ON SOME APPLICATIONS OF HARMONIC MEASURE IN THE GEOMETRIC THEORY OF ANALYTIC FUNCTIONS

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Summary. Let $\mathscr{P}$ denote the well-known class of functions of the form $p(z) = 1+q_1z+\ldots$ holomorphic in the unit disc $\mathbf{D}$ and fulfilling the condition $\operatorname{Re} p(z) > 0$ in $\mathbf{D}$. Let $0 \leqslant b < 1$, $b < B$, $0 < \alpha < 1$, be fixed real numbers and $\mathbf{F}$ a given measurable subset of the unit circle $\mathbf{T}$ of Lebesgue measure $2\pi\alpha$. For each $\tau \in \langle -\pi, \pi)$, denote by $\mathbf{F}_\tau = \{\xi \in \mathbf{T};\ e^{-i\tau}\xi \in \mathbf{F}\}$ the set arising by rotation of $\mathbf{F}$ through the angle τ. Denote by $\mathscr{P}(B, b, \alpha; \mathbf{F})$ the class of functions $p \in \mathscr{P}$ satisfying the following condition: there exists $\tau \in \langle -\pi, \pi)$ such that $\operatorname{Re} p(e^{i\theta}) \geqslant B$ a.e. on $\mathbf{F}_\tau$ and $\operatorname{Re} p(e^{i\theta}) \geqslant b$ a.e. on $\mathbf{T} \setminus \mathbf{F}_\tau$.

In the paper the properties of the class $\mathscr{P}(B, b, \alpha; \mathbf{F})$ for different values of the parameters B, b, α and measurable sets $\mathbf{F}$ are examined. This article belongs to the series of papers ([4], [5], [6]) where different classes of functions defined by conditions on the circle $\mathbf{T}$ were studied. The results of papers [5], [6] are generalized.*

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1.

As usual, we shall denote by $\mathbf{C}$ the complex plane, by $\mathbf{D} = \{z \in \mathbf{C};\ |z| < 1\}$ the unit disc, by $\mathbf{T} = \{z \in \mathbf{C};\ |z| = 1\}$ the unit circle. In our further considerations, we shall treat $\mathbf{T}$ on the one hand as a subset of $\mathbf{C}$ with the induced topology, on the other hand as a set homeomorphic to $\mathbf{T}$, namely, as the subset $\langle -\pi, \pi)$ of the real line $\mathbf{R}$, endowed with the factor topology $\mathbf{R}/2\pi\mathbf{Z}$ where $\mathbf{Z}$ is the set of integers. Therefore we shall sometimes treat the function $f(e^{i\theta})\colon \mathbf{T} \to \mathbf{C}$ as a function $f(t)\colon \langle -\pi, \pi) \to \mathbf{C}$.

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Let $\mathscr{P}$ denote the class of functions of the form

$$p(z) = 1 + q_1 z + \ldots + q_n z^n + \ldots \tag{1}$$

holomorphic in the unit disc $\mathbf{D}$ with $\operatorname{Re} p(z) > 0$ in $\mathbf{D}$ ([2]).

Let us recall some properties of real parts of functions from $\mathscr{P}$, which will be essential in what follows:

(a) Every function $\operatorname{Re} p(z)$, $p \in \mathscr{P}$, has the Poisson representation by means of a unique positive measure ([3], pp. 21–24, [7], pp. 11-12)

$$\operatorname{Re} p(z) = \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \tag{2}$$

where $d\mu(t) \geqslant 0$ and $\int_{-\pi}^{\pi} d\mu(t) = 1$; conversely, every function p holomorphic in $\mathbf{D}$ whose real part is given by (2), where $d\mu(t) \geqslant 0$ and $\int_{-\pi}^{\pi} d\mu(t) = 1$, and for which $\operatorname{Im} p(0) = 0$, belongs to $\mathscr{P}$.

(b) Let $d\mu(t) = f(t)\frac{dt}{2\pi} + d\sigma(t)$ be the Lebesgue decomposition of the representing measure μ with respect to the normalized Lebesgue measure $\frac{dt}{2\pi}$ on $\langle -\pi, \pi \rangle$, i.e. $\int_{-\pi}^{\pi} f(t)\, dt < \infty$, $f \geqslant 0$ almost everywhere (a.e.) on $\langle -\pi, \pi \rangle$ with respect to $\frac{dt}{2\pi}$ and $d\sigma$ is singular. Then $\operatorname{Re} p(z)$ has nontangential limits a.e. on $\langle -\pi, \pi \rangle$ (to be denoted $\operatorname{Re} p(\cdot)$) and

$$\operatorname{Re} p(e^{i\theta}) = f(e^{i\theta}) \quad \text{a.e. on } \langle -\pi, \pi \rangle \tag{3}$$

(see [7], Chapter 1, Th. 5.3.).

In [5] the following subclass $\tilde{\mathscr{P}}(B, b; \alpha)$, $0 \leqslant b < 1$, $b < B$, $0 < \alpha < 1$, of $\mathscr{P}$ was introduced: $p \in \tilde{\mathscr{P}}(B, b; \alpha)$ if there exists an open arc $I_\alpha = I_\alpha(p)$ of $\mathbf{T}$ of length $2\pi\alpha$ such that

$$\underline{\lim}_{z \to z_0,\ z \in \mathbf{D}} \operatorname{Re} p(z) \geqslant B \quad \text{for each } z_0 \in I_\alpha \tag{4}$$

and

$$\underline{\lim}_{z \to z_0,\ z \in \mathbf{D}} \operatorname{Re} p(z) \geqslant b \quad \text{for each } z_0 \in \mathbf{T} \setminus \bar{I}_\alpha. \tag{5}$$

Among other results, the following properties of $\tilde{\mathscr{P}}(B, b; \alpha)$ were proved in [5]: 1) a necessary and sufficient condition on the parameters B, b, α for $\tilde{\mathscr{P}}(B, b; \alpha)$ to be nonvoid was given; 2) $\tilde{\mathscr{P}}(B, b; \alpha)$ is compact in the topology given by the uniform convergence on compact subsets of $\mathbf{D}$; 3) $\tilde{\mathscr{P}}(B, b; \alpha)$ is not convex.

In this paper we generalize all these results to the situation where arcs are replaced by measurable subsets of $\mathbf{T}$.

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We start with the following reformulation of conditions (4), (5).

Lemma 1. *Let $I_\alpha \subset \mathbf{T}$ be a given open* arc *and let $p \in \mathcal{P}$. The following conditions are equivalent:*

(c) p fulfils conditions (4) and (5);

(d) p fulfils the conditions

$$\operatorname{Re} p(e^{i\theta}) \geqslant B \quad \text{a.e. on } I_\alpha, \tag{6}$$

$$\operatorname{Re} p(e^{i\theta}) \geqslant B \quad \text{a.e. on } \mathbf{T} \setminus \bar{I}_\alpha. \tag{7}$$

Recall that $\operatorname{Re} p(e^{i\theta})$ *are nontangential limits of* $\operatorname{Re} p$ *which exists* a.e. *on* $\mathbf{T}$ *by* (b).

Proof. (c) $\Longrightarrow$ (d). Clear by (b).

(d) $\Longrightarrow$ (c). Assume there exists a point $z_0 \in I_\alpha$ for which $\varliminf_{z \to z_0,\, z \in \mathbf{D}} \operatorname{Re} p(z) = B' < B$. Denote $\mathbf{D}(z_0, r) = \{z \in \mathbf{C};\ |z - z_0| < r\}$. By the definition of limes inferior, there exists an $\varepsilon > 0$ such that $\operatorname{Re} p(z) < \frac{B+B'}{2} < B$ on $\mathbf{D}(z_0, \varepsilon) \cap \mathbf{D}$ and, simultaneously, the arc $\mathbf{D}(z_0, \varepsilon) \cap \mathbf{T}$ does not intersect the complementary arc $\mathbf{T} \setminus \bar{I}_\alpha$. Then (6) cannot be fulfilled on the subarc of I_α of measure $2 \cdot 2\pi\varepsilon > 0$. Similarly we proceed in (7). □

Now, we are in position to give our main definition.

Definition 1. Let $0 \leqslant b < 1$, $b < B$, $0 < \alpha < 1$, be fixed real numbers and $\mathbf{F}$ a given measurable subset of the unit circle $\mathbf{T}$ of Lebesgue measure $2\pi\alpha$. For each $\tau \in \langle -\pi, \pi)$, denote by $\mathbf{F}_\tau = \{\xi \in \mathbf{T};\ e^{-i\tau}\xi \in \mathbf{F}\}$ the set arising by rotation of $\mathbf{F}$ through the angle τ. Denote by $\mathcal{P}(B, b, \alpha; \mathbf{F})$ the class of functions $p \in \mathcal{P}$ satisfying the following conditions: there exists $\tau = \tau(p) \in \langle -\pi, \pi)$ such that

$$\operatorname{Re} p(e^{i\theta}) \geqslant B \quad \text{a.e. on } \mathbf{F}_\tau \tag{8}$$

and

$$\operatorname{Re} p(e^{i\theta}) \geqslant b \quad \text{a.e. on } \mathbf{T} \setminus \mathbf{F}_\tau. \tag{9}$$

It follows directly from Definition 1 that, for $B > 1$, the class $\mathcal{P}(B, b, \alpha; \mathbf{F})$ does not contain the function $p_0(z) \equiv 1$, $z \in \mathbf{D}$. If $B \leqslant 1$, then, clearly, $p_0 \in \mathcal{P}(B, b, \alpha; \mathbf{F})$ for arbitrary admissible values of the parameters b, α and the set $\mathbf{F}$.

In our further considerations, if it is not otherwise stated, we shall always assume that B, b, α, $\mathbf{F}$ and τ fulfil the conditions from Definition 1.

2.

Theorem 1. *If* $\mathscr{P}(B, b, \alpha; \mathbf{F}) \neq \emptyset$, *then*

$$1 \geqslant \alpha B + (1 - \alpha) b. \tag{10}$$

Proof. Let $p \in \mathscr{P}(B, b, \alpha; \mathbf{F})$. So, there exists $\tau = \tau(p) \in \langle -\pi, \pi)$ such that (8) and (9) are fulfilled. Let $\omega(\cdot; \mathbf{F}_\tau)$ be the harmonic measure of the set $\mathbf{F}_\tau$ with respect to $\mathbf{D}$, i.e.

$$\omega(z; \mathbf{F}_\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{\mathbf{F}_\tau}(e^{it}) \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \, dt \tag{11}$$

where $\chi_{\mathscr{A}}$ is the characteristic function of the set $\mathscr{A}$. Clearly, $0 < \omega(z, \mathbf{F}_\tau) < 1$ in $\mathbf{D}$ and, by (3), $\omega(e^{it}; \mathbf{F}_\tau) = 1$ a.e. on $\mathbf{F}_\tau$ and $\omega(e^{it}; \mathbf{F}_\tau) = 0$ a.e. on $\mathbf{T} \setminus \mathbf{F}_\tau$. Put

$$u_\tau(z) = b + (B - b)\omega(z; \mathbf{F}_\tau).$$

Then $u_\tau(z) = b$ a.e. on $\mathbf{F}_\tau$ and $u_\tau(z) = b$ a.e. on $\mathbf{T} \setminus \mathbf{F}_\tau$ (again in the sense of nontangential limits). Since, by (8) and (9), $\operatorname{Re}\left(p(e^{i\theta}) - u_\tau(e^{i\theta})\right) \geqslant 0$ a.e. on $\mathbf{T}$, we have, for each $z \in \mathbf{D}$, by (2) and (3),

$$\begin{aligned} \operatorname{Re} p(z) &= \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \geqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} p(e^{it}) \cdot \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \, dt \\ &\geqslant \frac{1}{2\pi} \int_{\mathbf{F}_\tau} B \cdot \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \, dt + \frac{1}{2\pi} \int_{\mathbf{T} \setminus \mathbf{F}_\tau} b \cdot \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} \, dt = u_\tau(z), \end{aligned}$$

hence

$$\operatorname{Re} p(z) \geqslant b + (B - b)\omega(z; \mathbf{F}_\tau), \quad z \in \mathbf{D}. \tag{12}$$

For $z = 0$ we obtain, with respect to (1) and $\omega(0; \mathbf{F}_\tau) = \alpha$, inequality (10). □

Remark 1. Inequality (12) corresponds to the well-known two-constant theorem for bounded holomorphic functions ([1], p. 39).

Theorem 2. *Let condition* (10) *hold. Then, for each* $\tau \in \langle -\pi, \pi)$, *there exists a function* $p_{\mathbf{F}_\tau} \in \mathscr{P}(B, b, \alpha; \mathbf{F})$ *such that* $\operatorname{Re} p_{\mathbf{F}_\tau}(e^{i\theta}) = B$ *a.e. on* $\mathbf{F}_\tau$ *and* $\operatorname{Re} p_{\mathbf{F}_\tau}(e^{i\theta}) = b$ *a.e. on* $\mathbf{T} \setminus \mathbf{F}_\tau$.

Proof. Since the disc $\mathbf{D}$ is simply connected, therefore the function

$$h(z; \mathbf{F}_\tau) = \omega(z; \mathbf{F}_\tau) + i\omega^*(z; \mathbf{F}_\tau), \tag{13}$$

where ω^*, $\omega^*(0) = 0$, is the harmonic conjugate of $\omega(z; \mathbf{F}_\tau)$, is holomorphic in $\mathbf{D}$ for each $\tau \in \langle -\pi, \pi)$.

Let the equality in (10) hold. Then

$$p_{\mathbf{F}_\tau}(z) = b + (B - b)h(z; \mathbf{F}_\tau), \quad z \in \mathbf{D}, \tag{14}$$

has the required property.

If $B\alpha + b(1 - \alpha) < 1$, then the function $\tilde{p}_{\mathbf{F}_\tau}(z) = b + (B - b)h(z; \mathbf{F}_\tau)$, $z \in \mathbf{D}$, fulfils (8) and (9) but does not belong to $\mathscr{P}(B, b, \alpha; \mathbf{F})$ since $\tilde{p}_{\mathbf{F}_\tau}(0) = B\alpha + b(1 - \alpha) < 1$. So, it is natural to achieve the required normalization by adding a proper multiple of $\frac{e^{i\gamma}+z}{e^{i\gamma}-z}$, γ real. Since we have $\frac{e^{i\gamma}+z}{e^{i\gamma}-z} = 0$ a.e. on $\mathbf{T}$, therefore, clearly,

$$p_{\mathbf{F}_\tau}(z) = \tilde{p}_{\mathbf{F}_\tau}(z) + (1 - \eta)\frac{e^{i\gamma} + z}{e^{i\gamma} - z}, \quad z \in \mathbf{D},$$

where

$$\eta = B\alpha + b(1 - \alpha), \tag{15}$$

is the required function. □

Corollary 1. *The class* $\mathscr{P}(B, b, \alpha; \mathbf{F})$ *is nonvoid if and only if inequality* (10) *holds. If in* (10) *the equality holds, then*

$$\mathscr{P}(B, b, \alpha; \mathbf{F}) = \{p_{\mathbf{F}_\tau};\ \tau \in \langle -\pi, \pi)\}$$

where $p_{\mathbf{F}_\tau}$ *is the function* (14).

Proof. The first assertion follows from Theorems 1 and 2. The second assertion follows from (1), (12) and the minimum principle for harmonic functions ([1], p. 39). □

Theorem 3. *The class $\mathscr{P}(B, b, \alpha; \mathbf{F})$, where B, b, α satisfy condition* (10), *is compact in the topology given by the uniform convergence on compact subsets of* $\mathbf{D}$.

Proof. Since $\mathscr{P}(B, b, \alpha; \mathbf{F}) \subset \mathscr{P}$ and the class $\mathscr{P}$ is compact, it suffices to prove that $\mathscr{P}(B, b, \alpha; \mathbf{F})$ is closed in $\mathscr{P}$. So, let $\{p_n\}_{n=1}^{\infty}$ be an arbitrary sequence of functions in $\mathscr{P}(B, b, \alpha; \mathbf{F})$ converging to a function $p \in \mathscr{P}$ uniformly on compact subsets of $\mathbf{D}$. For every p_n there exists $\tau_n \in \langle -\pi, \pi)$ such that (8) and (9) are fulfilled on $\mathbf{F}_{\tau_n}$ and $\mathbf{T} \setminus \mathbf{F}_{\tau_n}$, respectively. From $\{\tau_n\}_{n=1}^{\infty}$ we can select a subsequence $\{\tau_{n_k}\}_{k=1}^{\infty}$ converging to some $\tau \in \langle -\pi, \pi)$ for $k \to \infty$ (if $\tau_{n_k} \to \pi$, we put $\tau = -\pi$). Without loss of generality we can suppose $\tau = 0$ (by considering the functions $\tilde{p}_n(z) = p_n(z e^{-i\tau})$) and denote the subsequence by $\{\tau_n\}_{n=1}^{\infty}$ again. Since $\mathbf{F}_{\tau_n} = \{e^{i\tau_n}\xi;\ \xi \in \mathbf{F}\}$, we have $\chi_{\mathbf{F}_{\tau_n}}(\xi) = \chi_{\mathbf{F}}(e^{i\tau_n}\xi)$, for each $\xi \in \mathbf{T}$, and since $\chi_{\mathbf{F}}$ is integrable on $\mathbf{T}$, and thus, the mapping $\tau \to \psi_\tau$, where $\psi_\tau(t) = \chi_{\mathbf{F}}(e^{i(t+\tau)})$, is continuous in the $L^1(\langle -\pi, \pi))$-norm (see e.g. [10], Th. 9.5, p. 183), from

$$\chi_{\mathbf{F}_{\tau_n}}(e^{it}) - \chi_{\mathbf{F}}(e^{it}) = \chi_{\mathbf{F}}(e^{i(t-\tau_n)}) - \chi_{\mathbf{F}}(e^{it})$$

we obtain

$$\int_{-\pi}^{\pi} |\chi_{\mathbf{F}_{\tau_n}}(e^{it}) - \chi_{\mathbf{F}}(e^{it})|\, dt \to 0 \quad \text{as } n \to \infty. \tag{16}$$

Now, fix $z \in \mathbf{D}$. The harmonic measure of the set $\mathbf{F}_{\tau_n}$ with respect to the point $z \in \mathbf{D}$ is

$$\omega(z; \mathbf{F}_{\tau_n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{\mathbf{F}_{\tau_n}}(e^{it}) \operatorname{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) dt, \tag{17}$$

and so, by (12), we have

$$\operatorname{Re} p_n(z) \geqslant b + (B - b)\omega(t; \mathbf{F}_{\tau_n}). \tag{18}$$

We can take limits for $n \to \infty$ on both sides of (18). From the assumption we have that $p_n(z) \to p(z)$, $z \in \mathbf{D}$, as $n \to \infty$. On the other hand, by (16) we have

$$\begin{aligned} |\omega(z; \mathbf{F}) - \omega(z; \mathbf{F}_{\tau_n})| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (\chi_{\mathbf{F}}(e^{it}) - \chi_{\mathbf{F}_{\tau_n}}(e^{it})) \operatorname{Re}\left(\frac{e^{it} + z}{e^{it} - z}\right) dt \right| \\ &\leqslant \frac{1}{2\pi} \frac{1 + |z|}{1 - |z|} \int_{-\pi}^{\pi} |\chi_{\mathbf{F}_{\tau_n}}(e^{it}) - \chi_{\mathbf{F}}(e^{it})|\, dt \to 0 \\ &\quad \text{if } n \to \infty. \end{aligned} \tag{19}$$

Consequently, by (18) we have

$$\operatorname{Re} p(z) \geqslant b + (B - b)\omega(z; \mathbf{F}) \quad \text{for each } z \in \mathbf{D}. \tag{20}$$

Now, by (20), by property (b) and by the boundary properties of $\omega(z;\mathbf{F})$, we obtain for the function p inequalities (8) and (9) on $\mathbf{F}$ and $\mathbf{T}\setminus\mathbf{F}$, respectively, so $p\in\mathscr{P}(B,b,\alpha;\mathbf{F})$. □

Remark 2. It is evident that in the proof of Theorem 3 we have shown, more generally, that $\mathscr{P}(B,b,\alpha;\mathbf{F})$ is closed in $\mathscr{P}$ in the topology given by pointwise convergence. Moreover, from (19) it follows that $\omega(z;\mathbf{F}_{\tau_n})\to\omega(z;\mathbf{F}_\tau)$ uniformly on compact sets in $\mathbf{D}$ if $\tau_n\to\tau$.

3.

In this section we shall study sets $\mathbf{E}\mathscr{P}(B,b,\alpha;\mathbf{F})$ and $\operatorname{supp}\mathscr{P}(B,b,\alpha;\mathbf{F})$ of extreme points and support points of $\mathscr{P}(B,b,\alpha;\mathbf{F})$, respectively.

Let us recall ([8], p. 44, p. 91) that 1) a point $x\in A\subset X$ (X is a vector space over $\mathbf{C}$ or $\mathbf{R}$) is called an extreme point of A if $a,b\in A$, $0<\lambda<1$ and $x=\lambda a+(1-\lambda)b$ implies $a=b=x$; 2) if X is a topological vector space over $\mathbf{C}$ or $\mathbf{R}$, then a point $x\in A\subset X$ is called a support point of the set A if there is a continuous linear functional J on X such that $\operatorname{Re}J$ is nonconstant on A and $\operatorname{Re}J(x)=\max\{\operatorname{Re}J(\sigma);\ \sigma\in A\}$.

For this purpose we denote, for a fixed $\tau\in\langle-\pi,\pi)$, by $\mathscr{P}(B,b,\alpha;\mathbf{F},\tau)$ the set of all functions from $\mathscr{P}(B,b,\alpha;\mathbf{F})$ satisfying (8) and (9) on $\mathbf{F}_\tau$ and $\mathbf{T}\setminus\mathbf{F}_\tau$, respectively. Clearly, $\mathscr{P}(B,b,\alpha;\mathbf{F},\tau)$ is convex, compact and

$$\mathscr{P}(B,b,\alpha;\mathbf{F})=\bigcup_{\tau\in\langle-\pi,\pi)}\mathscr{P}(B,b,\alpha;\mathbf{F},\tau). \tag{21}$$

Proposition 1. (i) $\mathscr{P}(B,b,\alpha;\mathbf{F},\tau)=\{p_{\mathbf{F}_\tau}+(1-\eta)p;\ p\in\mathscr{P}\}$ *where* $p_{\mathbf{F}_\tau}$ *is the function* (14) *and* η *is given by* (15).

(ii) *For every* τ *the correspondence* $p\to p_{\mathbf{F}_\tau}+(1-\eta)p$ *between the classes* $\mathscr{P}$ *and* $\mathscr{P}(B,b,\alpha;\mathbf{F},\tau)$ *is one-to-one.*

(iii) $p\in\mathscr{P}(B,b,\alpha;\mathbf{F},\tau_1)$ *if and only if* $\tilde p(z)\equiv p(\mathrm{e}^{\mathrm{i}\tau}z)\in\mathscr{P}(B,b,\alpha;\mathbf{F},\tau_1+\tau)$.

Proof. (i) For $\eta=1$ this follows from the second assertion of Corollary 1. So, let $0<\eta<1$ and take $q\in\mathscr{P}(B,b,\alpha;\mathbf{F},\tau)$. Put $p(z)=\frac{1}{1-\eta}(q(z)-p_{\mathbf{F}_\tau}(z))$, $z\in\mathbf{D}$. By (12), (13), (14) and the minimum principle we have $\operatorname{Re}p(z)\geqslant 0$, $z\in\mathbf{D}$, and since $q(0)=1$ and $p_{\mathbf{F}_\tau}(0)=\eta$, therefore $p(0)=1$, so $p\in\mathscr{P}$ and $q=p_{\mathbf{F}_\tau}+(1-\eta)p$.

(ii) and (iii) are obvious. □

Now, denote by $\mathscr{E}(B,b,\alpha;\mathbf{F},\tau)$ the set of all $p(z;\gamma,\mathbf{F}_\tau)\in\mathscr{P}(B,b,\alpha;\mathbf{F},\tau)$ of the form

$$p(z;\gamma,\mathbf{F}_\tau)=b+(B-b)h(z;\mathbf{F}_\tau)+(1-\eta)\frac{\mathrm{e}^{\mathrm{i}\gamma}+z}{\mathrm{e}^{\mathrm{i}\gamma}-z},\quad \gamma\text{-real},\ z\in\mathbf{D}, \tag{22}$$

and by $\mathscr{S}(B,b,\alpha;\mathsf{F},\tau)$ the set of all $s(z;\mathsf{F}_\tau)\in\mathscr{P}(B,b,\alpha;\mathsf{F},\tau)$ of the form

$$s(z;\mathsf{F}_\tau)=b+(B-b)h(z;\mathsf{F}_\tau)+(1-\eta)\sum_{k=1}^{m}\lambda_k\frac{1+x_kz}{1-x_kz},\quad z\in\mathbf{D}, \tag{23}$$

where $\lambda_k\geqslant 0$, $\sum\limits_{k=1}^{m}\lambda_k=1$ and $|x_k|=1$; $m=1,2,\ldots$.

From Proposition 1, from the description of extreme points and support points of $\mathscr{P}$ given in [8] (p. 48 and p. 94) and from (21) we immediately obtain

Corollary 2. *For arbitrary admissible B, b, α, F, τ, we have*

$$\mathsf{E}\mathscr{P}(B,b,\alpha;\mathsf{F},\tau)=\mathscr{E}(B,b,\alpha;\mathsf{F},\tau), \tag{24}$$

$$\operatorname{supp}\mathscr{P}(B,b,\alpha;\mathsf{F},\tau)=\mathscr{S}(B,b,\alpha;\mathsf{F},\tau) \tag{25}$$

$$\mathsf{E}\mathscr{P}(B,b,\alpha;\mathsf{F})\subset\bigcup_{\tau\in(-\pi,\pi)}\mathscr{E}(B,b,\alpha;\mathsf{F},\tau). \tag{26}$$

P r o o f. Assertions (24) and (25) follow immediately from Proposition 1, from formulae (22) and (23) and from the description of extreme points and support points of $\mathscr{P}$ given in [8], Theorem 5.2 and Theorem 7.3, respectively. Formula (26) follows directly from (21), (24) and the obvious inclusion $\mathsf{E}\bigcup A_j\subset\bigcup\mathsf{E}A_j$. □

Theorem 4. *Let $p\in\mathscr{P}(B,b,\alpha;\mathsf{F})$ have an expansion* (1) *in $\mathbf{D}$. Then, for $n=1,2,\ldots$,*

$$|q_n|\leqslant 2\left[(B-b)\frac{1}{2\pi}\left|\int_{\mathsf{F}}\mathrm{e}^{-int}\,\mathrm{d}t\right|+1-\eta\right]. \tag{27}$$

This estimate is sharp and is attained only for functions (22) *where*

$$\gamma=-\frac{1}{n}(\arg\int_{\mathsf{F}_\tau}\mathrm{e}^{-int}\,\mathrm{d}t+2k\pi),\quad k\in\mathbf{Z}$$

(for $\int_{\mathsf{F}_\tau}\mathrm{e}^{-int}\,\mathrm{d}t=0$ we put $\arg\int_{\mathsf{F}_\tau}\mathrm{e}^{-int}\,\mathrm{d}t=0$).

P r o o f. Since it is sufficient to verify estimate (27) for the extreme points of $\mathscr{P}(B,b,\alpha;\mathsf{F})$ (see e.g. [8], Th. 4.6, p. 45), by (26) we have only to make sure that the estimate holds for all functions of the form (22) for all $\tau\in\langle-\pi,\pi)$ and is attained on some of them. So, we have only to write the Taylor expansion of the functions (22). Since

$$\frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}=1+2\sum_{n=1}^{\infty}\mathrm{e}^{-\mathrm{i}nt}z^n,\quad z\in\mathbf{D},$$

and the series converges uniformly in $\langle -\pi, \pi\rangle$ for $|z| < \varrho < 1$, we can integrate term by term and obtain by elementary calculations (cf. (11), (13), (15), (22))

$$\begin{aligned} p(z;\gamma,\mathsf{F}_\tau) &= b + (B-b)\int_{\mathsf{F}_\tau} \frac{e^{it}+z}{e^{it}-z}\frac{dt}{2\pi} + (1-\eta)\frac{e^{i\gamma}+z}{e^{i\gamma}-z} \\ &= 1 + 2\sum_{n=1}^{\infty}\left[(B-b)\int_{\mathsf{F}_\tau} e^{-int}\,dt + (1-\eta)e^{-in\gamma}\right]z^n, \end{aligned}$$

hence

$$q_n = 2\left[(B-b)\frac{1}{2\pi}\int_{\mathsf{F}_\tau} e^{-int}\,dt + (1-\eta)e^{-in\gamma}\right].$$

Denoting $\varphi_\tau = \arg\int_{\mathsf{F}_\tau} e^{-int}\,dt$ if $\int_{\mathsf{F}_\tau} e^{-int}\,dt \neq 0$ and putting $\varphi_\tau = 0$ in the opposite case, we have

$$q_n = 2\left[(B-b)\frac{1}{2\pi}\left|\int_{\mathsf{F}_\tau} e^{-int}\,dt\right| + (1-\eta)e^{-i(n\gamma+\varphi_\tau)}\right]e^{i\varphi_\tau},$$

hence

$$\begin{aligned} |q_n| &= 2\left|(B-b)\frac{1}{2\pi}\left|\int_{\mathsf{F}_\tau} e^{-int}\,dt\right| + (1-\eta)e^{-i(n\gamma+\varphi_\tau)}\right| \\ &= 2\left|(B-b)\frac{1}{2\pi}\left|\int_{\mathsf{F}} e^{-int}\,dt\right| + (1-\eta)e^{-i(n\gamma+\varphi_\tau)}\right|. \end{aligned}$$

Since the first term of the sum is nonnegative, we obtain the estimate (27). □

Remark 3. Using definition (15) of η, we can rewrite the estimate (27) as follows:

$$(28)\qquad |q_n| \leqslant 2\left[1 - b - (B-b)\left(\alpha - \frac{1}{2\pi}\left|\int_{\mathsf{F}} e^{-int}\,dt\right|\right)\right].$$

Remark 4. When the parameters B, b, α lie on the boundary of the three-dimensional set given by the inequalities in Definition 1 and inequality (10), we obtain the following limiting cases:

$\mathscr{P}(0,0,\alpha;\mathsf{F}) = \mathscr{P}, \quad \alpha \in (0,1),$
$\mathscr{P}(B,B,\alpha;\mathsf{F}) = \mathscr{P}_B, \quad \alpha \in (0,1),\ B \in (0,1),$
$\mathscr{P}(B,b,\alpha;\mathsf{F}) = \{\mathscr{P}_{\mathsf{F}_\tau} \text{ of the form (14)}, \tau \in \langle -\pi,\pi)\}, \quad \alpha \in (0,1),\ \eta = 1,$
$\mathscr{P}(B,b,0;\mathsf{F}) = \mathscr{P}_b, \quad 0 \leqslant b < 1,\ b < B,$
$\mathscr{P}(B,b,1;\mathsf{F}) = \mathscr{P}_B, \quad 0 \leqslant b \leqslant B < 1,$
$\mathscr{P}(1,1,\alpha;\mathsf{F}) = \{p_0\}.$

Here $\mathscr{P}_\beta$, $0 \leqslant \beta < 1$, is the class of functions (1) with values in the half plane $\operatorname{Re} z > \beta$. Passing suitably to the limits, we obtain form (28) the well-known coefficient estimates in the classes $\mathscr{P}_b$ ([9]) and $\mathscr{P}$ ([2]).

4.

As we have mentioned above, the classes $\mathscr{P}(B, b, \alpha; \mathsf{F}, \tau)$ are convex and hence also connected. In this section we shall discuss the question of convexity and connectedness of the class $\mathscr{P}(B, b, \alpha; \mathsf{F})$. In the sequel we denote by $\ell(A)$ the normalized Lebesgue measure on $\mathbf{T}$ ($\ell(\mathbf{T}) = 1$). We shall need the following geometric lemma.

Lemma 2. *Let $\mathsf{F} \subset \mathbf{T}$ be a closed set, $\ell(\mathsf{F}) = \alpha$, $0 < \alpha < 1$. Then for each $\tau \in (-\pi, \pi)$ there exists $\delta > 0$ such that $\ell(\mathsf{F}_{\tau+h} \cap \mathsf{F}_\tau) < \ell(\mathsf{F}_\tau)$ for each h, $0 < |h| < \delta$.*

Proof. Without loss of generality we can choose $\tau_0 = 0$ and F_0 to be perfect (because the set of isolated points of F_0 is countable and hence a set of Lebesgue measure zero). Denote by D_{F_0} the set of density points of F_0 (i.e. $\xi \in D_{\mathsf{F}_0}$ if and only if $\lim\limits_{r \to 0} \frac{\ell(\mathsf{F}_0 \cap B(\xi, r))}{2r} = 1$ where $B(\xi, r)$ is the subarc of $\mathbf{T}$ with centre at the point ξ and $\ell(B(\xi, r)) = 2r$). Then (see [10], Exercise 11, p. 177) $\ell(D_{\mathsf{F}_0}) = \ell(\mathsf{F}_0) = \alpha > 0$. Hence each interval containing a point of F_0 contains a point of D_{F_0}. Denote $G_0 = \mathbf{T} \setminus \mathsf{F}_0$, so $\ell(G_0) = 1 - \ell(F_0) = 1 - \alpha > 0$. G_0 is an open subset of $\mathbf{T}$, hence G_0 is the sum of a nonvoid finite or countable family of mutually disjoint open arcs $G_i \subset \mathbf{T}$. Let $\ell(G_{i_0}) \geqslant \ell(G_i)$ for every i and put $\delta = \ell(G_{i_0})$. The endpoints ξ_0, ξ_1 of G_{i_0} are lying in F_0. So, by rotating F_0 through any angle h, $|h| < \delta$, $G_{i_0} \cap \mathsf{F}_0$ contains ξ_0 and ξ_1, and so, in any case, a point $\xi \in D_{\mathsf{F}_0}$ and an arc $B(\xi, r_0)$. Take $0 < r < r_0$ such that $\ell(\mathsf{F}_0 \cap B(\xi, r)) > \frac{1}{2}\ell(B(\xi, r)) = r$. Then $\mathsf{F}_0 \cap \mathsf{F}_h \subset \mathsf{F}_0 \setminus (\mathsf{F}_0 \cap B(\xi, r))$, so $\ell(\mathsf{F}_0 \cap \mathsf{F}_h) \leqslant \ell(\mathsf{F}_0) - r < \ell(\mathsf{F}_0)$, q.e.d. □

Theorem 5. *If $0 < \alpha < 1$, then $\mathscr{P}(B, b, \alpha; \mathsf{F})$ is not convex.*

Proof. By Theorem 2 there exists a non-constant function $p \in \mathscr{P}(B, b, \alpha; \mathsf{F})$ such that $\operatorname{Re} p(e^{i\theta}) = B$ a.e. on F, $\operatorname{Re} p(e^{i\theta}) = b$ a.e. on $\mathbf{T} \setminus \mathsf{F}$ (cf. (12)). By similar arguments as in the proof of Lemma 2 there exists $\tau \neq 0$ so that $\ell(\mathsf{F}_\tau \cap \mathsf{F}) < \alpha$. Define $p_\tau(z) = p(e^{-2\pi i \tau} z)$, $z \in \mathbf{D}$. Obviously, $p_\tau \in \mathscr{P}(B, b, \alpha; \mathsf{F}, \tau)$. Join p, p_τ by the segment $p_\lambda = \lambda p_\tau + (1 - \lambda) p$, $0 \leqslant \lambda \leqslant 1$. Clearly, $p_\lambda(0) = 1$. One has $\operatorname{Re} p_\lambda(\xi) \leqslant \lambda b + (1 - \lambda) B < B$ on $\mathbf{T} \setminus \mathsf{F}_\tau$ for $\lambda > 0$ and $\operatorname{Re} p_\lambda(\xi) \leqslant \lambda B + (1 - \lambda) b < B$ on $\mathbf{T} \setminus \mathsf{F}$ for $\lambda < 1$. So, $\operatorname{Re} p_\lambda(\xi) \geqslant B$ can be fulfilled only a.e. on $\mathsf{F}_\tau \cap \mathsf{F}$. Hence, for each $\lambda \in (0, 1)$, p_λ does not belong to $\mathscr{P}(B, b, \alpha; \mathsf{F})$. □

Theorem 6. *$\mathscr{P}(B,b,\alpha;\mathsf{F})$ is arcwise connected (and thus connected).*

Proof. Let $p_1,p_2\in\mathscr{P}(B,b,\alpha;\mathsf{F})$. Then there exist $\tau_1,\tau_2\in\langle-\pi,\pi)$ such that $p_k\in\mathscr{P}(B,b,\alpha;\mathsf{F},\tau_k)$, $k=1,2$. Since the classes $\mathscr{P}(B,b,\alpha;\mathsf{F},\tau)$ are convex, we can join p_1, p_2 by a segment with $p_{\mathsf{F}_{\tau_1}}+1-\eta$, $p_{\mathsf{F}_{\tau_2}}+1-\eta$, respectively, and then $p_{\mathsf{F}_{\tau_1}}+1-\eta$ with $p_{\mathsf{F}_{\tau_2}}+1-\eta$ by the arc $\tau\to p_{\mathsf{F}_\tau}+1-\eta$, $\tau_1\leqslant\tau\leqslant\tau_2$ (cf. Remark 2). □

Remark 5. In the case $B\leqslant 1$, the assertion of Theorem 6 is obvious because $p_0\in\mathscr{P}(B,b,\alpha;\mathsf{F})$ for each $\tau\in\langle-\pi,\pi)$.

Remark 6. All the properties of the class $\mathscr{P}(B,b,\alpha;\mathsf{F})$ which we have examined up to now (i.e. compactness, convexity and connectedness) require non-trivial means from real analysis for their proofs, but can be proved almost trivially if we restrict our attention to the classes $\mathscr{P}(B,b,\alpha;\mathsf{F},\tau)$. In this context, the following properties can be of some interest.

Lemma 3. *For each $\tau\in\langle-\pi,\pi)$ we have*

$$\lim_{h\to0}\ell(\mathsf{F}_{\tau+h}\cap\mathsf{F}_\tau)=\ell(\mathsf{F}_\tau).$$

Proof. We can suppose $\tau=0$ and write $\mathsf{F}_\tau=\mathsf{F}_0$. Since $\chi_{\mathsf{F}_h\cap\mathsf{F}_0}=\chi_{\mathsf{F}h}\cdot\chi_{\mathsf{F}_0}$, we have

$$\begin{aligned}\ell(\mathsf{F}_0)-\ell(\mathsf{F}_h\cap\mathsf{F}_0)&=\int_{-\pi}^{\pi}(\chi_{\mathsf{F}_0}-\chi_{\mathsf{F}_0}\cdot\chi_{\mathsf{F}_h})\frac{\mathrm{d}t}{2\pi}=\int_{-\pi}^{\pi}(\chi_{\mathsf{F}_0}^2-\chi_{\mathsf{F}_0}\cdot\chi_{\mathsf{F}_h})\frac{\mathrm{d}t}{2\pi}\\&=\int_{-\pi}^{\pi}\chi_{\mathsf{F}_0}(\chi_{\mathsf{F}_0}-\chi_{\mathsf{F}_h})\frac{\mathrm{d}t}{2\pi}\leqslant\int_{-\pi}^{\pi}|\chi_{\mathsf{F}_0}-\chi_{\mathsf{F}_h}|\frac{\mathrm{d}t}{2\pi}\\&=\int_{-\pi}^{\pi}|\psi_{\mathsf{F}_0}(t+h)-\psi_{\mathsf{F}_0}(t)|\frac{\mathrm{d}t}{2\pi}\end{aligned}$$

where we denoted $\psi_{\mathsf{F}_0}(t)=\chi_{\mathsf{F}_0}(\mathrm{e}^{\mathrm{i}t})$. But $\lim_{h\to0}\int_{-\pi}^{\pi}|\psi_{\mathsf{F}_0}(t+h)-\psi_{\mathsf{F}_0}(t)|\,\mathrm{d}t=0$ (see e.g. [10], Th. 9.5, p. 183) and $\lim_{h\to0}\ell(\mathsf{F}_0\cap\mathsf{F}_h)=\ell(\mathsf{F}_0)$, q.e.d. □

Theorem 7. *Let $\eta<1$. Then there exist $\tau_i=\tau_i(\mathsf{F})$, $i=1,2$, such that for each $\tau\in(-\tau_i,\tau_i)$, $i=1,2$, we have*

(i) $\mathscr{P}(B,b,\alpha;\mathsf{F},\tau)\neq\mathscr{P}(B,b,\alpha;\mathsf{F},0)$ for $0<|\tau|<\tau_1$, if F is closed,

(ii) $\mathscr{P}(B,b,\alpha;\mathsf{F},\tau)\cap\mathscr{P}(B,b,\alpha;\mathsf{F},0)\neq\emptyset$ for $|\tau|<\tau_2$.

Proof. (i) By Lemma 2 there exists $\tau_1>0$ such that, for each $\tau\in(-\tau_1,\tau_1)$, one has $\ell(\mathsf{F}\cap\mathsf{F}_\tau)<\ell(\mathsf{F})$. Hence the function $\tilde p_{\mathsf{F}}(z)=b+(B-b)h(z;\mathsf{F})+(1-\eta)\frac{\mathrm{e}^{\mathrm{i}\gamma}+z}{\mathrm{e}^{\mathrm{i}\gamma}-z}$,

γ real, $z \in \mathbf{D}$, does not belong to $\mathscr{P}(B, b, \alpha; \mathsf{F}, \tau)$ since $\operatorname{Re} \tilde{p}_{\mathsf{F}}(e^{i\theta}) = b < B$ a.e. on $\mathsf{F}_\tau \setminus \mathsf{F}$, and $\ell(\mathsf{F}_\tau \setminus \mathsf{F}) = \ell(\mathsf{F}_\tau) - \ell(\mathsf{F}_\tau \cap \mathsf{F}) = \ell(\mathsf{F}) - \ell(\mathsf{F} \cap \mathsf{F}_\tau) > 0$.

(ii) Define

$$f(e^{it}) = B \quad \text{on } \mathsf{F} \cup \mathsf{F}_\tau,$$
$$f(e^{it}) = b \quad \text{on } \mathbf{T} \setminus (\mathsf{F} \cup \mathsf{F}_\tau)$$

and

$$p(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \frac{e^{it} + z}{e^{it} - z} \, dt, \quad z \in \mathbf{D}.$$

Then $\operatorname{Re} p(e^{i\theta}) = B$ a.e. on $\mathsf{F} \cup \mathsf{F}_\tau$, $\operatorname{Re} p(e^{i\theta}) = b$ a.e. on $\mathbf{T} \setminus (\mathsf{F} \cup \mathsf{F}_\tau)$. It is clear that $\operatorname{Re} p$ fulfils condition (8) a.e. on F and F_τ and condition (9) a.e. on $\mathbf{T} \setminus \mathsf{F}$ and $\mathbf{T} \setminus \mathsf{F}_\tau$. An easy calculation gives

$$p(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \, dt = \eta + (B - b)[\ell(\mathsf{F}) - \ell(\mathsf{F} \cup \mathsf{F}_\tau)]. \tag{29}$$

However, by Lemma 3, $\lim_{\tau \to 0} \ell(\mathsf{F} \cup \mathsf{F}_\tau) = \ell(\mathsf{F})$. Hence, by (29) and on account of $\eta < 1$, there exists $\tau_2 > 0$ such that $p(0) < 1$ for $|\tau| < \tau_2$. Then the function

$$\tilde{p}(z) = p(z) + (1 - p(0)) \frac{e^{i\gamma} + z}{e^{i\gamma} - z}, \quad \gamma \text{ real}, \ z \in \mathbf{D},$$

belongs to $p(B, b, \alpha; \mathsf{F}, \tau) \cap \mathscr{P}(B, b, \alpha; \mathsf{F}, 0)$ for each $\tau \in (-\tau_2, \tau_2)$. □

5.

Next, we introduce

Definition 2. Let $0 \leqslant b < 1$, $b < B$, $0 < \alpha < 1$, be fixed real numbers. Denote by $\mathscr{P}(B, b, \alpha)$ the class of functions $p \in \mathscr{P}$ such that the exists a measurable subset F of $\mathbf{T}$ of Lebesgue measure $2\pi\alpha$ such that $p \in \mathscr{P}(B, b, \alpha; \mathsf{F})$.

It follows directly form Definition 2 that

$$\mathscr{P}(B, b, \alpha) = \bigcup_{\mathsf{F}} \mathscr{P}(B, b, \alpha; \mathsf{F}) \tag{30}$$

where $\mathsf{F} \subset \mathbf{T}$ satisfies the conditions mentioned above.

Let $p \in \mathscr{P}(B, b, \alpha)$ have the expansion (1). Then form (30) and Theorem 4 we obtain

$$|q_n| \leqslant 2(1 - \eta) + \frac{1}{\pi}(B - b)Q_n, \quad n = 1, 2, \ldots, \tag{31}$$

where

$$Q_n = \sup_{\mathsf{F}\in F}\left\{Q_n(\mathsf{F}) := \left|\int_{\mathsf{F}} e^{-int}\,dt\right|\right\}, \quad n = 1, 2, \ldots, \tag{32}$$

and F is the st of all measurable subsets of the circle $\mathbf{T}$ of Lebesgue measure $2\pi\alpha$. Estimating roughly the real and imaginary parts of $\int_{\mathsf{F}} e^{-int}\,dt$, we obtain $Q_n(\mathsf{F}) \leqslant 4\pi\alpha$ for each $\mathsf{F} \in F$, therefore $Q_n < \infty$, $n = 1, 2, \ldots$.

In view of the rotation-invariance of the Lebesgue measure on $\mathbf{T}$ we have $\int_{\mathsf{F}} e^{-int}\,dt = e^{in\tau}\int_{\mathsf{F}_\tau} e^{-int}\,dt$, $\mathsf{F} \in F$, $\tau \in \langle -\pi, \pi)$. Taking $\tau = \tau^* = \frac{1}{n}\arg\int_{\mathsf{F}} e^{-int}\,dt$, we obtain $Q_n(\mathsf{F}) = \int_{\mathsf{F}_{\tau^*}} \cos nt\,dt$. From here we easily see that

$$Q_n = \sup_{\mathsf{F}\in F}\int_{\mathsf{F}} \cos nt\,dt\,. \tag{33}$$

The following lemma is the clue for estimating Q_n. We write $m(A)$ for the Lebesgue measure of $A \subset \mathbf{R}$.

Lemma 4. *Let $a, b \in \mathbf{R}$ and let $E \subset \langle a, b\rangle$ be a measurable subset of the interval $\langle a, b\rangle$ and f a bounded nondecreasing function on $\langle a, b\rangle$. Then*

$$\int_a^{a+m(E)} f(t)\,dt \leqslant \int_E f(t)\,dt \leqslant \int_{b-m(E)}^b f(t)\,dt\,. \tag{34}$$

P r o o f. Choose $\varepsilon > 0$ arbitrarily. Clearly, it is sufficient to suppose $0 < m(E) < b - a$. By definition, there exist a compact set F and an open set G such that $F \subset E \subset G$ and $m(G \setminus F) < \varepsilon/(2M)$ where $0 < M < \infty$ is the upper bound for $|f|$ on $\langle a, b\rangle$. From $E\setminus F \subset G\setminus F$ and $G\setminus E \subset G\setminus F$ it follows that $m(E\setminus F) < \varepsilon/(2M)$ and $m(G \setminus E) < \varepsilon/(2M)$, respectively. G is the sum of disjoint open intervals I_k, $k = 1, 2, \ldots$.

Moreover, $\bigcup_{k=1}^{\infty} I_k \supset F$ and hence one can choose a finite set of disjoint intervals I_1, $I_2, \ldots, I_s$ such that $I = \bigcup_{k=1}^{s} I_k \supset F$. We order them in such a way that $a_1 < a_2 < \ldots < a_s$ where a_k is the left endpoint of I_k, $k = 1, \ldots, s$, and outside of $\langle a, b\rangle$ we put $f = 0$. Then $m(I \setminus E) \leqslant m(G \setminus E) < \varepsilon/(2M)$ and $m(E \setminus I) \leqslant m(E \setminus F) < \varepsilon/(2M)$. Since both the sets $E \cup (I \setminus E)$ and $I \cup (E \setminus I)$ are decompositions of $I \cup E$ in two disjoint sets, we have

$$\int_E f(t)\,dt + \int_{I\setminus E} f(t)\,dt = \int_I f(t)\,dt + \int_{E\setminus I} f(t)\,dt,$$

and so

$$\int_E f(t)\,\mathrm{dt} = \int_I f(t)\,\mathrm{dt} + \int_{E\setminus I} f(t)\,\mathrm{dt} + \int_{I\setminus E} [-f(t)]\,\mathrm{dt}.$$

However,

$$\int_{E\setminus I} f(t)\,\mathrm{dt} \leqslant M\cdot m(E\setminus I) < M\cdot\varepsilon/(2M) = \varepsilon/2$$

and

$$\int_{I\setminus E} [-f(t)]\,\mathrm{dt} \leqslant M\cdot m(I\setminus E) < M\cdot\varepsilon/(2M) = \varepsilon/2,$$

hence

$$\int_E f(t)\,\mathrm{dt} < \int_I f(t)\,\mathrm{dt} + \varepsilon. \tag{35}$$

It remains to estimate $\int_I f(t)\,\mathrm{dt} = \sum_{k=1}^{s} \int_{I_k} f(t)\,\mathrm{dt}$ from above. In view of the translation-invariance of m on F and since f is nondecreasing, we have, for any c, d, $a \leqslant c < d \leqslant b$, and for each $r \in (0, b-d)$,

$$\int_c^d f(t)\,\mathrm{dt} = \int_{c+r}^{d+r} f(t-r)\,\mathrm{dt} \leqslant \int_{c+r}^{d+r} f(t)\,\mathrm{dt}.$$

Let $c = a_k$, $d = b_k$, $r_k = b - b_k - \sum_{j=k+1}^{s} m(I_j)$ where $I_k = (a_k, b_k)$. Since $r_k \in \langle 0, b - b_k\rangle$, therefore

$$\int_{I_k} f(t)\,\mathrm{dt} \leqslant \int_{a_k+r_k}^{b_k+r_k} f(t)\,\mathrm{dt} = \int_{b-\sum_{j=k}^{s} m(I_j)}^{b-\sum_{j=k+1}^{s} m(I_j)} f(t)\,\mathrm{dt}, \quad \text{for } k = 1, 2, \ldots, s-1 \tag{36}$$

and also

$$\int_{I_s} f(t)\,\mathrm{dt} \leqslant \int_{b-m(I_s)}^{b} f(t)\,\mathrm{dt}. \tag{37}$$

By adding (36) and (37) we obtain

$$\int_I f(t)\,\mathrm{dt} \leqslant \int_{b-m(I)}^{b} f(t)\,\mathrm{dt},$$

and so, by (35),

$$\int_E f(t)\,\mathrm{dt} < \int_{b-m(I)}^{b} f(t)\,\mathrm{dt} + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and $F \subset I \subset G$, the right-hand side in inequality (34) is proved. The proof of the left-hand side in inequality (34) follows the same lines. □

R e m a r k 7. If g is nonincreasing on $\langle a, b\rangle$, then, applying Lemma 4 to the function $f = -g$, we obtain

$$\int_{b-m(E)}^{b} g(t)\,\mathrm{d}t \leqslant \int_{E} g(t)\,\mathrm{d}t \leqslant \int_{a}^{a+m(E)} g(t)\,\mathrm{d}t. \tag{38}$$

Theorem 8. *Let $p \in \mathscr{P}(B, b, \alpha)$ have the expansion (1) in $\mathbf{D}$. Then, for $n = 1, 2, \ldots$*

$$|q_n| \leqslant 2\Big[\frac{B-b}{\pi}\sin\alpha\pi + 1 - \eta\Big]. \tag{39}$$

The estimate (39) is sharp and is attained only on the function $p^*(z) = p(\varepsilon z; \mathsf{F})$, $|\varepsilon| = 1$, $z \in \mathbf{D}$, where

$$\mathsf{F} = \mathsf{F}_n = \bigcup_{k=1}^{n} \mathsf{F}_n^k \quad \text{and} \quad \mathsf{F}_n^k = \Big\{z \in \mathbf{T};\, z = \mathrm{e}^{\frac{2k\pi\mathrm{i}}{n}}\mathrm{e}^{\mathrm{i}\varrho},\ \frac{-\alpha\pi}{n} \leqslant \varrho \leqslant \frac{\alpha\pi}{n}\Big\},$$

and so

$$p(z; \mathsf{F}) = b + \frac{B-b}{2\pi}\sum_{k=1}^{n}\int_{(-\alpha+2k)\pi/n}^{(\alpha+2k)\pi/n} \frac{\mathrm{e}^{\mathrm{i}t}+z}{\mathrm{e}^{\mathrm{i}t}-z}\,\mathrm{d}t + (1+\eta)\frac{1+z}{1-z}, \quad z \in \mathbf{D}. \tag{40}$$

P r o o f. Fix n and divide the interval $\langle -\pi, \pi)$ into n disjoint subintervals

$$T_k = \Big\langle \pi\frac{-n+2(k-1)}{n}, \pi\frac{-n+2k}{n}\Big), \quad k = 1, 2, \ldots, n, \text{ if } n \text{ is odd}$$

and

$$T_k = \Big\langle \pi\frac{-n+2k-1}{n}, \pi\frac{-n+2k+1}{n}\Big), \quad k = 1, 2, \ldots, n-1,$$
$$T_n = \Big\langle \pi - \frac{\pi}{n}, \pi\Big) \cup \Big\langle -\pi, -\pi + \frac{\pi}{n}\Big), \quad \text{if } n \text{ is even.}$$

Then

$$Q_n(\mathsf{F}) = \sum_{k=1}^{n}\int_{\mathsf{F}\cap T_k} \cos ns\,\mathrm{d}s \tag{41}$$

(here and in the sequel, a subset of the unit circle $\mathbf{T}$ is identified with the set of the corresponding points of $\langle -\pi, \pi)$). Denote $m(\mathsf{F} \cap T_k) = 2\pi\alpha_k$. Since $\cos ns$ is increasing on the left half T_k^1 of T_k and decreasing on the second half T_k^2 of T_k (for n even, $k = n$: $T_n^1 = \langle \pi - \frac{\pi}{n}, \pi)$, $T_n^2 = \langle -\pi, -\pi + \frac{\pi}{n}))$, we obtain by (34) and (38)

$$0 \leqslant \int_{F \cap T_k} \cos ns \, ds \leqslant \int_{\pi\frac{-n+2k-1}{n} - 2\pi\alpha_k^{(1)}}^{\pi\frac{-n+2k-1}{n} + 2\pi\alpha_k^{(2)}} \cos ns \, ds, \quad k = 1, \ldots, n, \text{ for } n \text{ odd}$$

and

$$0 \leqslant \int_{F \cap T_k} \cos ns \, ds \leqslant \int_{\pi\frac{-n+2k}{n} - 2\pi\alpha_k^{(1)}}^{\pi\frac{-n+2k}{n} + 2\pi\alpha_k^{(2)}} \cos ns \, ds, \quad k = 1, 2, \ldots, n-1$$

and

$$0 \leqslant \int_{F \cap T_n} \cos ns \, ds \leqslant \int_{\pi - 2\pi\alpha_k^{(1)}}^{\pi} \cos ns \, ds, \quad + \int_{-\pi}^{-\pi + 2\pi\alpha_n^{(2)}} \cos ns \, ds$$

if n is even, where $2\pi\alpha_k^{(i)} = m(F \cap T_k^i)$, $i = 1, 2$. Since $\cos \pi(-n + 2k + 1) = 1$ for n odd; $\cos \pi(-n + 2k) = 1$, $\cos(\pm n\pi) = 1$ for n even and $\cos(-x) = \cos x$, these integrals are not greater than

$$\int_{\pi\frac{-n+2k-1}{n} - 2\pi\frac{\alpha_k^{(1)} + \alpha_k^{(2)}}{2}}^{\pi\frac{-n+2k-1}{n} + 2\pi\frac{\alpha_k^{(1)} + \alpha_k^{(2)}}{2}} \cos ns \, ds, \qquad \int_{\pi\frac{-n+2k}{n} - 2\pi\frac{\alpha_k^{(1)} + \alpha_k^{(2)}}{2}}^{\pi\frac{-n+2k}{n} + 2\pi\frac{\alpha_k^{(1)} + \alpha_k^{(2)}}{2}} \cos ns \, ds,$$

$$\int_{\pi - 2\pi\frac{\alpha_k^{(1)} + \alpha_k^{(2)}}{2}}^{\pi} \cos ns \, ds + \int_{-\pi}^{-\pi + 2\pi\frac{\alpha_k^{(1)} + \alpha_k^{(2)}}{2}} \cos ns \, ds,$$

respectively.

However, $\alpha_k^{(1)} + \alpha_k^{(2)} = \alpha_k$ and so we finally calculate

$$\int_{F \cap T_k} \cos ns \, ds \leqslant \frac{2}{n} \sin n\pi\alpha_k, \quad k = 1, 2, \ldots, n;$$

thus we have

$$\int_F \cos ns \, ds = \sum_{k=1}^{n} \int_{F \cap T_k} \cos ns \, ds \leqslant \frac{2}{n} \sum_{k=1}^{n} \sin n\pi\alpha_k. \tag{42}$$

Now, we notice that $2\pi\alpha_k \leqslant \frac{2\pi}{n}$ and, of course, $0 \leqslant \alpha_k \leqslant \alpha$, so $0 \leqslant \alpha_k \leqslant \min(\frac{1}{n}, \alpha)$. Hence (31), (32), (33), (41) and (42) imply

$$|q_n| \leqslant 2(1-\eta) + \frac{2(B-b)}{\pi} \sup\left\{\frac{1}{n}\sum_{k=1}^{n} \sin n\pi\alpha_k\right\},$$

where the upper bound is taken over all systems $(\alpha_1, \ldots, a_n)$ such that $0 \leqslant \alpha_k \leqslant \min(\frac{1}{n}, \alpha)$ and $\sum_{k=1}^{n} \alpha_k = \alpha$. But because $0 \leqslant \alpha_k \leqslant \frac{1}{n}$, we have $0 \leqslant n\pi\alpha_k \leqslant \pi$. On the interval $\langle 0, \pi \rangle$ the function $\sin x$ is concave and so

$$\frac{1}{n}\sum_{k=1}^{n} \sin n\pi\alpha_k \leqslant \sin \frac{1}{n}\sum_{k=1}^{n} n\pi\alpha_k = \sin \alpha\pi.$$

The equality holds if and only if $\alpha_k = \frac{\alpha}{n}$, $k = 1, 2, \ldots, n$ (notice that $\frac{\alpha}{n} < \frac{1}{n}$).

The form of the extremal functions is a consequence of the form of the set $\mathbf{F}$ shown in Theorem 8 and follows form formula (22). Since $\mathscr{P}(B, b, \alpha; \mathbf{F}_1 \cup \mathbf{F}_2) = \mathscr{P}(B, b, \alpha; \mathbf{F}_1)$ for an arbitrary measurable set $\mathbf{F}_1$ and an arbitrary set $\mathbf{F}_2$ of Lebesgue measure 0 therefore, for a fixed n, the function (40) is the only function realizing the maximum of $|q_n|$ in the class $\mathscr{P}(B, b, \alpha)$.

The theorem is proved. □

From Definitions 1 and 2, Lemma 1 and Theorem 8 we get

Corollary 3. *Let* $p \in \tilde{\mathscr{P}}(B, b; \alpha)$ *have the expansion* (1) *in* $\mathbf{D}$. *Then, for* $n = 1, 2, \ldots$,

$$|q_n| \leqslant 2\left[\frac{B-b}{\pi} \sin \alpha\pi + 1 - \eta\right]. \tag{43}$$

Remark 8. The estimate (43) for $n = 1$ is sharp. For $n = 2, 3, \ldots$, it is not sharp because the function (40) belongs to the class $\mathscr{P}(B, b, \alpha)$ but not to $\tilde{\mathscr{P}}(B, b; \alpha)$. The sharp estimate in the class $\tilde{\mathscr{P}}(B, b; \alpha)$ for $n = 2, 3, \ldots$ is ([6])

$$|q_n| \leqslant 2\left[\frac{B-b}{n\pi} |\sin n\alpha\pi| + 1 - \eta\right].$$

The estimate can also be obtained directly from (27).

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