Bohdan Zelinka

Locally regular graphs

*Mathematica Bohemica*, Vol. 125 (2000), No. 4, 481–484

Persistent URL: http://dml.cz/dmlcz/126271

# LOCALLY REGULAR GRAPHS

BOHDAN ZELINKA, Liberec

(Received November 11, 1998)

*Abstract.* A graph $G$ is called locally $s$-regular if the neighbourhood of each vertex of $G$ induces a subgraph of $G$ which is regular of degree $s$. We study graphs which are locally $s$-regular and simultaneously regular of degree $r$.

*Keywords*: regular graph, locally regular graph

*MSC 1991*: 05C75

At the Czechoslovak Symposium on Graph Theory in Smolenice in 1963 [1] A. A. Zykov suggested the problem to characterize graphs $H$ with the property that there exists a graph $G$ in which the neighbourhood of any vertex induces a subgraph isomorphic to $H$. This problem inspired many mathematical works and led also to a certain generalization, namely the study of local properties of graphs. A graph $G$ is said to have locally a property $P$, if the neighbourhood of each vertex of $G$ induces a subgraph having the property $P$. For locally connected graphs let us mention e.g. [2] and [4], for locally linear graphs e.g. [3]. A survey paper on local properties of graphs was written by J. Sedláček [5].

Here we will study locally $s$-regular graphs. A graph $G$ is called locally $s$-regular, where $s$ is a non-negative integer, if the neighbourhoods of all vertices of $G$ induce subgraphs which are regular of degree $s$, shortly $s$-regular. We consider finite undirected graphs without loops and multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$, the complement of $G$ by $\overline{G}$. If $A \subseteq V(G)$, then $G\langle A\rangle$ is the subgraph of $G$ induced by $A$. The symbol $G_1 + G_2$ denotes the disjoint union of two graphs $G_1, G_2$; the symbol $G_1 \oplus G_2$ denotes the Zykov sum of $G_1$ and $G_2$, i.e. the graph obtained from $G_1 + G_2$ by joining each vertex of $G_1$ with each vertex of $G_2$ by an edge. By $G_1 \times G_2$ the Cartesian product of $G_1$ and $G_2$ is denoted; its vertex set is $V(G_1) \times V(G_2)$ and two vertices $(u_1, u_2)$, $(v_1, v_2)$ are adjacent in it if and only if either $u_1 = v_1$ and $u_2, v_2$ are adjacent in $G_2$, or $u_1, v_1$ are adjacent in $G_1$ and

$u_2 = v_2$. The symbol $N_G(v)$ denotes the (open) neighbourhood of a vertex $v$ in a graph $G$, i.e. the set of all vertices which are adjacent to $v$ in $G$. By $C_n$ we denote the circuit of length $n$.

By $\mathrm{Locreg}(r, s)$, where $r$ is a positive integer and $s$ a non-negative integer, we denote the class of graphs which are simultaneously $r$-regular and locally $s$-regular.

**Proposition 1.** *If* $\mathrm{Locreg}(r, s) \neq \emptyset$, *then* $r \geqslant s+1$ *and at least one of the numbers* $r, s$ *is even.*

*This assertion is evident, because the conditions mentioned are the well-known necessary conditions for the existence of an* $s$*-regular graph with* $r$ *vertices.*

**Proposition 2.** *Let* $s, k$ *be positive integers, let* $k$ *be a divisor of* $s$. *Then* $\mathrm{Locreg}(s + k, s) \neq \emptyset$.

Proof. Let $G$ be the complement of the disjoint union of $s + 2$ copies of the complete graph $K_k$ with $k$ vertices. Then $G \in \mathrm{Locreg}(s + k, s)$. $\square$

**Corollary 1.** $\mathrm{Locreg}(s + 1, s) \neq \emptyset$ *for each integer* $s \geqslant 0$.

**Corollary 2.** $\mathrm{Locreg}(s + 2, s) \neq \emptyset$ *for each even integer* $s \geqslant 0$.

**Proposition 3.** *If* $\mathrm{Locreg}(r_1, s) \neq \emptyset$ *and* $\mathrm{Locreg}(r_2, s) \neq \emptyset$, *then also* $\mathrm{Locreg}(r_1 + r_2, s) \neq \emptyset$.

Proof. If $G_1 \in \mathrm{Locreg}(r_1, s)$ and $G_2 \in \mathrm{Locreg}(r_2, s)$, then the Cartesian product $G_1 \times G_2 \in \mathrm{Locreg}(r_1 + r_2, s)$. $\square$

Now we state two lemmas.

**Lemma 1.** *Let* $p, q$ *be positive integers such that* $q \geqslant p^2 - 1$. *Then there exist non-negative integers* $a, b$ *such that*

$$q = ap + b(p + 1).$$

Proof. Let $b$ be an integer such that $0 \leqslant b \leqslant p - 1$ and $b \equiv q (\mathrm{mod}\ p)$. Let $a = (q - b)/p - b$. We have $ap + b(p + 1) = q$. The number $a$ is an integer, because $q \equiv b (\mathrm{mod}\ p)$. Further we have $q \geqslant p^2 - 1 = (p + 1)(p - 1) \geqslant b(p + 1)$ and thus $a = (q - b)/p - b \geqslant (b(p + 1) - b)/p - b = 0$. This proves the assertion. $\square$

**Lemma 2.** *Let* $p, q$ *be positive integers such that* $q \geqslant (2p - 1)(p - 1)$. *Then there exist non-negative integers* $a, b$ *such that*

$$q = ap + b(2p - 1).$$

Proof. Let $b$ be an integer such that $0 \leqslant b \leqslant p-1$ and $b+q \equiv 0(\bmod p)$. Let $a = (q+b)/p - 2b$. The proof that $a$ is a non-negative integer is analogous to the proof of Lemma 1. □

Now we prove some theorems.

**Theorem 1.** *Let $r, s$ be positive integers such that $s$ is even and $r \geqslant s(s+2)$. Then* $\mathrm{Locreg}(r,s) \neq \emptyset$.

Proof. According to Lemma 1 there exist non-negative integers $a, b$ such that $r = a(s+1) + b(s+2)$. Then the assertion follows from the Corollaries 1 and 2 and from Proposition 3. □

**Theorem 2.** *Let $r, s$ be positive integers such that $r$ is even, $s$ is odd and $r \geqslant s(s-1)$. Then* $\mathrm{Locreg}(r,s) \neq \emptyset$.

Proof. Put $p = \frac{1}{2}(s = 1)$; then $\frac{1}{2}r \geqslant (2p-1)(p-1)$ and, as $r$ is even, according to Lemma 2 there exist non-negative integers $a, b$ such that $\frac{1}{2}r = ap + b(2p-1) = \frac{1}{2}a(s+1) + bs$ and thus $r = a(s+1) + 2bs$. According to Proposition 2 we have $\mathrm{Locreg}(s+1, s) \neq \emptyset$ and $\mathrm{Locreg}(2s, s) \neq 0$ and thus, by Proposition 3, also $\mathrm{Locreg}(r,s) \neq \emptyset$. □

Now we turn our attention to small values of $s$.

**Proposition 4.** *Let $0 \leqslant s \leqslant 2$ and $r \geqslant s+1$ and in the case of $s = 1$ let $r$ be even. Then* $\mathrm{Locreg}(r,s) \neq \emptyset$.

Proof. A graph from $\mathrm{Locreg}(r, 0)$ is an arbitrary $r$-regular graph without triangles, e.g. the complete bipartite graph $K_{r,r}$. A graph from $\mathrm{Locreg}(2,1)$ is $C_3$ and $\mathrm{Locreg}(n,1) \neq \emptyset$ follows from Theorem 1. Examples of graphs from $\mathrm{Locreg}(3,2)$, $\mathrm{Locreg}(4,2)$ and $\mathrm{Locreg}(5,2)$ are successively the graphs of regular polyhedra tetrahedron, octahedron, icosahedron. Every $s \geqslant 6$ is a sum of numbers from $\{3,4,5\}$ and thus Proposition 3 implies $\mathrm{Locreg}(r,2)$ for every $r \geqslant 6$. □

From these results it may seem that $\mathrm{Locreg}(r,s) \neq \emptyset$ for any $r, s$ which satisfy the condition of Proposition 1. We will show an example for which this is not true.

**Theorem 3.** *The class* $\mathrm{Locreg}(7,4) = \emptyset$.

Proof. Suppose the contrary and let $G \in \mathrm{Locreg}(7,4)$. Let $u$ be a vertex of $G$. The graph $G\langle N_G(u)\rangle$ is a 4-regular graph with seven vertices. Its complement is a 2-regular graph and therefore it is isomorphic either to $C_3 + C_4$, or to $C_1$. Hence $G\langle N_g(u)\rangle \cong \overline{C}_3 \oplus \overline{C}_4$ or $G\langle N_G(u)\rangle \cong \overline{C}_1$. Suppose the first case occurs. Denote the vertices of $G\langle N_G(u)\rangle$ by $v_1, v_2, v_3, w_1, w_2, w_3, w_4$ so that

$v_1v_2, v_2v_3, v_3v_1, w_1w_2, w_2w_3, w_3w_4, w_4w_1$ are edges of the complement of $G\langle N_a(u)\rangle$. Consider the graph $G\langle N_G(v_1)\rangle$. It contains the graph $\langle\{w_1, w_2, w_3, w_4\}\rangle \cong \overline{C}_4$ as an induced subgraph and therefore it cannot be isomorphic to $\overline{C}_7$. We have $G\langle N_G(v_1)\rangle \cong \overline{C}_3 \oplus \overline{C}_4$ and there exist vertices $x_1, x_2, x_3$ outside $N_G(u)$ which are pairwise non-adjacent and each of them is adjacent to $v_1, w_1, w_2, w_3, w_4$. (One of them is $u$.) But now $G\langle N_G(w_1)\rangle$ contains two disjoint independent triples $\{v_1, v_2, v_3\}$, $\{x_1, x_2, x_3\}$ and hence it is isomorphic neither to $\overline{C}_3 \oplus \overline{C}_4$ nor to $\overline{C}_7$, which is a contradiction. As $u$ was chosen arbitrarily, we have proved that the neighbourhood of any vertex of $G$ cannot induce $\overline{C}_3 \oplus \overline{C}_4$ and thus it must induce $\overline{C}_7$.

Thus let $G\langle N_G(u)\rangle \cong \overline{C}_7$. The vertices of $G\langle N_G(u)\rangle$ will be denoted by $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ in such a way that $v_iv_{i+1}$ for $i = 1, \ldots, 7$ are edges of the complement of $G\langle N_G(u)\rangle$; here and everywhere in the sequel the subscripts are taken modulo 7. Consider the graph $G\langle N_G(v_i)\rangle$ for an arbitrary $i \in \{1, \ldots, 7\}$. It contains the vertices $u$, $v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}$ and does not contain $v_{i+1}$ and $v_{i+6}$. As $G\langle N_G(v_i)\rangle \cong \overline{C}_7$, it contains vertices $w_i, x_i$ outside $N_6(u)$ such that $w_i$ is adjacent to $v_i, v_{i+3}, v_{i+4}, v_{i+5}, x_i$ and non-adjacent to $u$, $v_{i+2}$ while $x_i$ is adjacent to $v_i, v_{i+2}, v_{i+3}, v_{i+4}, w_i$ and non-adjacent to $u, v_{i+5}$. Consider the vertices $w_i$ for $i = 1, \ldots, 7$. Suppose that $w_i = w_j$ for some $i$ and $j$. This vertex is non-adjacent to $v_{j+2}$ and adjacent to $v_i, v_{i+3}, v_{i+4}, v_{i+5}$ and thus $j + 2 \notin \{i, i+3, i+4, i+5\}$, which implies $j \notin \{i+1, i+2, i+3, i+5\}$. Further, this vertex is non-adjacent to $v_{i+2}$ and adjacent to $v_j, v_{j+3}, v_{j+4}, v_{j+5}$; we have $i \notin \{j+1, j+2, j+3, j+5\}$, which implies $j \notin \{i+2, i+4, i+5, i+6\}$. Therefore $w_i = w_j$ implies $i = j$ and the vertices $w_1, \ldots, w_7$ are pairwise distinct. As $w_i$ is adjacent to $v_i, v_{i+3}, v_{i+4}, v_{i+5}$ for $i = 1, \ldots, 7$, the vertex $v_i$ is adjacent to $w_i, w_{i+2}, w_{i+3}, w_{i+4}$ for $i = 1, \ldots, 7$. Further, it is adjacent to $u, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}$ and thus its degree in $G$ is at least 9, which is a contradiction. This proves the assertion. □

### *References*


[1] Theory of Graphs and its Applications. Proc. Symp. Smolenice, June 1963, Academia, Praha, 1964.

[2] *G. Chartrand, R. J. Gould, A. D. Polimeni*: A note on locally connected and hamiltonian-connected graphs. Israel Math. J. *33* (1979), 5–8.

[3] *D. Fronček*: Locally linear graphs. Math. Slovaca *39* (1989), 3–6.

[4] *D. J. Oberly, D. P. Sumner*: Every connected, locally connected nontrivial graph with no induced claw is hamiltonian. J. Graph Theory *3* (1979), 351–356.

[5] *J. Sedláček*: Local properties of graphs. Časopis Pěst. Mat. *106* (1981), 290–298.



*Author's address*: *Bohdan Zelinka*, Department of Applied Mathematics, Technical University of Liberec, Voroněžská 13, 460 01 Liberec, Czech Republic.