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NOTE ON k-CHROMATIC GRAPHS

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Summary. In this paper we characterize k-chromatic graphs without isolated vertices and connected k-chromatic graphs having a minimal number of edges.

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Graphs, considered here, are finite and simple (without loops and multiple edges), and [1, 2] are followed for terminology and notation. Let $G = (V, E)$ be an undirected graph, with V the set of vertices and E the set of edges, such that $|V| = n$ and $|E| = m$. By *colouring* a graph we mean paiting the vertices of the graph with one or more distinct colours. By *properly colouring* a graph, we mean painting the vertices of the graph in such a way that no two adjacent vertices are painted with the same colour. The *chromatic number* $\gamma(G)$ of a graph G is the least number of distinct colours that can be used to colour the graph properly. A graph is said to be *complete*, if every two vertices of it are joined by an edge. We shall denote by K_n the complete graph on n vertices. If v is an arbitrary vertex of G, we shall denote by $G - v$ the subgraph obtained from G by deleting v together with its incident edges.

A set of vertices in a graph is said to be an *independent set* if no two vertices in it are adjacent.

For any real number x, we use $\lceil x \rceil$ to denote the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

Theorem 1. *If $G = (V, E)$ is a graph without isolated vertices and $\gamma(G) = k$, then*

$$m \geqslant \binom{k}{2} + \left\lceil \frac{n-k}{2} \right\rceil.$$

Proof. First, suppose that for each $v \in V$ the subgraph $G - v$ contains isolated vertices. Let w be an isolated vertex of $G - v$, that is, w is adjacent only to v in G. However, the subgraph $G - w$ also contains isolated vertices. Thus, $(v, w) \in E$ and vertices v, w are not adjacent to other vertices in G.

Repeating this reasoning, we obtain that if for each $v \in V$ the subgraph $G - v$ contains isolated vertices and G does not contain isolated vertices, then n is even, $\gamma(G) = 2$ and

$$m = \frac{n}{2} = \binom{2}{2} + \frac{n-2}{2}.$$

However, this number is the minimal number of edges of G, since G does not contain isolated vertices and, hence, the degree $d(v)$ of each vertex of G is at least equal to 1. Therefore, we have

$$2m = \sum_{v \in V} d(v) \geqslant n,$$

that is,

$$m \geqslant \left\lceil \frac{n}{2} \right\rceil.$$

Thus, in this case, the theorem is proved.

In the sequel, we shall prove the theorem by induction on n. So, suppose that the theorem is true for all graphs G having $n - 1$ vertices and the chromatic number equal to k ($k \leqslant n - 1$). Let G be a graph with n vertices. If $\gamma(G) = n$, then G is isomorphic to K_n, and the theorem is proved. Suppose that $\gamma(G) = k \leqslant n - 1$. Let $v \in V$ be such that $G - v$ does not contain isolated vertices. If such a vertex does not exist, we have seen above that the theorem is true. We have two cases.

(a) $\gamma(G - v) = k$. Thus, by the induction hypothesis, the minimal number of edges of the subgraph $G - v$ is equal to

$$\binom{k}{2} + \left\lceil \frac{n-k-1}{2} \right\rceil.$$

But v is not an isolated vertex. Thus, $d(v) \geqslant 1$ and, therefore, the number of edges of G is greater than or equal to

$$\binom{k}{2} + \left\lceil \frac{n-k-1}{2} \right\rceil + 1 \geqslant \binom{k}{2} + \left\lceil \frac{n-k}{2} \right\rceil.$$

We obtain equality, that is,

$$m = \binom{k}{2} + \left\lceil \frac{n-k}{2} \right\rceil,$$

only if $n - k$ is odd, $d(v) = 1$ and the subgraph $G - v$ has a minimal number of edges.

(b) $\gamma(G - v) = k - 1$. In this case there exists a partition of V consisting of independent sets in the form $\{v\}$, $C_1, C_2, \ldots, C_{k-1}$, and v is joined by an edge to at least one vertex from each class $C_1, C_2, \ldots, C_{k-1}$. Thus $d(v) \geqslant k - 1$, as otherwise $\gamma(G) \leqslant k - 1$, which contradicts the hypothesis, that is, the fact that $\gamma(G) = k$. Hence, the number of edges of $G - v$ plus $k - 1$ is a lower bound for m and, by the induction hypothesis, we have

$$m \geqslant \binom{k-1}{2} + \left\lceil \frac{n - 1 - (k-1)}{2} \right\rceil + k - 1 = \binom{k}{2} + \left\lceil \frac{n-k}{2} \right\rceil.$$

The equality holds only if $d(v) = k - 1$ and the subgraph $G - v$ has a minimal number of edges. □

Following the above proof and the cases when inequalities become equalities, we obtain, by induction, the characterization of graphs G without isolated vertices, with n vertices and $\gamma(G) = k$, which have a minimal number of edges, as follows.

If $n - k$ is even, the graph G with a minimal number of edges is unique (up to an isomorphism) and consists of a subgraph K_k and $n - k$ vertices which are pairwise joined by $\frac{1}{2}(n - k)$ edges.

If $n - k$ is odd, then there are two types of non-isomorphic graphs which have a minimal number of edges: a graph consisting of subgraph K_k, $n - k - 1$ vertices which are pairwise joined by $\frac{1}{2}(n - k - 1)$ edges, and another vertex which is joined by an edge to an arbitrary vertex of K_k. The other type consists of a subgraph K_k, $n - k - 1$ vertices which are pairwise joined by $\frac{1}{2}(n - k - 1)$ edges, and another vertex which is joined by an edge to a vertex which does not belong to K_k. Obviously, for $k = 2$, these two types of graphs coincide.

Indeed, in case (a), in order to obtain the minimal value of m, the number $n - k - 1$ must be even. Thus, the subgraph $G - v$ having a minimal number of edges is unique, and for v we have two possibilities of joining it by an edge such that $d(v) = 1$.

In case (b), the vertex v is joined to all vertices of the subgraph K_{k-1} of $G - v$ which has a minimal number of edges, as otherwise we obtain $\gamma(G) < k$, contradicting the hypothesis ($\gamma(G) = k$). Hence, the minimal graph must have necessarily the above indicated structure. If G has n vertices, $\gamma(G) = k$ and no restriction is imposed on G, then the minimal number of edges is equal to $\binom{k}{2}$, since between two arbitrary classes of a partition of V consisting of k independent sets there exists at least one edge, as otherwise $\gamma(G) < k$, contradicting the hypothesis ($\gamma(G) = k$). It is easy to show similarly, by induction on n, that the single graph having this minimal number of edges consists of a subgraph K_k and $n - k$ isolated vertices. Thus, we have obtained

$$m \geqslant \frac{k^2 - k}{2} + \frac{n - k}{2}$$

or

$$k^2 - 2k + n - 2m \leqslant 0,$$

wherefrom

$$k \leqslant 1 + \sqrt{2m - n + 1}.$$

Corollary. *If $G = (V, E)$ is a graph without isolated vertices, then*

$$\gamma(G) \leqslant 1 + \sqrt{2m - n + 1}.$$

It is easy to see that this inequality becomes equality, for example, if G is isomorphic to K_n.

According to [3], if G is connected, then

$$\gamma(G) \leqslant \left\lfloor \frac{3 + \sqrt{9 + 8(m - n)}}{2} \right\rfloor.$$

Thus, if G is connected and $\gamma(G) = k$, we have

$$m \geqslant \binom{k}{2} + n - k.$$

The connected graph having this minimal number of edges is not unique. For example, it consists of a subgraph K_k and $n - k$ vertices, each of them being joined by an edge to a vertex of K_k, or it consists of a subgraph K_k and a path with $n - k$ vertices which is joined by an edge to a vertex of K_k.

For $k = 2$, these graphs are trees with n vertices. For $k = 3$, such a minimal connected graph is composed by an odd cycle with p vertices ($3 \leqslant p \leqslant n$), such that the other $n - p$ vertices either are joined to a vertex of the cycle or form paths joined by an edge to a vertex of the cycle. More generally, we have

Theorem 2. *The minimal number of edges of a connected graph G with n vertices and $\gamma(G) = k$ ($2 \leqslant k \leqslant n$) is equal to*

$$\binom{k}{2} + n - k.$$

The graphs having this minimal number of edges are of the following kind:

(1) for $k = 2$, they are trees with n vertices;

(2) for $k = 3$, they consist of an odd cycle with p vertices ($3 \leqslant p \leqslant n$) and $n - p$ vertices such that if the vertices of the cycle are identified to a single vertex, then the resulting graph is a tree;

(3) *for* $k \geqslant 4$, *they consist of a subgraph* K_k *and* $n - k$ *vertices such that if the vertices of* K_k *are identified to a single vertex, the resulting graph is a tree.*

Proof. Obviously, for $k = 2$, the theorem is true. The connected graph G with n vertices and $\gamma(G) = 2$ which has a minimal number of edges is a tree with $n - 1$ edges, since the existence of a cycle is in contradiction with the hypothesis of minimality for the number of edges. For $k \geqslant 3$, we proceed by induction on n. Obviously, for $n = 2, 3$, the theorem is true. So, suppose that the theorem is true for all graphs with $n - 1$ vertices and let G be a connected graph with n vertices and $\gamma(G) = k$. For $n \geqslant 3$, there exists a vertex v such that the subgraph $G - v$ is connected as well. We have two cases.

(a) If $\gamma(G - v) = k$, then, by the induction hypothesis, the minimal number of edges of $G - v$ is equal to

$$\binom{k}{2} + n - k - 1,$$

and $G - v$ is of one of the above kinds. Thus, $\binom{k}{2} + n - k$ is a lower bound for the number of edges of G since, G being connected, we must have $d(v) \geqslant 1$.

The connected graph G has a minimal number of edges only if $G - v$ has a minimal number of edges and $d(v) = 1$. Hence, G is of a kind specified in the theorem.

(b) If $\gamma(G - v) = k - 1$, then G has a colouring consisting of classes $\{v\}$, C_1, C_2, ..., C_{k-1}, and v is joined by an edge to at least one vertex of each independent set C_1, C_2, ..., C_{k-1}. Thus, $d(v) \geqslant k - 1$. Then

$$\binom{k-1}{2} + n - k + k - 1 = \binom{k}{2} + n - k$$

is a lower bound for the number of edges of G, and G has a minimal number of edges only if $d(v) = k - 1$ and the connected graph $G - v$ has a minimal number of edges.

If $k \geqslant 5$, then by the induction hypothesis, the minimal connected subgraph $G - v$ is of kind 3. Thus, the vertex v is joined to each vertex of the subgraph K_{k-1} of $G - v$, as otherwise we obtain $\gamma(G) = k - 1$, contradicting the hypothesis ($\gamma(G) = k$). Hence, in this case, G is also of kind 3.

If $k = 4$, the minimal subgraph $G - v$ consists of a triangle and $n - 4$ vertices which form trees which are joined by an edge to a variable vertex of the triangle, and the vertex v is joined to all vertices of the triangle due to the fact that $d(v) = 3$, since, otherwise, $\gamma(G) = 3$. In this case, the minimal graph G is of kind 3.

If $k = 3$, the subgraph $G - v$ is a tree with $n - 1$ vertices and $d(v) = 2$. Thus, the graph G contains a single odd cycle since $\gamma(G) = 3$, the other vertices being vertices of some trees which are joined by an edge to a variable vertex of the odd cycle. In this case, the minimal connected graph G is of kind 2. □

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