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LINEAR OPERATORS IN THE SPACE OF REGULATED FUNCTIONS

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Summary. Representation of bounded and compact linear operators in the Banach space of regulated functions is given in terms of Perron-Stieltjes integral.

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This note deals with some functional analytic properties of linear operators on spaces of regulated functions. The results are based on the recent work [2] of Milan Tvrdý where the fundamental properties of the Perron-Stieltjes integral are considered and used for studying certain concepts of functional analysis on the space of regulated functions. Our goal is to give a representation theorem for bounded and compact linear operators defined on the space $G_L(a,b)$ of regulated functions on $[a,b]$ which are continuous from the left in the open interval (a,b), and with values in the space $G(c,d)$ of regulated functions on $[a,b]$. Let us recall some fundamental concepts which form the background of our subsequent consideration. The notation introduced in [2] is used.

Assume that $-\infty < a < b < +\infty$. A function $f\colon [a,b] \to \mathbf{R}$ is said to be regulated on $[a,b]$ if the onesided limits

$$f(t+) = \lim_{\tau \to t+} f(\tau), \ t \in [a,b),$$
$$f(t-) = \lim_{\tau \to t-} f(\tau), \ t \in (a,b]$$

exist. The set of all regulated functions on $[a,b]$ is denoted by $G(a,b)$. $G(a,b)$ is a linear space. Given $f \in G(a,b)$ we define

$$\|f\|_{G(a,b)} = \sup_{t \in [a,b]} |f(t)| < \infty.$$

$f \mapsto \|f\|_{G(a,b)}$ is a norm on $G(a,b)$ and it is known (see e.g. [1]) that $G(a,b)$ with this norm is a Banach space.

The subset $G_L(a,b)$ consisting of all regulated functions f on $[a,b]$ which are continuous from the left on (a,b) forms a closed linear subset in $G(a,b)$, consequently $G_L(a,b)$ with the norm $\|.\|_{G_L(a,b)}$ given by

$$\|f\|_{G_L(a,b)} = \|f\|_{G(a,b)} \quad \text{for} \quad f \in G_L(a,b)$$

is also a Banach space.

Let us denote by $S(a,b) \subset G(a,b)$ the set of all finite step functions on $[a,b]$. It is known (see [1]) that $S(a,b)$ is dense in $G(a,b)$, i.e. $G(a,b)$ is the closure of $S(a,b)$ with respect to the topology given by the norm $\|.\|_{G_L(a,b)}$. This yields that

the set $S(a,b) \cap G_L(a,b)$ *is dense in* $G_L(a,b)$.

Indeed, if $f \in G_L(a,b) \subset G(a,b)$ then for every $\varepsilon > 0$, $s \in (a,b)$ there is a $\delta(s) > 0$ such that

$$|f(\sigma) - f(s)| < \varepsilon$$

for $\sigma \in (s-\delta(s), s) \cap (a,b)$ and by the density of $S(a,b)$ in $G(a,b)$ there is $\varphi \in S(a,b)$ such that

$$|f(s) - \varphi(s)| \leqslant \|f - \varphi\|_{G(a,b)} < \varepsilon$$

for every $s \in [a,b]$. Then

$$|\varphi(s) - \varphi(\sigma)| = |\varphi(s) - f(s) + f(s) + f(\sigma) - f(\sigma) - \varphi(\sigma)| \leqslant$$

$$\leqslant |\varphi(s) - f(s)| + |f(s) - f(\sigma)| + |f(\sigma) - \varphi(\sigma)| < 3\varepsilon$$

for every $s \in (a,b), \sigma \in (s - \delta(s), s) \cap (a,b)$, i.e.

$$|\varphi(s) - \varphi(s-)| \leqslant 3\varepsilon$$

for $s \in (a,b)$.

Define

$$\psi(s) = \varphi(s-) \quad \text{for} \quad s \in (a,b), \ \psi(a) = \varphi(a), \psi(b) = \varphi(b).$$

Then evidently $\psi \in S(a,b) \cap G_L(a,b)$ and we have

$$|f(s) - \psi(s)| \leqslant |f(s) - \varphi(s)| + |\varphi(s) - \varphi(s-)| \leqslant 4\varepsilon$$

and also $\|f - \psi\|_{G(a,b)} = \|f - \psi\|_{G_L(a,b)} \leqslant 4\varepsilon$, i.e. $S(a,b) \cap G_L(a,b)$ is dense in $G_L(a,b)$. A set $M \subset G(a,b)$ is called *equiregulated* if for every $\varepsilon > 0$ and $s \in [a,b]$ there is a $\delta(s) > 0$ such that

$$|f(\sigma) - f(s+)| < \varepsilon \quad \text{for} \quad \sigma \in (s, s + \delta(s)) \cap [a,b],$$
$$|f(\sigma) - f(s-)| < \varepsilon \quad \text{for} \quad \sigma \in (s - \delta(s), s) \cap [a,b]$$

whenever $f \in M$.

The following result is well known (see e.g. [1]):

Proposition 1. *A set $M \subset G(a,b)$ is conditionally compact in $G(a,b)$ if and only if M is equiregulated and the set*

$$M(s) = \{f(s) \in \mathbf{R};\ f \in M\}$$

is bounded for every $s \in [a,b]$.

Taking into account the topology in $G_L(a,b)$, by Proposition 1 we immediately obtain the following

Corollary 1. *A set $M \subset G_L(a,b)$ is conditionally compact in $G_L(a,b)$ if and only if M is equiregulated, the set*

$$M(s) = \{f(s) \in \mathbf{R};\ f \in M\}$$

is bounded for every $s \in [a,b]$ and M is equicontinuous from the left at every point $s \in (a,b)$, i.e. for every $\varepsilon > 0$ and $s \in (a,b)$ there is a $\delta(s) > 0$ such that

$$|f(\sigma) - f(s)| < \varepsilon \quad \text{for} \quad \sigma \in (s - \delta(s), s) \cap [a,b].$$

Proposition 2. *Assume that $h_n\colon [a,b] \to \mathbf{R}$, $n \in \mathbf{N}$ is a sequence of functions such that*

$$\operatorname{var}_a^b\ h_n \leqslant L = const.$$

and

$$\lim_{n\to\infty} h_n(t) = 0,\ t \in [a,b].$$

If $g \in G(a,b)$ then $\int_a^b h_n(t)\,dg(t)$ exists for every $n \in \mathbf{N}$ and

$$\lim_{n\to\infty} \int_a^b h_n(t)\,dg(t) = 0.$$

Proof. Assume that $\psi\colon [a,b] \to \mathbf{R}$ is a finite step function, i.e. $\psi \in S(a,b)$. Then ψ is a finite linear combination of characteristic functions of intervals of the form

$$[a,\tau], [a,\tau), [\tau,b], (\tau,b]$$

or of a single point $[\tau]$,$\tau \in [a,b]$. If e.g. $\chi_{[a,\tau]}$ is the characteristic function of an interval $[a,\tau] \subset [a,b]$ then

$$\int_a^b h_n(t)\,d\chi_{[a,\tau]}(t) = -h_n(\tau) \quad \text{if} \quad \tau < b$$

and

$$\int_a^b h_n(t)\,d\chi_{[a,\tau]}(t) = 0 \quad \text{if} \quad \tau = b$$

by the results given in Proposition 2.3 in [2]. Hence if $\tau < b$ then

$$\lim_{n\to\infty} \int_a^b h_n(t)\,d\chi_{[a,\tau]}(t) = -\lim_{n\to\infty} h_n(\tau) = 0$$

and similarly for the remaining characteristic functions mentioned above. Therefore we have

$$(*) \qquad \lim_{n\to\infty} \int_a^b h_n(t)\,d\psi(t) = 0$$

for every finite step function $\psi \in S(a,b)$.

Let $g \in G(a,b)$. Since $S(a,b)$ is dense in $G(a,b)$, for every $\eta > 0$ there exists $\psi \in S(a,b)$ such that

$$\|g - \psi\|_{G(a,b)} = \sup_{t\in[a,b]} |g(t) - \psi(t)| < \eta.$$

Denote $\varphi = g - \psi$. Then $g = \varphi + \psi$ where $\varphi \in G(a,b)$ is such that $|\varphi(t)| < \eta$ for every $t \in [a,b]$, i.e. $\|\varphi\|_{G(a,b)} < \eta$ and $\psi \in S(a,b)$. Using the estimate given by M. Tvrdý in [2, 2.8.Theorem] we have

$$\left|\int_a^b h_n(t)\,d\varphi(t)\right| \leqslant (|h_n(a)| + |h_n(b)| + \operatorname{var}_a^b h_n)\|\varphi\|_{G(a,b)} \leqslant$$
$$\leqslant (|h_n(a)| + |h_n(b)| + L)\eta$$

Since the sequence $(h_n)_{n=1}^{\infty}$ tends pointwise to zero for $n \to \infty$ there is $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $|h_n(a)| + |h_n(b)| < L$ and therefore

$$\left|\int_a^b h_n(t)\,d\varphi(t)\right| < 2L\eta$$

for $n > n_0$.

Assume that $\varepsilon > 0$ is given. Let us set $\eta = \dfrac{\varepsilon}{2L+1}$ and assume that a fixed $\psi \in S(a,b)$ is given such that $\|g-\psi\|_{G(a,b)} < \eta$. Using $(*)$ we obtain that there is $n_1 \in \mathbb{N}$, $n_1 > n_0$ such that for $n > n_1$ we have $|\int_a^b h_n d\psi| < \varepsilon$ and finally also

$$\left|\int_a^b h_n\,dg\right| \leqslant \left|\int_a^b h_n\,d\varphi\right| + \left|\int_a^b h_n\,d\psi\right| < 2\varepsilon$$

for $n > n_1$ where $\varphi = g - \psi$. Hence $\lim_{n\to\infty} \int_a^b h_n(t)\,dg(t) = 0$ and the statement holds.

□

Theorem 1. *Let $T\colon G_L(a,b) \to G(c,d)$ be a bounded linear operator. Then there exists $r \in G(c,d)$ with $\|r\|_{G(c,d)} \leqslant \|T\|$ and $K\colon [c,d] \times [a,b] \to \mathbf{R}$ such that*

a) $K(s,.) \in BV(a,b)$ for every $s \in [c,d]$; $\operatorname{var}_a^b K(s,.) \leqslant \|T\|$ for every $s \in [c,d]$, $|K(s,a)| \leqslant \|T\|$;

b) $K(.,t) \in G(c,d)$ for every $t \in [a,b]$;

c) $(Tf)(s) = r(s)f(a) + \int_a^b K(s,t)\,df(t))$, $s \in [c,d]$, $f \in G_L(a,b)$, and

d) $\|T\| \leqslant \|r\|_{G(c,d)} + 2\sup_{s\in[c,d]} \|K(s,.)\|_{BV(a,b)}$.

Proof. For a given set $M \subset \mathbf{R}$ let us denote by χ_M the characteristic function of M. Define

$$r(s) = T(\chi_{[a,b]})(s) \quad \text{for } s \in [c,d] \tag{1}$$

and

$$\begin{aligned} K(s,t) &= T(\chi_{(t,b]})(s) \quad \text{for } t \in [a,b),\ s \in [c,d], \\ K(s,b) &= T(\chi_{[b]})(s) \quad \text{for } s \in [c,d]. \end{aligned} \tag{2}$$

Since all characteristic functions to which the operator T is applied in (1) and (2), evidently belong to $G_L(a,b)$ we get $r \in G(c,d)$ and also $K(.,t) \in G(c,d)$ for every $t \in [a,b]$.

Hence

$$\|r\|_{G(c,d)} = \sup_{s\in[c,d]} |r(s)| = \|T\chi_{[a,b]}\|_{G(c,d)} \leqslant \|T\|$$

because $\|\chi_{[a,b]}\|_{G_L(a,b)} = 1$ and therefore we get

$$\|r\|_{G(c,d)} \leqslant \|T\|.$$

For a fixed $s \in [c,d]$ let us consider the variation of the function $K(s,.)$. Using the definition of K given by (2) and the properties of characteristic functions we have for an arbitrary division $D\colon a = t_0 < t_1 < \ldots < t_m = b$ of $[a,b]$ the equality

$$\begin{aligned} &\sum_{j=1}^{m} |K(s,t_j) - K(s,t_{j-1})| = \\ &= \sum_{j=1}^{m-1} |K(s,t_j) - K(s,t_{j-1})| + |K(s,t_m) - K(s,t_{m-1})| = \\ &= \sum_{j=1}^{m-1} |T(\chi_{(t_j,b]})(s) - T(\chi_{(t_{j-1},b]})(s)| + |T(\chi_{[b]})(s) - T(\chi_{(t_{m-1},b]})(s)| = \\ &= \sum_{j=1}^{m-1} |-T(\chi_{(t_{j-1},t_j]})(s)| + |-T(\chi_{(t_{m-1},b)})(s)| = \\ &= \sum_{j=1}^{m-1} c_j T(\chi_{(t_{j-1},t_j]})(s) + c_m T(\chi_{(t_{m-1},b)})(s) = \\ &= T\Big(\sum_{j=1}^{m-1} c_j \chi_{(t_{j-1},t_j]}) + c_m \chi_{(t_{m-1},b)}\Big)(s) = T(h)(s) \end{aligned} \tag{3}$$

where $c_j = \pm 1$, $j =, 1, 2, \ldots, m$ are chosen suitably and

$$h = \sum_{j=1}^{m-1} c_j \chi_{(t_{j-1}, t_j]} + c_m \chi_{(t_{m-1}, b)}.$$

It is easy to see that this function h is an element of $G_L(a,b)$ and $\|h\|_{G_L(a,b)} = 1$ for every choice of the division D. Hence the boundedness of the operator $T\colon G_L(a,b) \to G(c,d)$ yields

$$\sum_{j=1}^{m} |K(s,t_j) - K(s,t_{j-1})| = T(h)(s) \leqslant |T(h)(s)|| \leqslant$$
$$\leqslant \sup_{s \in G(c,d)} |T(h)(s)| = \|T(h)\|_{G(c,d)} \leqslant \|T\| . \|h\|_{G_L(a,b)} \leqslant \|T\|$$

for every division D. Consequently

(4) $$\operatorname*{var}_a^b K(s,.) = \sup_D \sum_{j=1}^{m} |K(s,t_j) - K(s,t_{j-1})| \leqslant \|T\| < +\infty$$

for every $s \in [c,d]$.

Moreover, for every $s \in [c,d]$ we also have

$$|K(s,a)| = |T(\chi_{(a,b]})(s)| \leqslant \|T(\chi_{(a,b]})\|_{G(c,d)} \leqslant \|T\| . \|\chi_{(a,b]}\|_{G_L(a,b)} = \|T\|$$

because $\chi_{(a,b]} \in G_L(a,b)$ and $\|\chi_{(a,b]}\|_{G_L(a,b)} = 1$ and henceforth

(5) $$\|K(s,.)\|_{BV(a,b)} \leqslant 2\|T\| < +\infty.$$

The proof of c) is based on a density argument; we use Proposition 2.3 from [2] for the subsequent calculations.

For $f = \chi_{[a,b]}$ ve have $\int_a^b K(s,t)\,df(t) = 0$ and $r(s)f(a) = r(s) = T(\chi_{[a,b]})(s) = (Tf)(s)$, i.e.

(6) $$r(s)f(a) + \int_a^b K(s,t)\,df(t) = (Tf)(s)$$

in this case.

If $f = \chi_{[b]}$ then $\int_a^b K(s,t)\,df(t) = K(s,b) = T(\chi_{[b]})(s)$, $r(s)f(a) = 0$ and (6) is satisfied.

If $\tau \in [a,b)$ and $f = \chi_{(\tau,b]}$ then $\int_a^b K(s,t)\,df(t) = K(s,\tau) = T(\chi_{(\tau,b]})(s)$, $r(s)f(a) = 0$ and again (6) is satisfied.

Since every function belonging to $S(a,b) \cap G_L(a,b)$ is a finite linear combination of functions of the type $\chi_{(\tau,b]}$ with $\tau \in [a,b)$, $\chi_{[b]}$, $\chi_{[a,b]}$ the above results show that (6) is true for every $f \in S(a,b) \cap G_L(a,b)$.

If $f \in G_L(a,b)$ is arbitrary then by the density of $S(a,b) \cap G_L(a,b)$ in $G_L(a,b)$ there exists a sequence $f_n \in S(a,b) \cap G_L(a,b)$ such that $f_n \to f$ in $G_L(a,b)$ and by Corollary 2.10 in [2] (on the limiting behaviour of Perron-Stieltjes integrals) we obtain by the above result the equality

$$\lim_{n\to\infty} T(f_n)(s) = \lim_{n\to\infty} \Big[r(s)f_n(a) + \int_a^b K(s,t)\, df_n(t) \Big] = r(s)f(a) + \int_a^b K(s,t)\, df(t).$$

This together with the continuity of the operator T yields

$$(Tf)(s) = r(s)f(a) + \int_a^b K(s,t)\, df(t), \quad s \in [c,d],\ f \in G_L(a,b),$$

i.e. c) is satisfied.

For the norm of the operator T given by c) we have

$$\|T\| = \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \|Th\|_{G(c,d)} = \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \Big\| r(s)h(a) + \int_a^b K(s,t)\, dh(t) \Big\|_{G(c,d)} =$$

$$= \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \sup_{s \in [c,d]} \Big| r(s)h(a) + \int_a^b K(s,t)\, dh(t) \Big| \leqslant$$

$$\leqslant \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \sup_{s \in [c,d]} [|r(s)||h(a)| + (|K(s,a)| + |K(s,b)| + \operatorname*{var}_a^b K(s,.))\|h\|_{G_L(a,b)}] \leqslant$$

$$\leqslant \sup_{\|h\|_{G_L(a,b)} \leqslant 1} (\|r\|_{G(c,d)} + \sup_{s \in [c,d]} 2\|K(s,.)\|_{BV(a,b)})\|h\|_{G_L(a,b)}].$$

Since the operator T is represented in the form given by c) we obtain immediately the estimate of its norm presented in d). □

Theorem 2. *Assume that $r \in G(c,d)$ and that $K: [c,d] \times [a,b] \to \mathbf{R}$ satisfies a) and b) from Theorem 1 where $\|K(s,.)\|_{BV(a,b)} \leqslant M$, $M = const.$ for every $s \in [c,d]$. Then the formula*

$$\text{c)} \qquad (Tf)(s) = r(s)\Big(f(a) + \int_a^b K(s,t)\, df(t)\Big), \ s \in [c,d],\ f \in G_L(a,b),$$

from Theorem 1 defines a bounded linear operator from $G_L(a,b)$ to $G(c,d)$ and for its norm we have

$$\|T\| = \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \|Th\|_{G(c,d)} \leqslant \|r\|_{G(c,d)} + 2M.$$

Proof. By the results from [2] $(Tf)(s)$ given by c) is well defined for every $f \in G_L(a,b)$ and $s \in [c,d]$. The linearity of the mapping T is clear. Let us show that for $f \in G_L(a,b)$ we have $Tf \in G(c,d)$.

Since $r \in G(c,d)$ the first term on the right hand side of c) evidently belongs to $G(c,d)$. By the assumption b) from Theorem 1 the onesided limits $K(s-,t)$, $K(s+,t)$ exist for every $t \in [a,b]$, i.e. we have

$$\text{(7)} \qquad \lim_{\sigma \to s-} K(\sigma,t) = K(s-,t) \qquad \text{for every } t \in [a,b),\ s \in (c,d],$$
$$\lim_{\sigma \to s+} K(\sigma,t) = K(s+,t) \qquad \text{for every } t \in [a,b),\ s \in [c,d).$$

Since $\|K(s,.)\|_{BV(a,b)} \leqslant M$ is assumed, Helly's Choice Theorem implies that $K(s-,.)$, $K(s+,.) \in BV(a,b)$ and therefore the integrals

$$\int_a^b K(s-,t)\,df(t),\ \int_a^b K(s+,t)\,df(t)$$

exist for $s \in (c,d]$, $s \in [c,d)$, respectively. Applying Proposition 2 and (7) we obtain

$$\lim_{\sigma \to s-} \int_a^b [K(\sigma,t) - K(s-,t)]\,df(t) = 0,$$

i.e.

$$\lim_{\sigma \to s-} \int_a^b K(\sigma,t)\,df(t) = \int_a^b K(s-,t)\,df(t) \qquad \text{for } s \in (c,d],$$

and similarly also

$$\lim_{\sigma \to s+} \int_a^b K(\sigma,t)\,df(t) = \int_a^b K(s+,t)\,df(t) \qquad \text{for } s \in [c,d).$$

Hence the function

$$s \in [c,d] \longmapsto \int_a^b K(s,t)\,df(t) \in \mathbf{R}$$

possesses onesided limits in $[c,d]$ and belongs therefore to $G(c,d)$. Moreover we have

$$|(Tf)(s)| \leqslant |r(s)||f(a)| + \left|\int_a^b K(s,t)\,df(t)\right|.$$

The inequality $\|K(s,.)\|_{BV(a,b)} \leqslant M$, $s \in [c,d]$ and the estimate given in Theorem 2.8 in [2] yields

$$|(Tf)(s)| = \left|r(s)h(a) + \int_a^b K(s,t)\,dh(t)\right| \leqslant$$
$$\leqslant |r(s)||f(a)| + (|K(s,a)| + |K(s,b)| + \operatorname*{var}_a^b K(s,.))\|f\|_{G_L(a,b)} \leqslant$$
$$\leqslant \|r\|_{G(c,d)}.\|f\|_{G_L(a,b)} + (|K(s,a)| + |K(s,b)| + \operatorname*{var}_a^b K(s,.))\|f\|_{G_L(a,b)} \leqslant$$
$$\leqslant (\|r\|_{G(c,d)} + 2\|K(s,.)\|_{BV(a,b)}).\|f\|_{G_L(a,b)} \leqslant$$
$$\leqslant (\|r\|_{G(c,d)} + 2M).\|f\|_{G_L(a,b)}.$$

Hence

$$\|Tf\|_{G(c,d)} = \sup_{s \in [c,d]} |(Tf)(s)| \leqslant (\|r\|_{G(c,d)} + 2M).\|f\|_{G_L(a,b)}$$

and the operator T is bounded. □

Denote by $B([c,d] \times [a,b])$ the set of all functions $K: [c,d] \times [a,b] \to \mathbf{R}$ for which

$$K(s,.) \in BV(a,b) \text{ for every } s \in [c,d],$$
$$\|K(s,.)\|_{BV(a,b)} \leqslant M, \ M = const. \quad \text{for every } s \in [c,d]$$

and

$$K(.,t) \in G(c,d) \text{ for every } t \in [a,b]$$

hold. It is easy to see that $B([c,d] \times [a,b])$ is a linear space and that

$$\|K\| = \sup_{s \in [c,d]} |K(s,a)| + \operatorname*{var}_a^b K(s,.)$$

defines a norm in $B([c,d] \times [a,b])$. Let us denote by $\mathcal{B}([c,d] \times [a,b])$ the normed linear space $B([c,d] \times [a,b])$ with the norm given above.

Using Theorems 1 and 2 the following can be stated.

Corollary 2. *For a given pair $(r,K) \in G(c,d) \times \mathcal{B}([c,d] \times [a,b])$ denote*

$$T_{(r,K)}(f)(s) = r(s)f(a) + \int_a^b K(s,t)\,df(t)), \ s \in [c,d], \ f \in G_L(a,b)$$

and let $\mathbf{B}(G_L(a,b); G(c,d))$ be the Banach space of all bounded linear operators from $G_L(a,b)$ to $G(c,d)$. The mapping

$$\Phi: (r,K) \in G(c,d) \times \mathcal{B}([c,d] \times [a,b]) \mapsto T_{(r,K)} \in \mathbf{B}(G_L(a,b); G(c,d))$$

is an isomorphism.

Theorem 3. *Let $T: G_L(a,b) \to G(c,d)$ be a compact linear operator. Then there exists $r \in G(c,d)$ and $K: [c,d] \times [a,b] \to \mathbf{R}$ such that a) for every $s \in [c,d]$ we håve $K(s,.) \in BV(a,b)$, i.e.*

$$\|K(s,.)\|_{BV(a,b)} = |K(s,a)| + \operatorname*{var}_a^b K(s,.) < \infty, \ s \in [c,d];$$

b) the mapping

$$m_K: [c,d] \to BV(a,b)$$

given by

$$m_K(s) = K(s,.), \ s \in [c,d]$$

is regulated, i.e. the limits

$$m_K(s+) = \lim_{\substack{\sigma \to s+ \\ \sigma \in [c,d]}} m_K(\sigma) \quad \text{and} \quad m_K(s-) = \lim_{\substack{\sigma \to s- \\ \sigma \in [c,d]}} m_K(\sigma)$$

exist and

$$\|K\| = \sup_{s\in[c,d]} \|m_K(s)\|_{BV(a,b)} = \sup_{s\in[c,d]} |K(s,a)| + \operatorname*{var}_a^b K(s,.) < \infty;$$

c) $$(Tf)(s) = r(s)\Big(f(a) + \int_a^b K(s,t)\,df(t)\Big),\ s \in [c,d],\ f \in G_L(a,b);$$

d) $$\|K\| \leqslant 2 \sup_{\|h\|_{G(a,b)}\leqslant 1} \|Th\|_{G(c,d)} = 2\|T\|,$$

$$\|r\| \leqslant \|T\|.$$

Proof. Since the operator T is bounded, the conclusion of Theorem 1 holds. The only thing we have to prove is b). Let us consider the mapping m_K given by

$$m_K : s \in [c,d] \to K(s,.).$$

By the results of Theorem 1 we have $K(s,.) \in BV(a,b)$ for every $s \in [c,d]$. We show that the mapping $m_K : [c,d] \to BV(a,b)$ is regulated as a BV-valued function. For $s_1, s_2 \in [c,d]$ we have

(8) $$\|m_K(s_2) - m_K(s_1)\|_{BV(a,b)} = \|K(s_2,.) - K(s_1,.)\|_{BV(a,b)} =$$
$$= |K(s_2,a) - K(s_1,a)| + \operatorname*{var}_a^b(K(s_2,.) - K(s_1,.)).$$

Since $T(\chi_{(a,b]}) \in G(c,d)$ the onesided limits of this function exist at every point $s \in [c,d]$, i.e. by the Bolzano-Cauchy condition for the existence of these limits for every $\varepsilon > 0$, $s \in [c,d]$ there is a $\delta(s) > 0$ such that

(9) $$|K(s_2,a) - K(s_1,a)| = |T(\chi_{(a,b]})(s_2) - T(\chi_{(a,b]})(s_1)| < \varepsilon$$

provided $s_1, s_2 \in (s, s+\delta(s)) \cap [c,d]$ or $s_1, s_2 \in (s-\delta(s), s) \cap [c,d]$ (cf. the definition (2) of K).

Let us consider the second term on the right hand side of (8). Assume that $D\colon a = t_0 < t_1 < \ldots < t_m = b$ is an arbitrary division of $[a,b]$. By (2) and by the properties of characteristic functions we get

(10) $$\sum_{j=1}^{m-1} |K(s_2,t_j) - K(s_2,t_{j-1}) - K(s_1,t_j) + K(s_1,t_{j-1})| =$$
$$= \sum_{j=1}^{m-1} |T(\chi_{(t_j,b]})(s_2) - T(\chi_{(t_{j-1},b]})(s_2) - T(\chi_{(t_j,b]})(s_1) + T(\chi_{(t_{j-1},b]})(s_1)| =$$
$$= \sum_{j=1}^{m-1} |-T(\chi_{(t_{j-1},t_j]})(s_2) + T(\chi_{(t_{j-1},t_j]})(s_1)| =$$
$$= \sum_{j=1}^{m-1} c_j[T(\chi_{(t_{j-1},t_j]})(s_2) - T(\chi_{(t_{j-1},t_j]})(s_1)] =$$
$$= T\Big(\sum_{j=1}^{m-1} c_j\chi_{(t_{j-1},t_j]}\Big)(s_2) - T\Big(\sum_{j=1}^{m-1} c_j\chi_{(t_{j-1},t_j]}\Big)(s_1)$$

where $c_j = \pm 1$, $j = 1, \ldots, m-1$,

$$\begin{aligned}|K(s_2,t_m) - K(s_2,t_{m-1}) - K(s_1,t_m) + K(s_1,t_{m-1})| =\\ = |K(s_2,b) - K(s_2,t_{m-1}) - K(s_1,b) + K(s_1,t_{m-1})| =\\ = |T(\chi_{[b]})(s_2) - T(\chi_{(t_{m-1},b]})(s_2) - T(\chi_{[b]})(s_1) + T(\chi_{(t_{(m-1},b]})(s_1)| =\\ = |-T(\chi_{(t_{m-1},b)})(s_2) + T(\chi_{(t_{m-1},b)})(s_1)| =\\ = c_m[T(\chi_{(t_{m-1},b)})(s_2) - T(\chi_{(t_{m-1},b)})(s_1)] =\\ = T(c_m\chi_{(t_{m-1},b)})(s_2) - T(c_m\chi_{(t_{m-1},b)})(s_1)\end{aligned}$$

with $c_m = 1$ or $c_m = -1$. Let us set

$$h_D = \sum_{j=1}^{m-1} c_j\chi_{(t_{j-1},t_j]} + c_m\chi_{(t_{m-1},b)};$$

then evidently $h_D \in G_L(a,b)$ and $\|h_D\|_{G_L(a,b)} = 1$. Using (10) and the above result we have

$$\begin{aligned}(11)\qquad \sum_{j=1}^{m-1} |K(s_2,t_j) - K(s_2,t_{j-1}) - K(s_1,t_j) + K(s_1,t_{j-1})| =\\ = T(h_D)(s_2) - T(h_D)(s_1) \leqslant |T(h_D)(s_2) - T(h_D)(s_1)|.\end{aligned}$$

Since for every division D of $[a,b]$ the corresponding function h_D belongs to the closed unit ball in $G_L(a,b)$ and the operator T is compact, the elements $T(h_D)$ belong to a conditionally compact set in $G(c,d)$, i.e. the set of functions of the form $T(h_D)$ is equiregulated by Proposition 1. This means that for every $\varepsilon > 0$, $s \in [c,d]$ there is $\delta(s) > 0$ such that

$$|T(h_D)(t) - T(h_D)(s+)| < \frac{\varepsilon}{2} \quad \text{for } t \in (s, s+\delta(s)) \cap [c,d]$$

and

$$|T(h_D)(t) - T(h_D)(s-)| < \frac{\varepsilon}{2} \quad \text{for } t \in (s-\delta(s), s) \cap [c,d]$$

i.e. independently of the choice of D we have

$$|T(h_D)(s_2) - T(h_D)(s_1)| < \varepsilon$$

whenever $s_1, s_2 \in (s-\delta(s), s) \cap [c,d]$ or $s_1, s_2 \in (s, s+\delta(s)) \cap [c,d]$. Using (9) we obtain that independently of the choice of the division D we have

$$\sum_{j=1}^{m-1} |K(s_2,t_j) - K(s_2,t_{j-1}) - K(s_1,t_j) + K(s_1,t_{j-1})| < \varepsilon$$

provided $s_1, s_2 \in (s - \delta(s), s) \cap [c, d]$ or $s_1, s_2 \in (s, s + \delta(s)) \cap [c, d]$. Passing to the supremum with respect to all divisions D we get

$$\operatorname*{var}_a^b(K(s_2, .) - K(s_1, .)) \leqslant \varepsilon$$

for $s_1, s_2 \in (s - \delta(s), s) \cap [c, d]$ or $s_1, s_2 \in (s, s + \delta(s)) \cap [c, d]$.

Using (8), (9) together with this last inequality we obtain that for any $\varepsilon > 0$ and $s \in [c, d]$ there is $\delta(s) > 0$ such that for $s_1, s_2 \in (s - \delta(s), s) \cap [c, d]$ or $s_1, s_2 \in (s, s + \delta(s)) \cap [c, d]$ we have

$$\|m_K(s_2) - m_K(s_1)\|_{BV(a,b)} < 2\varepsilon,$$

i.e. the function $m_K: [c, d] \to BV(a, b)$ is regulated and b) from the theorem is satisfied. This completes the proof of Theorem 3. □

Theorem 4. *Assume that $r \in G(c, d)$ and that $K: [c, d] \times [a, b] \to \mathbf{R}$ satisfies a) and b) from Theorem 3. Then*

c) $$(Tf)(s) = r(s)f(a) + \int_a^b K(s, t)\, df(t)), \quad s \in [c, d], \; f \in G_L(a, b)$$

defines a compact operator from $G_L(a, b)$ to $G(c, d)$ and

$$\|T\| = \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \|Th\|_{G(c,d)} \leqslant \|r\|_{G(c,d)} + 2\|K\|.$$

Proof. By the results from [2] $(Tf)(s)$ given by c) is well defined for every $f \in G_L(a, b)$ and $s \in [c, d]$.

For a given $f \in G_L(a, b)$ and $s_1, s_2 \in [c, d]$ we have

$$\begin{aligned}
&|(Tf)(s_2) - (Tf)(s_1)| = \\
&\quad = |(r(s_2) - r(s_1))f(a) + \int_a^b (K(s_2, t) - K(s_1, t))\, df(t)| \leqslant \\
&\quad \leqslant |r(s_2) - r(s_1)||f(a)| + |\int_a^b (K(s_2, t) - K(s_1, t))\, df(t)| \leqslant \\
&\quad \leqslant |r(s_2) - r(s_1)||f(a)| + [|K(s_2, a) - K(s_1, a)| + |K(s_2, b) - K(s_1, b)| + \\
&\qquad + \operatorname*{var}_a^b(K(s_2, .) - K(s_1, .))]\|f\|_{G_L(a,b)} \leqslant \\
&\quad \leqslant |r(s_2) - r(s_1)||f(a)| + 2\|K(s_2, .) - K(s_1, .)\|_{BV(a,b)}\|f\|_{G_L(a,b)} \leqslant \\
&\quad \leqslant [|r(s_2) - r(s_1)| + 2\|K(s_2, .) - K(s_1, .)\|_{BV(a,b)}]\|f\|_{G_L(a,b)}.
\end{aligned}$$

Since $r \in G(c, d)$ and K satisfies b) we obtain that for every $\varepsilon > 0$, $s \in [c, d]$ there is $\delta(s) > 0$ such that

(12) $$|(Tf)(s_2) - (Tf)(s_1)| \leqslant \varepsilon\|f\|_{G_L(a,b)}$$

provided $s_1, s_2 \in (s, s+\delta(s)) \cap [c,d]$ or $s_1, s_2 \in (s-\delta(s), s) \cap [c,d]$, i.e. the onesided limits $(Tf)(s+)$, $(Tf)(s-)$ exist for every $s \in [c,d)$, $s \in (c,d]$, respectively, and therefore $Tf \in G(c,d)$.

For the norm of the operator T given by c) we have

$$\begin{aligned}
\|T\| &= \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \|Th\|_{G(c,d)} = \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \|r(s)h(a) + \int_a^b K(s,t)\, dh(t)\|_{G(c,d)} = \\
&= \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \sup_{s\in[c,d]} |r(s)h(a) + \int_a^b K(s,t)\, dh(t)| \leqslant \\
&\leqslant \sup_{\|h\|_{G_L(a,b)} \leqslant 1} \sup_{s\in[c,d]} [|r(s)||h(a)| + (|K(s,a)| + |K(s,b)| + \operatorname{var}_a^b K(s,.))\|h\|_{G_L(a,b)}] \leqslant \\
&\leqslant \sup_{\|h\|_{G_L(a,b)} \leqslant 1} (\|r\|_{G(c,d)} + \sup_{s\in[c,d]} 2\|K(s,.)\|_{BV(a,b)})\|h\|_{G_L(a,b)}] \leqslant \\
&\leqslant \|r\|_{G(c,d)} + 2\|K\|.
\end{aligned}$$

Hence if $f \in G_L(a,b)$ is such that $\|f\|_{G_L(a,b)} \leqslant 1$ then

$$|(Tf)(s)| \leqslant \|Tf\|_{G(c,d)} \leqslant \|T\|\|f\|_{G_L(a,b)} \leqslant \|T\|$$

for every $s \in [c,d]$ and by (12) we have

$$|(Tf)(s_2) - (Tf)(s_1)| \leqslant \varepsilon$$

provided $s_1, s_2 \in (s, s+\delta(s)) \cap [c,d]$ or $s_1, s_2 \in (s-\delta(s), s) \cap [c,d]$. Therefore by Proposition 1 the set $M = \{g \in G(c,d); g = Tf, f \in G_L(a,b), \|f\|_{G_L(a,b)} \leqslant 1\}$ is conditionally compact in $G(c,d)$ and consequently the operator T given by the relation c) is compact. □

Denote by $\mathcal{K}([c,d] \times [a,b])$ the set of all functions $K\colon [c,d] \times [a,b] \to \mathbf{R}$ for which a) and b) from Theorem 1 hold. It is easy to see that by $\|K\|$ from b) in Theorem 3 a norm in $\mathcal{K}([c,d] \times [a,b])$ is given.

Using Theorems 1 and 2 we obtain the following result.

Corollary 3. *For a given pair $(r,K) \in G(c,d) \times \mathcal{K}([c,d] \times [a,b])$ denote*

$$T_{(r,K)}(f)(s) = r(s)f(a) + \int_a^b K(s,t)\, df(t), \ \ s \in [c,d], \ f \in G_L(a,b)$$

and let $\mathbf{K}(G_L(a,b); G(c,d))$ be the Banach space of all compact operators from $G_L(a,b)$ to $G(c,d)$. The mapping

$$\Phi\colon (r,K) \in G(c,d) \times \mathcal{K}([c,d] \times [a,b]) \mapsto T_{(r,K)} \in \mathbf{K}(G_L(a,b); G(c,d))$$

is an isomorphism.

Using the criterion for conditional compactness of a set in $G_L(c,d)$ stated in Corollary 1 the proof of Theorems 3 and 4 can be repeated in order to obtain the following statement.

Theorem 5. *Let $T: G_L(a,b) \to G_L(c,d)$ be a compact linear operator. Then there exist $r \in G_L(c,d)$ and $K: [c,d] \times [a,b] \to \mathbf{R}$ such that a), c), d) from Theorem 3 hold and instead of b) the following condition holds:*

b–) the function $m_K: [c,d] \to BV(a,b)$ given in b) from Theorem 3 is a $BV(a,b)$ valued G_L-function, i.e. the conditions given in b) are satisfied with the additional continuity from the left on (c,d) of this function.

On the other hand, if $r \in G_L(c,d)$ and $K: [c,d] \times [a,b] \to \mathbf{R}$ satisfies a) and b–) then c) defines a compact operator $T: G_L(a,b) \to G_L(c,d)$.

The inequalities for the norms of the operators given in Theorems 1 and 2 remain unchanged in this case and a statement analogous to Corollary 3 holds.

References

[1] *Ch.S.Hönig*: Volterra-Stieltjes integral equations, North-Holland, Amsterdam, 1975.

[2] *M. Tvrdý*: Regulated functions and the Perron-Stieltjes integral, Časopis pěst. mat. *114* (1989), 187–209.

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