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TOLERANCES ON POSET ALGEBRAS

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Summary. To every partially ordered set a certain groupoid is assigned. A tolerance on it is defined similarly as a congruence, only the requirement of transitivity is omitted. Some theorems concerning these tolerances are proved.

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Partially ordered sets (shortly posets) may be studied by means of algebraic methods. Here we will investigate poset algebras. They are defined analogously to graph algebras which were introduced by G. F. McNulty and C. R. Shallon [1] and described by R. Pöschel [2].

Let a poset $(P, \leqslant)$ be given. The poset algebra $A(P)$ assigned to P is a commutative groupoid with the support $P \cup \{\infty\}$, where ∞ is an element not contained in P, and which has one commutative binary operation $\circ$ defined in the following way. Let a, b be two elements of $P \cup \{\infty\}$. If $a = \infty$ or $b = \infty$, than $a \circ b = \infty$. If $a \in P$, $b \in P$, $a \leqslant b$, then $a \circ b = b$. If $a \in P$, $b \in P$, $a \parallel b$, then $a \circ b = \infty$.

On poset algebras we shall study tolerances. A tolerance on a groupoid G with a binary operation $\circ$ is a reflexive and symmetric binary relation T on G with the property that $(a_1, b_1) \in T$ and $(a_2, b_2) \in T$ imply $(a_1 \circ a_2, b_1 \circ b_2) \in T$. If a tolerance T on G is transitive, it is called a congruence on G.

On $A(P)$ we define an important relation Q in the following way. We have $(a, b) \in Q$ if and only if either $a = b$, or a and b are comparable, the interval determined by them is a chain and each $x \in P$ not belonging to that interval is comparable with a if and only if it is comparable with b.

Theorem 1. *The relation Q is a congruence on $A(P)$.*

Proof. The reflexivity and symmetry of Q are evident. We shall prove the transitivity. Let $(a,b) \in Q$, $(b,c) \in Q$. If $a = b$ or $b = c$, then evidently $(a,b) \in Q$; thus suppose that $a \neq b$, $b \neq c$. The elements a, b must be comparable and so must b, c. Suppose $a \leqslant b$, $b \leqslant c$. Then $a \leqslant c$ and the elements a, c are comparable. The intervals $[a,b]$, $[b,c]$ are chains and so is their union. Suppose that the interval $[a,c]$ contains an element y which is not in $[a,b] \cup [b,c]$. Then $y \parallel b$. As $(a,b) \in Q$, $(b,c) \in Q$, this implies $y \parallel a$, $y \parallel c$, which is a contradiction with the assumption that $y \in [a,c]$. Hence $[a,c] = [a,b] \cup [b,c]$ and it is a chain. Now let $x \in P - [a,c]$. Then $x \in P - [a,b]$ and thus x is comparable with a if and only if it is comparable with b. As also $x \in P - [b,c]$, the element x is comparable with b if and only if it is comparable with c. Hence x is comparable with a if and only if it is comparable with c; we have $(a,c) \in Q$. If $a \geqslant b$, $b \geqslant c$, the proof is analogous. Suppose $a \leqslant b$, $b \geqslant c$. If $c \in [a,b]$, then c is comparable with a and $[a,c] \subseteq [a,b]$, therefore $[a,c]$ is a chain. If $c \notin [a,b]$, then, as $(a,b) \in Q$ and c is comparable with b, also c is comparable with a. As $[a,b]$ is a chain, $c \notin [a,b]$ and $c \leqslant b$, we have $c \leqslant a$. As $[c,a] \subseteq [c,b]$, the interval $[c,a]$ is a chain. We have $(a,c) \in Q$. If $a \geqslant b$, $b \leqslant c$, the proof is analogous. We have proved that Q is an equivalence relation.

Now let $(a_1,b_1) \in Q$, $(a_2,b_2) \in Q$. If $a_1 \parallel a_2$, then $a_1 \parallel b_2$, $b_1 \parallel b_2$ and $(a_1 \circ a_2, b_1 \circ b_2) = (\infty,\infty) \in Q$. If $a_1 \leqslant a_2$, then a_1 is comparable with b_2 and b_2 is comparable with b_1. If $b_1 \leqslant b_2$, then $(a_1 \circ a_2, b_1 \circ b_2) = (a_2,b_2) \in Q$. If $b_1 \geqslant b_2$, then $(a_1 \circ a_2, b_1 \circ b_2) = (a_2,b_1)$. We shall investigate this pair of elements. As a_2 is comparable with a_1, it is comparable also with b_1. If $a_2 \leqslant b_1$, then we have the interval $[a_2,b_1]$. This interval is a subset of $[a_1,b_1]$; as $[a_1,b_1]$ is a chain, so is $[a_2,b_1]$. Let $x \in P - [a_2,b_1]$. If $x \parallel b_1$, then $x \in P - [a_1,b_1]$ and also $x \parallel a_1$. The comparability of x with a_2 would imply its comparability with a_1 or b_1, which would be a contradiction. If $x > b_1$, then also $x > a_2$. If $x < b_1$, then either $x \leqslant a_1 \leqslant a_2$, or (as $x \in P - [a_2,b_1]$ and $[a_1,b_1]$ is a chain) $a_1 \leqslant x < a_2$. We have proved that x is comparable with a_2 if and only if it is comparable with b_1 and $(a_2,b_1) \in Q$. If $b_1 \leqslant a_2$, we have the interval $[b_1,a_2]$. This interval is a subset of b_2, a_2; as $[b_2,a_2]$ is a chain, so is $[b_1,a_2]$. If $x \in P - [b_1,a_2]$, we can prove that x is comparable with a_2 if and only if it is comparable with b_1; we do it analogously to the preceding case. We proceed analogously also in the case when $a_1 \geqslant a_2$. Thus it is proved that Q is a congruence on $A(P)$. □

Theorem 2. *Let T be a tolerance on $A(P)$ such that $(x,\infty) \in T$ implies $x = \infty$. Then $T \subseteq Q$.*

Proof. Suppose that there exist elements a, b of P such that $(a,b) \in T$ and $(a,b) \in Q$. Then either one of the elements a, b is ∞ while the order is not, or $a \in P$, $b \in P$, $a \parallel b$, or $a \in P$, $b \in P$, a and b are comparable and the interval determined by them is not a chain, or $a \in P$, a and b are comparable and there exists an element $x \in P$ comparable with one of them and not with the other. In the first case it is clear that T does not fulfil the condition of the theorem. If $a \parallel b$, then from $(a,b) \in T$,

$(a,a) \in T$ we obtain $(a \circ a, b \circ a) = (a, \infty) \in T$. As $a \in P$, we see that T does not fulfil the condition of the theorem. Now suppose that the third case occurs. Without loss of generality let $a < b$. Then the interval $[a,b]$ contains two incomparable elements x, y. From $(a,b) \in T$, $(x,x) \in T$ we obtain $(a \circ x, b \circ x) = (x,b) \in T$ and analogously $(y,b) \in T$. Now from $(x,b) \in T$, $(y,b) \in T$ we have $(x \circ y, b \circ b) = (\infty, b) \in T$ and also $(b,\infty) \in T$. Again T does not fulfil the condition of the theorem. In the fourth case let again $a < b$; then $[a,b]$ is a chain. Let there exist $x \in P$ such that x is comparable with a and not with b. Then $a \circ x = a$ or $a \circ x = x$, while $b \circ x = \infty$. We can proceed analogously in the case when x is comparable with b and not with a. Again T does not fulfil the condition of the theorem. Hence $T \subseteq Q$. □

Corollary 1. *Let C be a congruence on $A(P)$, one of whose classes is $\{\infty\}$. Then all other classes of C are intervals on P such that each element of P not belonging to that interval is either comparable with all of its elements, or incomparable with all of them.*

If a, b are two elements of $P \cup \{\infty\}$, then by $T(a,b)$ (or $C(a,b)$) we denote the intersection of all tolerances (or congruences, respectively) on $A(P)$ which contain the pair (a,b). By Δ we denote the identity relation on $P \cup \{\infty\}$.

Theorem 3. *Let $a \in P$. Then $T(a,\infty)$ consists of Δ and of all pairs (x,∞), (∞,x) with $x \geqslant a$.*

P r o o f is easy. □

Corollary 2. *Let $a \in P$. Then one class of $C(a,\infty)$ consists of all elements greater that or equal to a, and all its other classes are one-element sets.*

Theorem 4. *Let a, b be two incomparable elements of P. Then $T(a,b)$ consists of Δ, of the pairs (a,b), (b,a), and of the pairs (x,∞), (∞,x) for all $x \geqslant a$ and for all $x \geqslant b$.*

P r o o f. From $(a,b) \in T(a,b)$, $(a,a) \in T(a,b)$ we obtain $(a \circ a, b \circ b) = (a,\infty) \in T(a,b)$ and, by symmetry, $(\infty,a) \in T(a,b)$. Analogously (b,∞) and (∞,b) belong to $T(a,b)$. If $x \geqslant a$, then from $(a,\infty) \in T(a,b)$, $(x,x) \in T(a,b)$ we have $(a \circ x, \infty \circ x) = (x,\infty) \in T(a,b)$; analogously for $x \geqslant b$. Therefore $T(a,b)$ contains all the pairs mentioned. Now it is easy to verify that the relation described is really a tolerance on $A(P)$. □

Corollary 3. *Let a, b be two incomparable elements of P. Then one class of $C(a,b)$ consists of all elements $x \geqslant a$ and all elements $x \geqslant b$, and all its other classes are one-element sets.*

Theorem 5. *Let a, b be two elements of P such that $a < b$, $(a,b) \in Q$. Then $P(a,b)$ consists of Δ and of all pairs (x,b), (b,x) for $x \in [a,b]$.*

Proof. Let $x \in [a,b]$. Then from $(a,b) \in T(a,b)$, $(x,x) \in T(a,b)$ we have $(a \circ x, b \circ x) = (x,b) \in T(a,b)$ and, by symmetry, $(b,x) \in T(a,b)$. Now it is easy to verify that the relation descirbed is really a tolerance on $A(P)$. □

Corollary 4. *Let a, b be two elements of P such that $a < b$, $(a,b) \in Q$. Then one class of $C(a,b)$ is $[a,b]$ and all its other classes are one-element sets.*

We will not describe $T(a,b)$ for the other cases. We mention only that in the case when $a < b$ and $(a,b) \in Q$ the tolerance $T(a,b)$ contains all pairs described in Theorem 5 and, moreover, some pairs of the form (x,∞) and (∞,x).

In the end we will define another important relation R on P. If a, b are two elements of P, then $(a,b) \in R$ if and only if either $a = b$, or $a \parallel b$ and each element $x \in P - \{a,b\}$ is comparable with a if and only if it is comparable with b.

Theorem 6. *The relation R is an equivalence on P.*

Proof. The reflexivity and symmetry of R are obvious. Let $(a,b) \in R$, $(b,c) \in R$. If $a = b$ or $b = c$ or $c = a$, then evidently $(a,c) \in R$. Suppose that a, b, c are pairwise distinct. We have $c \in P - \{a,b\}$ and $c \parallel b$, therefore also $c \parallel a$. Now let $x \in P - \{a,c\}$. If $x \neq b$, then $x \in P - \{a,b\}$ and x is comparable with a if and only if it is comparable with b. Also $x \in P - \{b,c\}$ and x is comparable with b if and only if it is comparable with c. Hence x is comparable with a if and only if it is comparable with c. If $x = b$, then $x \parallel a$, $x \parallel c$. Therefore $(a,c) \in R$ is transitive. □

By R^* we denote the relation on $P \cup \{\infty\}$ consisting of R, of the pair (∞,∞) and of all pairs (x,∞), (∞,x) with x such that there exist elements y, z of P such that $y \leqslant z$, $y \parallel x$ and $(y,z) \in R$.

Theorem 7. *The relation R^* is a tolerance on $A(P)$.*

Proof. The relation R^* is evedently reflexive and symmetric. Let $(a_1,b_1) \in R^*$, $(a_2,b_2) \in R^*$. If some of the elements a_1, a_2, b_1, b_2 is ∞, then it is clear that $(a_1 \circ a_2, b_1 \circ b_2) \in R^*$. Thus suppose that all these elements are in P. Also the cases $a_1 = b_1$ and $a_2 = b_2$ are simple; thus suppose $a_1 \neq b_1$, $a_2 \neq b_2$. If $a_2 = b_1$, then $(a_1 \circ a_2, b_1 \circ b_2) = (\infty,\infty) \in R^*$. If $b_1 = b_2$, then $(a_1 \circ a_2, b_1 \circ b_2) = (\infty, b_1) \in R^*$. Similarly in the cases $a_1 = a_2$ and $a_1 = b_2$. Now suppose that a_1, a_2, b_1, b_2 are pairwise distinct. If $a_1 \parallel a_2$, then $a_1 \parallel b_2$, $b_2 \parallel b_1$ and $(a_1 \circ a_2, b_1 \circ b_2) = (\infty,\infty) \in R^*$. If $a_1 > a_2$, then a_1 is comparable with b_2 and a_2 is comparable with b_1. We have $a_1 > b_2$; otherwise $a_2 < a_1 < b_2$ and a_2, b_2 would be comparable. Further, b_1 is comparable with b_2 and also $b_1 > b_2$; otherwise $b_1 < b_2 < a_1$. Therefore $(a_1 \circ a_2, b_1 \circ b_2) = (a_1, b_1) \in R^*$. Analogously if $a_1 < a_2$. Hence R^* is a tolerance on $A(P)$. □

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