}w_s; \quad t \geqslant 0, \tag{3.14}$$

For $\varepsilon > 0$ choose $\delta(\varepsilon) > 0$ and $k_0$ such that $E(|\tilde{x}_t^k - \tilde{x}_t|^2) \leqslant \delta(\varepsilon)$ if $k \geqslant k_0$. Then the $L^2$-u.s. yields $\sup_{t\geqslant 0} E(|\tilde{x}_t^k - \tilde{x}|_t^2) \leqslant \varepsilon$ and

$$\begin{aligned} d_{BL}(\tilde{P} \circ x_{t+r_k}^{-1}, \tilde{P} \circ \tilde{x}_t^{-1}) &= d_{BL}(\tilde{P} \circ [\tilde{x}_t^k]^{-1}, \tilde{P} \circ \tilde{x}_t^{-1}) \\ &\leqslant \{E(|\tilde{x}_t^k - \tilde{x}_t|^2)\}^{1/2} \leqslant \varepsilon^{1/2} \quad \text{for } t \geqslant 0,\ k \geqslant 0, \end{aligned}$$

so that the mapping $t \to \tilde{P} \circ x_t^{-1}$ is a.a.p. by Corollary 2.4. Next we prove that for every sequence $\{r_{k'}\} \subset E_\theta^+$ we find a subsequence $\{r_k\}$ and an a.p. function $\hat{\nu}_t$: $\mathbf{R} \to \Pr(\mathbf{R}^d)$ satisfying (3.10) and such that for every $\alpha \in \mathbf{R}$

$$\sup_{t\geqslant\alpha} d_{BL}(\tilde{P} \circ x_{t+r_k}^{-1}, \hat{\nu}_t) \to 0 \quad \text{as } k \to \infty. \tag{3.15}$$

For every $q \in E_\theta^+$ we take a subsequence $\{r_{k''}^q\} \subset \{r_{k'}\}$, $\{r_{k''}^{q+1}\} \subset \{r_{k''}^q\}$ and $\hat{x}_{k''}^q, \hat{x}^q \in M$ such that

(a$_1$) $\hat{x}_{k''}^q \to \hat{x}^q$ $\tilde{P}$-a.s. In particular,

$$\lim_{k''\to\infty} E(|\hat{x}_{k''}^q - \hat{x}^q|^2) = 0;$$

(a$_2$) $\tilde{P} \circ [\hat{x}_{k''}^q]^{-1} = \tilde{P} \circ x_{r_{k''}-q}^{-1}$.

By the diagonal procedure we select a subsequence $\{r_k\} \subset \{r_{k'}\}$ such that

$$\lim_{k\to\infty} E(|\hat{x}_k^q - \hat{x}^q|^2) = 0 \quad \text{for all } q \in E_\theta^+. \tag{3.16}$$

Let $(\hat{x}_t^{q,k})_{t\geqslant -q}$ be the solution of the equation

$$x_t = \hat{x}_k^q + \int_{-q}^t F(s, x_s)\, ds + \int_{-q}^t G(s, x_s)\, dw_s; \quad t \geqslant -q. \tag{3.17}$$

Consider a continuous adapted process $(\hat{x}_t^q)_{t\geqslant -q}$ such that

$$\hat{x}_t^q = \hat{x}^q + \int_{-q}^t F(s, \hat{x}_s^q)\, ds + \int_{-q}^t G(s, \hat{x}_s^q)\, dw_s; \quad t \geqslant -q. \tag{3.18}$$

By $L^2$-u.s. we have

$$\lim_{k\to\infty} \sup_{t\geqslant -q} E(|\hat{x}_t^{q,k} - \hat{x}_t^q|^2) = 0. \tag{3.19}$$

Next we show that if $t \geqslant -q > -(q+1)$ then

$$\tilde{P} \circ [\hat{x}_t^{q+1}]^{-1} = P \circ [\hat{x}_t^q]^{-1}. \tag{3.20}$$

Indeed, we have

$$\tilde{P} \circ x_{r_k-q}^{-1} \Rightarrow \tilde{P} \circ [\hat{x}^q]^{-1} \quad (\text{by } a_1,\ a_2), \tag{3.21}$$

$$\tilde{P} \circ [x_{-q}^{q+1,k}]^{-1} \Rightarrow \tilde{P} \circ [\hat{x}_{-q}^{q+1}]^{-1} \quad (\text{by } (3.19)), \tag{3.22}$$

$$\hat{x}_{-q}^{q+1,k} = \hat{x}_k^{q+1} + \int\limits_{-q-1}^{-q} F(s, \hat{x}_s^{q+1,k})\, ds + \int\limits_{-q-1}^{-q} G(s, \hat{x}_s^{q+1,k})\, dw_s, \tag{3.23}$$

$$\tilde{P} \circ [\hat{x}_k^{q+1}]^{-1} = P \circ x_{r_k-q-1}^{-1}, \tag{3.24}$$

$$x_{r_k-q} = x_{r_k-q-1} + \int\limits_{-q-1}^{-q} F(s, x_{r_k+s})\, ds + \int\limits_{-q-1}^{-q} G(s, x_{r_k+s})\, dw_s. \tag{3.25}$$

From (3.23)–(3.25) and from the uniqueness in distribution we deduce that

$$\tilde{P} \circ [\hat{x}_{-q}^{q+1,k}]^{-1} = \tilde{P} \circ x_{r_k-q}^{-1}. \tag{3.26}$$

Now from (3.21), (3.22), (3.26) we obtain (3.20).

We can define $\hat{\nu}_t = \tilde{P} \circ [\hat{x}_t^q]^{-1}$ if $t \geqslant -q$.

For $\alpha \in \mathbf{R}$ we have $\alpha \geqslant -q$ for some $q \in E_\theta^+$, hence

$$\begin{aligned}\sup_{t \geqslant \alpha} d_{BL}(\tilde{P} \circ x_{t+r_k}^{-1}, \hat{\nu}_t) &\leqslant \sup_{t \geqslant -q} d_{BL}(\tilde{P} \circ [\hat{x}_t^{q,k}]^{-1}, \tilde{P} \circ [\hat{x}_t^q]^{-1}) \\ &\leqslant \sup_{t \geqslant -q} \{E(|\hat{x}_t^{q,k} - \hat{x}_t^q|^2)\}^{1/2} \to 0 \quad \text{as } k \to \infty \text{ (by (3.19))},\end{aligned}$$

and this proves (3.15). We show that $\hat{\nu}$ is a.p. For $\varepsilon > 0$ let $T(\varepsilon)$, $m(\varepsilon)$ be such that any interval of the length $m(\varepsilon)$ contains a $\tau$ with

$$d_{BL}(\tilde{P} \circ x_{t+\tau}^{-1}, \tilde{P} \circ x_t^{-1}) \leqslant \varepsilon \quad \text{if} \quad t \geqslant T(\varepsilon), \quad t + \tau \geqslant T(\varepsilon)$$

($\tilde{P} \circ x_t^{-1}$ is a.a.p.).

Then

$$d_{BL}(\tilde{P} \circ x_{t+\tau+r_k}^{-1}, \tilde{P} \circ x_{t+r_k}^{-1}) \leqslant \varepsilon \quad \text{if} \quad t, t + \tau \geqslant T(\varepsilon) - r_k.$$

For arbitrary $r \in \mathbf{R}$ we have $t, t + \tau \geqslant T(\varepsilon) - r_k$ for $k$ large enough, so that $d_{BL}(\tilde{P} \circ x_{t+\tau+r_k}^{-1}, \tilde{P} \circ x_{t+r_k}^{-1}) \leqslant \varepsilon$ and hence $d_{BL}(\hat{\nu}_{t+\tau}, \hat{\nu}_t) \leqslant \varepsilon$, i.e. $\hat{\nu}$ is a.p. By Fatou's lemma we obtain (3.10) for $\hat{\nu}$.

Since we have

$$\hat{x}_{t-r_k}^{q-r_k} = \hat{x}_0^{q+r_k} + \int_{-q}^{t} F(s, \hat{x}_{s-r_k}^{q+r_k})\, ds + \int_{-q}^{t} G(s, \hat{x}_{s-r_k}^{q+r_k})\, d(w_{s-r_k} - w_{-r_k})$$

for $t \geqslant -q$, we obtain as above an a.p. function $\hat{\mu}: \mathbf{R} \to \Pr(\mathbf{R}^d)$ such that

$$(3.27) \qquad \lim_{t\to\infty} \sup_{t\geqslant -q} d_{BL}(\hat{\nu}_{t-r_k}, \hat{\mu}_t) = 0 \quad \text{for all } q \in E_\theta^+$$

(if necessary we take a subsequence of $\{r_k\}$) and such that for $t \geqslant -q$, $(\hat{\mu}_t)_{t\geqslant -q}$ is generated by a solution of the equation

$$(3.28) \qquad \hat{y}_t^q = \hat{y}^q + \int_{-q}^t F(s, \hat{y}_s^q)\,\mathrm{d}s + \int_{-q}^t G(s, \hat{y}_s^q)\,\mathrm{d}w_s; \quad t \geqslant -q$$

in the sense that $\tilde{P} \circ [\hat{y}_t^q]^{-1} = \hat{\mu}_t$ if $t \geqslant -q$. By Fatou's lemma it is clear that (3.10) holds for $\hat{\mu}$.

From (3.28) we have

$$\hat{y}_t^q = \hat{y}_s^q + \int_s^t F(u, \hat{y}_u^q)\,\mathrm{d}u + \int_s^t G(u, \hat{y}_u^q)\,\mathrm{d}w_u; \quad \text{for } t \geqslant s \geqslant -q,$$

where form we obtain (3.11) by the Chapman-Kolmogorov relation. Finally, by the $L^2$-u.s. we deduce that

$$\sup_{t\geqslant r_k} d_{BL}(\tilde{P} \circ x_t^{-1}, \hat{\mu}_t) \leqslant \sup_{t\geqslant r_k} d_{BL}(\tilde{P} \circ [x_{t-r_k}^{0,k}]^{-1}, \quad \tilde{P} \circ [\hat{y}_t^0]^{-1} \to 0$$

as $k \to \infty$. The proof is now complete. $\square$

Proof of Theorem 3.5. Assume that (3.2) is $L^2$-u.a.s. $M$. Choose a subsequence $\{r_{k(l)}\}_l = \{m_{k(l)}\theta\}_l$ of $\{r_k\}$ and $\hat{z}^1$, $\hat{z}$ such that $\lim_{l\to\infty} E(|\hat{z}^1 - \hat{z}|^2) = 0$ and $\tilde{P} \circ [\hat{z}^l]^{-1} = \tilde{P} \circ \left[\hat{y}_{r_{k(l)}}^0\right]^{-1}$ for all $l$. Let $l$, $q$ be such that

$$E(|\hat{z}^l - \hat{z}^q|^2) \leqslant \delta_0, \qquad k(q) > k(l),$$

where $\delta_0$ is given by the $L^2$-u.a.s. Let $(\hat{z}_t^l)_{t\geqslant 0}$ be the solution of the equation

$$\hat{z}_t^l = \hat{z}^l + \int_0^t F(s, \hat{z}_s^l)\,\mathrm{d}s + \int_0^t G(s, \hat{z}_s^l)\,\mathrm{d}w_s$$

and $m = k(q) - k(l)$. Then we have

$$\begin{aligned} d_{BL}(\hat{\mu}_{t+m\theta+k(l)\theta}, \hat{\mu}_{t+k(l)\theta}) &= d_{BL}(\tilde{P} \circ [\hat{z}_t^q]^{-1}, \tilde{P} \circ [\hat{z}_t^l]^{-1}) \\ &\leqslant \{E(|\hat{z}_t^q - \hat{z}_t^l|^2)\}^{1/2} \to 0 \quad \text{as } t \to \infty \text{ by the } L^2\text{-u.a.s.,} \end{aligned}$$

since $E(|\hat{z}_0^q - \hat{z}_0^l|^2) = E(|\hat{z}^q - \hat{z}^l|^2) \leqslant \delta_0$.

Because $\hat{\mu}$ is a.p. we obtain from Lemma 2.7 that $\hat{\mu}_t = \hat{\mu}_{t+m\theta}$ for all $t \in \mathbf{R}$, i.e., $\hat{\mu}$ is $m\theta$-periodic.

Assume now that (3.2) is $L^2$-u.a.s. Reasoning as above we obtain that $\lim_{t\to\infty} d_{BL}(\hat{\mu}_{t+\theta}, \hat{\mu}_t) = 0$, where from $\hat{\mu}_t = \hat{\mu}_{t+\theta}$ for all $t \in \mathbf{R}$. The proof is finished. $\square$

**Corollary 3.6.** *Under the assumptions of Theorem 3.4 the functions $t \to E(x_t)$, $t \to \operatorname{cov}(x_t)$ are a.a.p. and the functions $t \to E(\hat{\mu}_t)$, $t \to \operatorname{cov}(\hat{\mu}_t)$ are a.p. Moreover, if (3.2) is $L^2$-u.a.s. ($L^2$-u.a.s.l.) then the mappings $t \to E(\hat{\mu}_t)$, $t \to \operatorname{cov}(\hat{\mu}_t)$ are $m\theta$-periodic ($\theta$-periodic) for some integer $m > 0$.*

Proof. It follows from Theorems 3.4 and 3.5 and the continuity of the mappings

$$\mu \in \hat{M} = \left\{\mu \in \Pr(\mathbf{R}^d); \int |x|^p \,\mathrm{d}\mu(x) \leqslant K\right\} \to E(\mu),$$
$$\mu \in \tilde{M} \to \operatorname{cov}(\mu).$$

□

Remark 3.7. For $s \in \mathbf{R}$ fixed and for all $t \geqslant s$ the flow $\hat{\mu}$ from Theorems 3.4, 3.5 is generated by solution $\{\hat{x}_t(s, \hat{x})\}_{t \geqslant s}$, where $\hat{x}$ is a random variable defined on $J$.

Remark 3.8. Sufficient conditions in terms of Lyapunov functions for the boundedness of solutions of Ito equations are given in [6].

Remark 3.9. The existence of a.a.p. solutions for Ito equations with a.a.p. coefficients is established in [12] under a stronger concept of stability (called total stability).

Remark 3.10. A careful inspection of the proofs of Theorems 3.4, 3.5 shows that in fact it is sufficient to assume the stability properties with the $L^p$-metric replaced by $d_{BL}$ (stability in distribution). It seems that for problems concerning the one-dimensional distributions (considered here) such stability in distribution is more natural and allows to consider Ito equations under the hypotheses on the existence and uniqueness in distribution of weak solutions (a wide class of solutions).

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