Mathematica Bohemica

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Mathematica Bohemica, Vol. 116 (1991), No. 1, 20–59

Persistent URL: http://dml.cz/dmlcz/126195

116 (1991) MATHEMATICA BOHEMICA No. 1, 20–59

REGULATED FUNCTIONS

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(Received August 18, 1988)

Summary. The first section consists of auxiliary results about nondecreasing real functions. In the second section a new characterization of relatively compact sets of regulated functions in the sup-norm topology is brought, and the third section includes, among others, an analogue of Helly's Choice Theorem in the space of regulated functions.

Keywords: regulated function, linear prolongation along an increasing function, ε-variation

AMS classification: 27A45, 46E15

INTRODUCTION

When investigating integral equations in the space of regulated functions there is a need to clarify some questions concerning the pointwise convergence of regulated functions. While the uniform convergence of regulated functions has been met with in classical literature and further interesting results have been brought e.g. by Ch. S. Hönig in [3], [4], the pointwise convergence has not been studied in a sufficient measure so far.

During the study of the pointwise convergence it has appeared fruitful to introduce a method of a prolongation along an increasing function, which is useful also for establishing new properties of regulated functions.

1. PRELIMINARIES. REAL MONOTONE FUNCTIONS

1.1. The symbol $\mathbb{N}$ will denote the set of all positive integers. For $N \in \mathbb{N}$ the symbol $\mathbb{R}^N$ denotes the N-dimensional Euclidean space with the norm $|\cdot|$. In case $N = 1$ we write $\mathbb{R}^1 = \mathbb{R}$.

The set of all continuous functions defined on an interval $[a, b]$ and with values in $\mathbb{R}^N$ is denoted by $\mathscr{C}_N[a, b]$. In case $[a, b] = [0, 1]$ we write $\mathscr{C}_N[0, 1] = \mathscr{C}_N$.

The symbol $(a_n)_{n=1}^{\infty}$ denotes the sequence $\{a_1, a_2, a_3, \ldots\}$.

The symbol $y \circ v$ denotes the composed function $y(v(t))$, provided it is well-defined. If Y is a set of functions then $Y \circ v = \{y \circ v;\ y \in Y\}$. If V is also a set of functions then $Y \circ V = \{y \circ v;\ y \in Y,\ v \in V\}$.

For any bounded function $x: [a, b] \to R^N$ we denote $\|x\|_{[a,b]} = \sup\{|x(t)|; t \in [a, b]\}$. If there is no danger of misunderstanding, we write shortly $\|x\|$.

The symbol $BV_N[a, b]$ denotes the set of all functions $x: [a, b] \to R^N$ with bounded variation; $BV_N[0, 1] = BV_N$.

1.2. The function $x: [a, b] \to R^N$ is *regulated* if for every $t \in [a, b)$ the right-sided limit $\lim_{\tau \to t+} x(\tau) = x(t+)$ exists and is finite, and for every $t \in (a, b]$ the left-sided limit $\lim_{\tau \to t-} x(\tau) = x(t-)$ exists and is finite.

The linear space of all regulated functions from $[a, b]$ to R^N will be denoted by $\mathscr{R}_N[a, b]$; we write $\mathscr{R}_N[0, 1] = \mathscr{R}_N$. It is usual to define the topology of uniform convergence on $\mathscr{R}_N[a, b]$, which is given by the sup-norm $\|\cdot\|_{[a,b]}$.

If a sequence of regulated functions $(x_n)_{n=1}^{\infty} \subset \mathscr{R}_N[a, b]$ converges uniformly to a function x_0, we write $x_n \rightrightarrows x_0$.

1.3. A set $\mathscr{A} \subset \mathscr{R}_N[a, b]$ has *uniform one-sided limits* at a point $t_0 \in [a, b]$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in \mathscr{A}$ and $t \in [a, b]$ we have: If $t_0 < t < t_0 + \delta$ then $|x(t) - x(t_0+)| < \varepsilon$; if $t_0 - \delta < t < t_0$ then $|x(t_0-) - x(t)| < \varepsilon$.

A set $\mathscr{A} \subset \mathscr{R}_N[a, b]$ is called *equiregulated*, if it has uniform one-sided limits at every point $t_0 \in [a, b]$.

1.4. Often it is useful to identify such regulated functions which have the same one-sided limits, and to deal e.g. only with left-continuous functions (see [3], p. 20 or [4], Def. 1.5): For $x \in \mathscr{R}_N[a, b]$ let us define $x^-(t) = x(t-)$ for $t \in (a, b]$, $x^-(a) = x(a+)$. The set

$$\mathscr{R}_N^-[a, b] = \{x \in \mathscr{R}_N[a, b];\ x^- = x\}$$

is a closed linear subspace of $\mathscr{R}_N[a, b]$. Two functions $x, y \in \mathscr{R}_N[a, b]$ are considered equivalent if $x^- = y^-$; the class of equivalence of any function $x \in \mathscr{R}_N[a, b]$ contains precisely one function from $\mathscr{R}_N^-[a, b]$.

Let us recall several properties of regulated functions:

1.5. A function $x: [a, b] \to R^N$ is regulated if and only if it is a uniform limit of a sequence of piecewise constant functions ([1], 7.3.2.1).

1.6. Every regulated function has an at most countable number of points of discontinuity ([1], 7.3.2.1).

1.7. Every regulated function from a compact interval $[a, b]$ to R^N is bounded by a constant (a consequence of 1.5).

1.8. The normed linear space $(\mathscr{R}_N[a, b]; \|\cdot\|)$ is a Banach space (a consequence of [1], 7.3.2.1 (2)).

1.9. Proposition. *A function $x: [a, b] \to R^N$ is regulated if and only if for every $\varepsilon > 0$ there is a finite sequence*

$$a = t_0 < t_1 < \ldots < t_n = b$$

such that

(1.1) *if* $t_{i-1} < t' < t'' < t_i$ *then* $|x(t'') - x(t')| < \varepsilon$

holds for every $i = 1, 2, \ldots, n$.

Proof. (i) Assume that x is regulated. Let $\varepsilon > 0$ be given. Let us denote by C the set of all $\tau \in (a, b]$ such that there is a finite sequence $a = t_0 < t_1 < \ldots < t_k = \tau$ satisfying (1.1) with k instead of n.

Since the limit $x(a+)$ exists, there is $\tau > a$ such that $|x(t) - x(a+)| < \varepsilon/2$ for $t \in (a, \tau)$. Then for every $a < t' < t'' < \tau$ we have

$$|x(t'') - x(t')| \leqq |x(t'') - x(a+)| + |x(t') - x(a+)| < \varepsilon .$$

Consequently $\tau \in C$. Denote $c = \sup C$; we have $c > a$.

Since the limit $x(c-)$ exists, there is $\delta > 0$ such that $|x(t) - x(c-)| < \varepsilon/2$ for every $t \in (c - \delta, c)$. Let us find a point $\tau \in C \cap (c - \delta, c)$. Since $\tau \in C$, there is a finite sequence $a = t_0 < t_1 < \ldots < t_k = \tau$ such that (1.1) holds with k instead of n. If we denote $t_{k+1} = c$, then (1.1) holds also for $n = k + 1$, since

$$|x(t'') - x(t')| \leqq |x(t'') - x(c-)| + |x(t') - x(c-)| < \varepsilon$$

provided $t_k = \tau < t' < t'' < c = t_{k+1}$. Hence $c \in C$. Similarly as at the beginning of this proof it can be shown that if $c < b$ then there is $t > c$ which belongs to C. This is impossible, hence $c = b$.

(ii) Let $t \in [a, b]$ and $\varepsilon > 0$ be given. Assume that there is a finite sequence $a = t_0 < t_1 < \ldots < t_n = b$ such that (1.1) holds.

In case that $t = t_i$ for some $i \in \{1, 2, \ldots, n - 1\}$, denote $\delta = \min \{t_{i+1} - t_i, t_i - t_{i-1}\}$.

In case $t = a$ we denote $\delta = t_1 - t_0$; if $t = b$ then $\delta = t_n - t_{n-1}$. If $t \in (t_{i-1}, t_i)$ for some $i \in \{1, 2, \ldots, n\}$, we denote

$$\delta = \min \{t_i - t,\ t - t_{i-1}\} .$$

In any of the cases listed above we have the following:

(1.2) If $t', t'' \in [a, b]$ and their $t - \delta < t' < t'' < t$ or $t < t' < t'' < t + \delta$ then $|x(t'') - x(t')| < \varepsilon$.

The Bolzano-Cauchy Theorem implies that if for every $\varepsilon > 0$ there is $\delta > 0$ such that (1.2) holds, then the limits $x(t-)$, $x(t+)$ exist.

1.10. Definition. *For every nondecreasing function* $f\colon [a, b] \to [c, d]$ *such that* $f(a) = c, f(b) = d$ *and* $a < b, c < d$ *let us define an "inverse function"* $f_{-1}\colon [c, d] \to [a, b]$ *by the formula*

$$f_{-1}(s) = \inf \{t \in [a, b];\ f(t-) \leqq s \leqq f(t+)\} \quad \textit{for} \quad s \in (c, d) ;$$

$$f_{-1}(c) = a , \quad f_{-1}(d) = b$$

(*we assume that* $f(a-) = f(a), f(b+) = f(b)$).

1.11. Proposition. *Assume that* $f: [a, b] \to [c, d]$ *is a nondecreasing function,* $f(a) = c, f(b) = d$. *Then*

(i) *the function* $f_{-1}: [c, d] \to [a, b]$ *is nondecreasing and left-continuous on* (c, d);

(ii) *if f is left-continuous on* (a, b) *then* $(f_{-1})_{-1} = f$;

(iii) f_{-1} *is continuous on* $[c, d]$ *if and only if f is increasing on* $[a, b]$;

(iv) *if f is increasing on* $[a, b]$ *then* $f_{-1}(f(t)) = t$ *for* $t \in [a, b]$.

Proof. (i) 1. For every nondecreasing function $\varphi: [\alpha, \beta] \to [\gamma, \delta]$ such that $\varphi(\alpha) = \gamma$, $\varphi(\beta) = \delta$ let us define a set

$$\Psi_\varphi = \{(t, s) \in \mathbb{R}^2 ;\quad t \in [\alpha, \beta] ,\quad \varphi(t-) \leqq s \leqq \varphi(t+)\} .$$

We will prove that Ψ_φ has the following properties:

(a) If $(t_1, s_1) \in \Psi_\varphi$ and $(t_2, s_2) \in \Psi_\varphi$, then either $t_1 \leqq t_2$ and $s_1 \leqq s_2$, or $t_1 \geqq t_2$ and $s_1 \geqq s_2$.

(b) Ψ_φ is a compact subset of $\mathbb{R}^2$.

ad (a): For $t_1 < t_2$ we have $\varphi(t_1+) \leqq \varphi(t_2-)$. Then $s_1 \leqq \varphi(t_1+) \leqq \varphi(t_2-) \leqq \leqq s_2$. Similarly, if $t_1 > t_2$ then $s_1 \geqq s_2$. In case $t_1 = t_2$ it is evident that either $s_1 \leqq s_2$ or $s_1 \geqq s_2$.

(b) To prove that Ψ_φ is compact, it is sufficient to verify that it is closed, because the boundedness is evident.

Assume that Ψ_φ is not closed. Then there is a sequence of pairs $(t_n, s_n)_{n=1}^\infty$ from Ψ_φ such that $(t_n, s_n) \to (t_0, s_0)$ and $(t_0, s_0) \notin \Psi_\varphi$. It is possible to find a subsequence $(t_{n_k})_{k=1}^\infty$ which is monotone.

If (t_{n_k}) is a nondecreasing sequence and $t_{n_k} < t_0$ for every integer k, then $\varphi(t_{n_k}) \to \to \varphi(t_0-)$ for $k \to \infty$. Since $\varphi(t_{n_k}-) \leqq s_{n_k} \leqq \varphi(t_0-)$, we get $s_{n_k} \to \varphi(t_0-)$. Taking into account that $s_n \to s_0$, we obtain the equality $s_0 = \varphi(t_0-)$ which implies that the pair $(t_0, s_0) = (t_0, \varphi(t_0-))$ belongs to Ψ_φ. We have got a contradiction with $(t_0, s_0) \notin \Psi_\varphi$. Similarly, if the subsequence (t_{n_k}) is nonincreasing and $t_{n_k} > t_0$ for every k, then $(t_0, s_0) = (t_0, \varphi(t_0+)) \in \Psi_\varphi$.

If there is k_0 such that $t_{nk_0} = t_0$, we have $t_{n_k} = t_0$ for every $k \geqq k_0$. Then $\varphi(t_0-) \leqq \leqq s_{n_k} \leqq \varphi(t_0+)$ holds for any $k \geqq k_0$; consequently $\varphi(t_0-) \leqq s_0 \leqq \varphi(t_0+)$. We conclude that $(t_0, s_0) \in \Psi_\varphi$ which is a contradiction with $(t_0, s_0) \notin \Psi_\varphi$.

(c) Assume that Ψ_φ is not connected. Then there are two open disjoint sets $A, B \subset \subset \mathbb{R}^2$ such that $\Psi_\varphi \cap A \neq \emptyset$, $\Psi_\varphi \cap B \neq \emptyset$ and $\Psi_\varphi \subset A \cup B$. Fot instance assume that $(\beta, \varphi(\beta)) \in B$. Let us denote

$$(1.3)\qquad t_A = \sup \{t \in [\alpha, \beta];\ \text{there is } s \text{ such that } (t, s) \in \Psi_\varphi \cap A\} ;$$

$$s_A = \sup (\{s \in [\gamma, \delta];\ (t_A, s) \in \Psi_\varphi \cap A\} \cup \{\varphi(t_A -)\}) .$$

If $s_A = \varphi(t_A-)$ then $(t_A, s_A) \in \Psi_\varphi$. If $s_A > \varphi(t_A-)$ then there is $s \geqq \varphi(t_A-)$ such that $(t_A, s) \in \Psi_\varphi \cap A$. For any $s \geqq \varphi(t_A-)$ such that $(t_A, s) \in \Psi_\varphi \cap A$ we have

$\varphi(t_A-) \leqq s \leqq \varphi(t_A+)$; hence also $\varphi(t_A-) \leqq s_A \leqq \varphi(t_A+)$ and we conclude that $(t_A, s_A) \in \Psi_\varphi$. Either $(t_A, s_A) \in A$, or $(t_A, s_A) \in B$. First assume that $(t_A, s_A) \in \Psi_\varphi \cap A$. Since A is open, there is $\varepsilon > 0$ such that if $(t, s) \in \mathbb{R}^2$, $|t - t_A| < \varepsilon$, $|s - s_A| < \varepsilon$, then $(t, s) \in A$.

In case that $s_A < \varphi(t_A+)$, every $s \in (s_A, \varphi(t_A+)) \cap (s_A, s_A + \varepsilon)$ satisfies $(t_A, s) \in A$. At the same time $(t_A, s) \in \Psi_\varphi$, and we get a contradiction with (1.3). The case $s_A = \varphi(t_A+)$ implies that $t_A \neq \beta$, because $(\beta, \varphi(\beta)) \in B$. There is $\delta > 0$ such that $\delta \leqq \varepsilon$ and if $t_A < t < t_A + \delta$ then $\varphi(t_A+) \leqq \varphi(t) < \varphi(t_A+) + \varepsilon$, and consequently $(t, \varphi(t)) \in A$. This is a contradiction with (1.3).

Now let us assume that $(t_A, s_A) \in \Psi_\varphi \cap B$. (t_A, s_A) is different from $(\alpha, \varphi(\alpha))$, because $\Psi_\varphi \cap A \neq \emptyset$. There is $\eta > 0$ such that if $(t, s) \in \mathbb{R}^2$, $|t - t_A| < \eta$, $|s - s_A| < \eta$, then $(t, s) \in B$. In case $s_A > \varphi(t_A-)$ we have $(t_A, s) \in \Psi_\varphi \cap B$ for any $s \in [\varphi(t_A-), s_A) \cap (s_A - \eta, s_A)$; this contradicts (1.3). In case $s_A = \varphi(t_A-)$ the point t_A is different from α, and there is $\lambda > 0$ such that $\lambda \leqq \eta$ and if $t_A - \lambda < t < t_A$ then $\varphi(t_A-) - \eta < \varphi(t-) \leqq \varphi(t+) \leqq \varphi(t_A-)$. Then $(t, s) \in \Psi_\varphi \cap B$ holds for every $(t, s) \in \Psi_\varphi$ such that $t_A - \lambda < t < t_A$. This contradicts (1.3). Since all the possibilities lead to a contradiction, we conclude that Ψ_φ is connected.

2. Let a nonempty, connected and compact set $\Psi \subset \mathbb{R}^2$ be given such that

(1.4) if $(t_1, s_1) \in \Psi$ and $(t_2, s_2) \in \Psi$, then either $t_1 \leqq t_2$ and $s_1 \leqq s_2$, or $t_1 \geqq t_2$ and $s_1 \geqq s_2$.

The following properties of Ψ are evident:

(1.5) If $(t, s_1) \in \Psi$, $(t, s_2) \in \Psi$ and $s_1 < s_2$, then the relations $(t', s) \in \Psi$ and $s_1 < s < s_2$ imply that $t' = t$.

(1.6) If $(t, s_1) \in \Psi$, $(t, s_2) \in \Psi$ and $s_1 < s_2$, then $(t, s) \in \Psi$ for every $s_1 \leqq s \leqq s_2$.

Let us denote

$$\alpha = \inf\{t \in \mathbb{R};\ \text{there is } s \in \mathbb{R} \text{ such that } (t, s) \in \Psi\}\,.$$

$$\beta = \sup\{t \in \mathbb{R};\ \text{there is } s \in \mathbb{R} \text{ such that } (t, s) \in \Psi\}\,.$$

Then $-\infty < \alpha \leqq \beta < \infty$ and

(1.7) for every $t \in [\alpha, \beta]$ the set $\{s \in \mathbb{R};\ (t, s) \in \Psi\}$ is nonempty and compact.

In the sequel assume that $\alpha < \beta$.

Let us define

(1.8) $\varphi(t) = \inf\{s \in \mathbb{R};\ (t, s) \in \Psi\}$ for $t \in [\alpha, \beta)$,

$$\varphi(t) = \sup\{s \in R;\ (t, s) \in \Psi\} \quad \text{for} \quad t = \beta .$$

We will show that the function φ is nondecreasing on $[\alpha, \beta]$ and left-continuous on (α, β).

If $\alpha \leqq t_1 < t_2 \leqq \beta$, then for every s_1, s_2 such that $(t_1, s_1) \in \Psi$, $(t_2, s_2) \in \Psi$ we have $s_1 \leqq s_2$, because Ψ satisfies (1.4). Consequently $\varphi(t_1) \leqq \varphi(t_2)$ which means that φ is nondecreasing.

Since Ψ is compact, for every $t \in [a, b]$ the pair $(t, \varphi(t))$ belongs to Ψ. The compactness yields also $(t, \varphi(t-)) \in \Psi$ and $(t, \varphi(t+)) \in \Psi$. For any $t \in (a, b)$ we have $\varphi(t-) \leqq \varphi(t)$ because φ is nondecreasing; at the same time $\varphi(t) = \inf\{s; (t, s) \in \Psi\} \leqq$ $\leqq \varphi(t-)$ because $(t, \varphi(t-)) \in \Psi$. Consequently $\varphi(t-) = \varphi(t)$ for any $t \in (\alpha, \beta)$.

Let us prove that if for the given set Ψ we define φ by (1.8) then $\Psi = \Psi_\varphi$. If $(t, s) \in \Psi_\varphi$ then $\alpha \leqq t \leqq \beta$ and $\varphi(t-) \leqq s \leqq \varphi(t+)$. Since $(t, \varphi(t-)) \in \Psi$ and $(t, \varphi(t+)) \in \Psi$, by (1.6) we have $(t, s) \in \Psi$. Hence $\Psi_\varphi \subset \Psi$.

Assume that there is $(t, s) \in \Psi \setminus \Psi_\varphi$. In case $t < \beta$ the definition (1.8) implies that $\varphi(t) \leqq s$. By the assumption $(t, s) \notin \Psi_\varphi$ we get $s > \varphi(t+)$. Then there is $t' > t$ such that $\varphi(t+) \leqq \varphi(t') < s$; we have two pairs $(t, s), (t', \varphi(t'))$ which both belong to Ψ, however $t < t'$ and $s > \varphi(t')$. This contradicts (1.4). Hence $\Psi = \Psi_\varphi$.

3. For a set $\Psi \subset R^2$ let us denote $\Psi_{-1} = \{(s, t) \in R^2; (t, s) \in \Psi\}$.

Now we can prove Proposition 1.11:

(i) Assume that a function f: $[a, b] \to [c, d]$ is given such that f is nondecreasing on $[a, b]$, and $f(a) = c$, $f(b) = d$ and $a < b$, $c < d$. Let us consider the set Ψ_f. It is evident that the set $(\Psi_f)_{-1}$ has the same properties as Ψ_f – it is connected, compact and (a) holds with $(\Psi_f)_{-1}$ instead of Ψ_f. Similarly as in (1.8) we can define such function φ that $(\Psi_f)_{-1} = \Psi_\varphi$, replacing $[\alpha, \beta]$ by $[c, d]$. The function φ is nondecreasing on $[c, d]$ and left-continuous on (c, d). Taking into account the definition of the inverse function f_{-1}, we immediately see that $f_{-1} = \varphi$.

(ii) Assuming that f is left-continuous on (a, b), from the evident equality $((\Psi_f)_{-1})_{-1} = \Psi_f$ we get $(f_{-1})_{-1} = f$.

(iii) The function f_{-1} is increasing if and only if for every $t \in [a, b]$ there is precisely one s such that $(t, s) \in \Psi_f$. The latter means that Ψ_f is the graph of a continuous function, namely f.

(iv) is evident.

1.12. Lemma. (i) *For every $n = 0, 1, 2, \ldots$ let a nondecreasing function $f_n \in$ $\in \mathscr{R}_1^-[a, b]$ be given, and assume that*

$$(1.9) \qquad f_n(t) \to f_0(t) \quad \textit{for every} \quad t \in [a, b] \quad \textit{and} \quad f_n(t+) \to f_0(t+) \quad \textit{for every} \quad t \in (a, b) .$$

Then the sequence of functions $f_n(t)$ converges to $f_0(t)$ uniformly on $[a, b]$.

(ii) *If the function f_0 is continuous, then the assumption*

(1.9)′ $\quad f_n(t) \to f_0(t)$ *for every* $t \in [a, b]$

implies (1.9).

Proof. (i) It is sufficient to prove that the functions f_n, $n \in \mathbb{N}$ are equiregulated. Then they will belong to a compact set in $\mathscr{R}_1^-[a, b]$ and consequently $f_n \rightrightarrows f_0$.

Let $t_0 \in [a, b]$ and $\varepsilon > 0$ be given. There is such $\delta > 0$ that for every $t \in [a, b]$ we have: If $t_0 - \delta \leqq t < t_0$ then $f_0(t_0) - f_0(t) < \varepsilon$; if $t_0 < t \leqq t_0 + \delta$ then $f_0(t) - f_0(t_0+) < \varepsilon$. By (1.9) there is an integer n_0 such that

$$|f_n(t_0 - \delta) - f_0(t_0 - \delta)| < \varepsilon\,, \quad |f_n(t_0) - f_0(t_0)| < \varepsilon\,,$$
$$|f_n(t_0+) - f_0(t_0+)| < \varepsilon \quad \text{and} \quad |f_n(t_0 + \delta) - f_0(t_0 + \delta)| < \varepsilon$$
$$\text{for every} \quad n \geqq n_0\,.$$

If $t \in [a, b]$ is such that $t_0 - \delta \leqq t < t_0$, then we have for every $n \geqq n_0$

$$0 \leqq f_n(t_0) - f_n(t) \leqq f_n(t_0) - f_n(t_0 - \delta) = [f_n(t_0) - f_0(t_0)] + $$
$$+ [f_0(t_0) - f_0(t_0 - \delta)] + [f_0(t_0 - \delta) - f_n(t_0 - \delta)] < 3\varepsilon\,.$$

If $t \in [a, b]$ and $t_0 < t \leqq t_0 + \delta$, then we have for every $n \geqq n_0$

$$0 \leqq f_n(t) - f_n(t_0+) \leqq f_n(t_0 + \delta) - f_n(t_0+) =$$
$$= [f_n(t_0 + \delta) - f_0(t_0 + \delta)] + [f_0(t_0 + \delta) - f_0(t_0+)] +$$
$$+ [f_0(t_0+) - f_n(t_0+)] < 3\varepsilon\,.$$

(ii) Assume that f_0 is continuous. Let $t \in [a, b)$ and $\varepsilon > 0$ be given. Let us find such $\delta > 0$ that $f_0(t + \delta) - f_0(t) < \varepsilon$. There is an integer n_0 such that

$$|f_n(t + \delta) - f_0(t + \delta)| < \varepsilon \quad \text{and} \quad |f_n(t) - f_0(t)| < \varepsilon$$
$$\text{for every} \quad n \geqq n_0\,.$$

For $n \geqq n_0$ we have

$$f_n(t+) - f_0(t+) \leqq f_n(t + \delta) - f_0(t) =$$
$$= [f_n(t + \delta) - f_0(t + \delta)] + [f_0(t + \delta) - f_0(t)] < 2\varepsilon\,;$$
$$f_n(t+) - f_0(t+) \geqq f_n(t) - f_0(t) > -\varepsilon\,.$$

Consequently $f_n(t+) \to f_0(t+) = f_0(t)$.

1.13. Proposition. *Assume that for every* $n = 0, 1, 2, \ldots$ *a nondecreasing function* $f_n\colon [a, b] \to [c, d]$ *is given,* $f_n(a) = c$, $f_n(b) = d$, f_n *is left-continuous on* (a, b).

(i) *If* $f_n(t) \to f_0(t)$ *for every* $t \in [a, b]$ *at which* f_0 *is continuous, then* $(f_n)_{-1}(s) \to (f_0)_{-1}(s)$ *for every* $s \in [c, d]$ *at which* $(f_0)_{-1}$ *is continuous, and vice versa.*

(ii) *If, moreover,* f_0 *is increasing on* $[a, b]$ *then* $(f_n)_{-1} \rightrightarrows (f_0)_{-1}$.

Proof. (i) We will prove that the condition

(1.10) $\quad f_n(t) \to f_0(t)$ for every $t \in [a, b]$ such that f_0 is continuous at t

is satisfied if and only if

(1.11) $$\operatorname{dist}(\Psi_{f_0}, \Psi_{f_n}) = \sup_{(t,s)\in\Psi_{f_0}} \inf_{(\tau,\sigma)\in\Psi_{f_n}} \{|t - \tau| + |s - \sigma|\} \to 0 \quad \text{with} \quad n \to \infty .$$

Assume that (1.10) holds. Let $\varepsilon > 0$ be given. By Proposition 1.9 there is a finite sequence $a = t_0 < t_1 < \ldots < t_k = b$ such that (1.1) holds for $i = 1, 2, \ldots, k$. Assume that $t_i - t_{i-1} < \varepsilon/2$ for $i = 1, 2, \ldots, k$. For every $i = 1, 2, \ldots, k$ let us find $\tau_i \in (t_{i-1}, t_i)$ such that f_0 is continuous at τ_i. Denote $\tau_0 = a$, $\tau_{k+1} = b$. Then $\tau_i - \tau_{i-1} < \varepsilon$ for $i = 1, 2, \ldots, k + 1$.

Since $f_n(\tau_i) \to f_0(\tau_i)$ with $n \to \infty$ for every $i = 0, 1, \ldots, k + 1$, there is an an integer n_0 such that

(1.12) $$|f_n(\tau_i) - f_0(\tau_i)| < \varepsilon \quad \text{for every} \quad i = 0, 1, \ldots, k + 1 , \quad n \geqq n_0 .$$

Let a pair $(\tilde{t}, \tilde{s}) \in \Psi_{f_0}$ be given. We want to show that

(1.13) $$\inf\{|\tilde{t} - t| + |\tilde{s} - s|;\ (t, s) \in \Psi_{f_n}\} < 2\varepsilon \quad \text{for every} \quad n \geqq n_0 .$$

There is $i \in \{1, 2, \ldots, k + 1\}$ such that $\tau_i \leqq \tilde{t} \leqq \tau_{i+1}$.

Let $n \geqq n_0$ be fixed. In case that $f_n(\tau_i) \leqq \tilde{s} \leqq f_n(\tau_{i+1})$ let us denote $s = \tilde{s}$. In case $\tilde{s} < f_n(\tau_i)$ denote $s = f_n(\tau_i)$; if $\tilde{s} > f_n(\tau_{i+1})$, let us denote $s = f_n(\tau_{i+1})$.

In the case $\tilde{s} < f_n(\tau_i)$ we have the inequalities

$$0 < s - \tilde{s} = f_n(\tau_i) - \tilde{s} \leqq f_n(\tau_i) - f_0(\tau_i) < \varepsilon \quad \text{(we have used (1.12))}$$

and

$$f_0(\tau_i) \leqq f_0(\tilde{t}) \leqq \tilde{s} \leqq f_0(\tilde{t}+)) .$$

Similarly in the case $\tilde{s} > f_n(\tau_{i+1})$ we have

$$0 < \tilde{s} - s = \tilde{s} - f_n(\tau_{i+1}) \leqq f_0(\tau_{i+1}) - f_n(\tau_{i+1}) < \varepsilon .$$

Consequently in each of the three cases mentioned we have

(1.14) $$|\tilde{s} - s| < \varepsilon .$$

Let us denote $t = (f_n)_{-1}(s)$. The inequality $f_n(\tau_i) \leqq s \leqq f_n(\tau_{i+1})$ implies that $\tau_i \leqq t \leqq \tau_{i+1}$. By virtue of the inequalities $\tau_i \leqq \tilde{t} \leqq \tau_{i+1}$ and $\tau_{i+1} - \tau_i < \varepsilon$ we get $|\tilde{t} - t| < \varepsilon$, which together with (1.14) yields (1.13). Then (1.11) holds.

Now let us assume that (1.11) holds. Let $t_0 \in (a, b)$ be given such that f_0 is continuous at t (we are not concerned with $t = a$, $t = b$ since the values $f_n(a), f_n(b)$ are fixed).

For a given $\varepsilon > 0$ let us find $\delta > 0$ such that

(1.15) $$\text{if} \quad |t - t_0| \leqq \delta \quad \text{then} \quad |f_0(t) - f_0(t_0)| < \varepsilon .$$

Denote $t' = t_0 - \delta$, $t'' = t_0 + \delta$. By (1.11) there is such an integer n_0 that

$\inf\{|\tau - t| + |\sigma - s|;\ (\tau, \sigma) \in \Psi_{f_n}\} < \delta$ for every $n \geqq n_0$, $(t, s) \in \Psi_{f_0}$. Let $n \geqq n_0$ be fixed. Then there are $(\tau', \sigma'), (\tau'', \sigma'') \in \Psi_{f_n}$ such that

$$(1.16) \qquad |\tau' - t'| + |\sigma' - f_0(t')| < \delta\,, \quad |\tau'' - t''| + |\sigma'' - f_0(t'')| < \delta\,.$$

We have $\tau' < t' + \delta = t_0$, $\tau'' > t'' - \delta = t_0$; hence $\tau' < t_0 < \tau''$. Using (1.16), we get

$$f_0(t') - \delta < \sigma' \leqq f_n(\tau'+) \leqq f_n(t_0) \leqq f_n(\tau'') \leqq \sigma'' < f_0(t'') + \delta\,.$$

By (1.15) we have

$$f_0(t_0) - 2\varepsilon < f_0(t') - \varepsilon \leqq f_0(t') - \delta \leqq$$
$$\leqq f_n(t_0) < f_0(t'') + \delta < f_0(t_0) + 2\varepsilon\,.$$

Consequently $|f_0(t_0) - f_n(t_0)| < 2\varepsilon$ for every $n \geqq n_0$.

Since evidently $\operatorname{dist}(\Psi_{f_0}, \Psi_{f_n}) = \operatorname{dist}((\Psi_{f_n})_{-1}, (\Psi_{f_n})_{-1})$, the equivalence of (1.10), (1.11) immediately yields part (i) of Proposition 1.11.

(ii) If f_0 is increasing, then $(f_0)_{-1}$ is continuous by Proposition 1.11 (iii). Lemma 1.12 implies that $(f_n)_{-1} \rightrightarrows (f_0)_{-1}$.

1.14. Let us denote by Λ the set of all continuous increasing functions $\lambda\colon [0, 1] \to \to [0, 1]$ such that $\lambda(0) = 0$, $\lambda(1) = 1$. In [2], Chap. 6, § 5 we can find a metric space

$$\mathscr{D} = \{x \in \mathscr{R}_N;\ x(t) = x(t+) \ \text{ for every } \ t \in [0, 1),\ x(1-) = x(1)\}$$

with the metric

$$\varrho(x, y) = \inf\{\|x - y \circ \lambda\| + \|\mathrm{id} - \lambda\|;\ \lambda \in \Lambda\}\,,$$

where $\mathrm{id}(t) = t$. The same metric can be introduced also in $\mathscr{R}_N^-$, only replacing the right-continuity in $\mathscr{D}$ by the left-continuity in $\mathscr{R}_N^-$.

It is evident that *a sequence $(f_n)_{n=1}^{\infty} \subset \mathscr{R}_N^-$ converges to $f_0 \in \mathscr{R}_N^-$ in the metric ϱ, if and only if there is a sequence*

$$(\lambda_n)_{n=1}^{\infty} \subset \Lambda \quad \textit{such that} \quad \lambda_n \rightrightarrows \mathrm{id} \quad \textit{and} \quad f_n \circ \lambda_n \rightrightarrows f_0\,.$$

1.15. Lemma. *Let sequences $(x_n)_{n=0}^{\infty} \subset \mathscr{R}_N^-$ and $(\lambda_n)_{n=1}^{\infty} \subset \Lambda$ be given such that $\lambda_n(t) \to t$ for every $t \in [0, 1]$. If $x_n \circ \lambda_n \rightrightarrows x_0$ on $[0, 1]$, then $x_n(t) \to x_0(t)$ holds for every $t \in (0, 1)$ at which the function x_0 is continuous.*

Proof. Assume that x_0 is continuous at $t \in (0, 1)$. For a given $\varepsilon > 0$ there is $\delta > 0$ such that $|x_0(\tau) - x_0(t)| < \varepsilon$ for every $\tau \in (t - \delta, t + \delta)$.

By Proposition 1.13 (ii) the pointwise convergence $\lambda_n(t) \to t$ yields $\lambda_n^{-1} \rightrightarrows \mathrm{id}$. There is $n_0 \in \mathbb{N}$ such that

$$\|\lambda_n^{-1} - \mathrm{id}\| < \delta \quad \text{and} \quad \|x_n \circ \lambda_n - x_0\| < \varepsilon \quad \text{for every} \quad n \geqq n_0\,.$$

For any $n \geqq n_0$ we have the estimate

$$|x_n(t) - x_0(t)| = |(x_n \circ \lambda_n)(\lambda_n^{-1}(t)) - x_0(t)| \leqq$$
$$\leqq |(x_n \circ \lambda_n)(\lambda_n^{-1}(t)) - x_0(\lambda_n^{-1}(t))| + |x_0(\lambda_n^{-1}(t)) - x_0(t)| \leqq$$
$$\leqq \|x_n \circ \lambda_n - x_0\| + |x_0(\lambda_n^{-1}(t)) - x_0(t)| < 2\varepsilon .$$

1.16. Let us denote by Q the set of all functions q: $[0, 1] \to [0, 1]$ satisfying the following conditions:

(1.17) *q is nondecreasing on $[0, 1]$ and left-continuous on $(0, 1]$;*

(1.18) $0 \leqq q(t) \leqq t$ *for every* $t \in [0, 1]$; $q(1) = 1$;

(1.19) *if* $t \in (0, 1)$ *is such that* $q(t+) < t$ *then q is linear on some neighborhood of t.*

1.17. Lemma. *Let $q \in Q$ be given. If $t \in (0, 1)$ is a point such that $q(t) < t$, then there are $\alpha, \beta \in [0, 1]$ such that $\alpha < t \leqq \beta$ and*
(i) *q is linear on $(\alpha, \beta]$ with slope less than 1;*
(ii) $q(\alpha+) = \alpha \leqq q(t)$;
(iii) $q(\beta) < \beta$; *if* $\beta < 1$ *then* $q(\beta+) = \beta$.

Proof. Let us fix τ such that $q(t) < \tau < t$. We have $q(\tau+) \leqq q(t) < \tau$; by (1.19) the function q has the form $q(s) = q(\tau) + c(s - \tau)$ for s belonging to a neighborhood of τ. Denote

(1.20) $$\alpha = \inf \{\sigma \in [0, \tau] ;\ q(s) = q(\tau) + c(s - \tau) \text{ for every } s \in [\sigma, \tau]\} ;$$
$$\beta = \sup \{\sigma \in [\tau, 1] ;\ q(s) = q(\tau) + c(s - \tau) \text{ for every } s \in [\tau, \sigma]\} .$$

We have $\alpha < \tau < \beta$.

If $q(\alpha+) < \alpha$ then the function q should be linear on a neighbourhood of α, it will have the same form to the left as to the right. This contradicts (1.20), hence $q(\alpha+) = \alpha$. The same argument yields $q(\beta+) = \beta$ in case that $\beta < 1$.

Let us verify that $\alpha < t \leqq \beta$. The first inequality follows from $\alpha < \tau < t$. If $t > \beta$ then $q(t) \geqq q(\beta+)$; consequently $\beta = q(\beta+) \leqq q(t) < \tau$ which contradicts $\tau < \beta$.

From $\alpha < \tau < t$ we get $\alpha = q(\alpha+) \leqq q(t)$. Then (ii) holds.

Let us prove that $q(\beta) < \beta$. The function q has on $(\alpha, \beta]$ the form

$$q(s) = q(t) + \frac{q(t) - \alpha}{t - \alpha} (s - t) \quad \text{for} \quad s \in (\alpha, \beta] , \quad \text{where} \quad \frac{q(t) - \alpha}{t - \alpha} < 1 \text{ is the}$$

slope of the linear function.

Then

$$q(\beta) = q(t) + \frac{q(t) - \alpha}{t - \alpha}(\beta - t) < q(t) + 1 \cdot (\beta - t) < \beta .$$

1.18. Lemma. *Let a sequence $(q_n)_{n=1}^{\infty} \subset Q$ be given. Assume that there is a function $q_0 \in \mathscr{R}_1^-$ such that $q_0(1) = 1$ and $q_n(t) \to q_0(t)$ for every $t \in (0, 1)$ at which q_0 is continuous. Then $q_0 \in Q$.*

Proof. The function q_0 is evidently nondecreasing and satisfies $0 \leqq q_0(t) \leqq t$ for every $t \in [0, 1]$.

Let $t \in (0, 1)$ be given such that $q_0(t+) < t$. Let us fix σ such that $q_0(t+) < \sigma < t$. There are $\tau', \tau'' \in [0, 1]$ such that $\sigma < \tau' < t < \tau''$, q_0 is continuous at τ', τ'' and $q_0(s) < \sigma$ for every $s \in [\tau', \tau'']$. Since $q_n(\tau') \to q_0(\tau)$, $q_n(\tau'') \to q_0(\tau'')$, there is an integer n_0 such that

$$q_n(\tau') < \sigma \quad \text{and} \quad q_n(\tau'') < \sigma \quad \text{for every} \quad n \geqq n_0 .$$

For every $s \in [\tau', \tau'']$ and $n \geqq n_0$ we have $q_n(s) \leqq q_n(\tau'') < \sigma < s$. According to Lemma 1.17 the function q_n is linear on $[\tau', \tau'']$ for $n \geqq n_0$. Consequently also q_0 is linear on $[\tau', \tau'']$.

1.19. Lemma. *Assume that a sequence $(q_n)_{n=0}^{\infty} \subset Q$ is given such that $q_n(t) \to q_0(t)$ for every $t \in [0, 1]$ at which q_0 is continuous. Then there is a sequence of continuous increasing functions $(\lambda_n)_{n=1}^{\infty} \subset \Lambda$ such that $\lambda_n \rightrightarrows \mathrm{id}$ and $q_n \circ \lambda_n \rightrightarrows q_0$.*

Proof. For every $k \in \mathbb{N}$ there are finitely many points $t \in (0, 1)$ such that

$$q_0(t+) - q_0(t) \geqq 1/k .$$

Let us denote all these points by $\beta_1^k, \beta_2^k, \ldots, \beta_{m_k}^k$; further let us denote $\beta_0^k = 0$, $\beta_{m_k+1}^k = 1$, and assume that

$$0 = \beta_0^k < \beta_1^k < \ldots < \beta_{m_k+1}^k = 1 .$$

By Lemma 1.17 for every $i = 1, 2, \ldots, m_k$ there is α_i^k such that $\beta_{i-1}^k \leqq \alpha_i^k < \beta_i^k$ and q_0 is linear on $(\alpha_i^k, \beta_i^k]$, $q_0(\alpha_i^k+) = \alpha_i^k$. We have

$$\beta_i^k - \alpha_i^k = q_0(\beta_i^k+) - q_0(\alpha_i^k+) \geqq q_0(\beta_i^k+) - q_0(\beta_i^k) \geqq 1/k \tag{1.21}$$

for $i = 1, 2, \ldots, m_k$. Denote $\alpha_{m_k+1}^k = 1$.

Let us prove that

$$\text{if} \quad t \in (\beta_{i-1}^k, \alpha_i^k) \quad \text{for} \quad i = 1, 2, \ldots, m_k + 1 \quad \text{then} \quad t - q_0(t) < 1/k . \tag{1.22}$$

Assume that $t - q_0(t) \geqq 1/k$ for some $t \in (\beta_{i-1}^k, \alpha_i^k]$; then by Lemma 1.17 there is $t' \geqq t$ such that q_0 is linear on (t, t') and $q_0(t') < t' = q_0(t'+)$. By the definition of α_i^k we have $t' \leqq \alpha_i^k$. Then

$$q_0(t'+) - q_0(t') = t' - q_0(t') \geqq t - q_0(t) \geqq 1/k$$

and the point t' should belong to the set $\{\beta_1^k, \beta_2^k, \ldots, \beta_{m_k}^k\}$ which is not true.

For every $i = 1, 2, \ldots, m_k$ denote $t_i^k = \beta_i^k - 1/4k$, $\vartheta_i^k = \beta_i^k - 1/2k$, $s_i^k = \alpha_i^k + 1/4k$; by (1.21) we have $\vartheta_i^k > s_i^k$.

Let $i = 0, 1, \ldots, m_k$. In case $\beta_i^k = \alpha_{i+1}^k$ define $\tau_i^k = \beta_i^k + 1/4k = s_{i+1}^k$.

In case $\beta_i^k < \alpha_{i+1}^k$ let us find τ_i^k such that

$$\beta_i^k < \tau_i^k < \alpha_{i+1}^k, \quad \tau_i^k \leqq \beta_i^k + 1/4k,$$

and q_0 is continuous at τ_i^k.

There is an integer n_k^1 such that for every $n \geqq n_k^1$ the function q_n is linear on each of the intervals $[s_i^k, t_i^k]$, $i = 1, 2, \ldots, m_k$ and

$$(1.23) \qquad |q_n(t) - q_0(t)| < 1/4k \quad \text{for every} \quad t \in [s_i^k, t_i^k], \quad i = 1, 2, \ldots, m_k.$$

Denote $\tau_0^k = 0$, $s_{m_k+1}^k = 1$.

Let $i = 1, 2, \ldots, m_k + 1$. In case that $\beta_{i-1}^k < \alpha_i^k$, let us find a division

$$\tau_{i-1}^k = \sigma_{i0}^k < \sigma_{i1}^k < \ldots < \sigma_{ir_i^k}^k = s_i^k$$

such that $\sigma_{ij}^k - \sigma_{i,j-1}^k < 1/4k$ for $i = 1, 2, \ldots, r_i^k$ and q_0 is continuous at σ_{ij}^k, $j = 0, 1, \ldots, r_i^k$. In case $\beta_{i-1}^k = \alpha_i^k$ denote $r_i^k = 0$.

There is an integer n_k^2 such that $|q_n(\sigma_{ij}) - q_0(\sigma_{ij})| < 1/4k$ for every $n \geqq n_k^2$, $i = 1, 2, \ldots, m_k + 1$, $j = 0, 1, \ldots, r_i^k - 1$.

Let us denote $n_k = \max\{n_k^1, n_k^2\}$. For $n = n_k, n_k + 1, \ldots, n_{k+1} - 1$ let us define a function $\lambda_n \in \Lambda$ in the following way:

For every $i = 1, 2, \ldots, m_k$ we have

$$\begin{aligned} t_i^k - q_n(t_i^k) &= [t_i^k - q_0(t_i^k)] + [q_0(t_i^k) - q_n(t_i^k)] > [t_i^k - q_0(\beta)] - 1/4k = \\ &= [\beta_i^k - q_0(\beta_i^k)] + [t_i^k - \beta_i^k] - 1/4k \geqq 1/2k; \\ \tau_i^k - q_n(\tau_i^k) &= [\tau_i^k - q_0(\tau_i^k)] + [q_0(\tau_i^k) - q_n(\tau_i^k)] < [\tau_i^k - q_0(\tau_i^k)] + \\ &+ 1/4k \leqq [\tau_i^k - q_0(\beta_i^k+)] + 1/4k = [\tau_i^k - \beta_i^k] + 1/4k \leqq 1/2k. \end{aligned}$$

These inequalities yield

$$(1.24) \qquad t_i^k - q_n(t_i^k) > 1/2k > \tau_i^k - q_n(\tau_i^k).$$

Using Lemma 1.17, we can find $\gamma_{i,n} \geqq t_i^k$ such that q_n is linear on $[t_i^k, \gamma_{i,n}]$ and $q_n(\gamma_{i,n}+) > q_n(\gamma_{i,n})$. According to (1.24) it is impossible that q_n are linear on $[t_i^k, \tau_i^k]$. Hence

$$t_i^k \leqq \gamma_{i,n} < \tau_i^k.$$

Let us define $\lambda_n(\beta_i^k) = \gamma_{i,n}$, $\lambda_n(\vartheta_i^k) = \vartheta_i^k$, $\lambda_n(\tau_i^k) = \tau_i^k$, λ_n being linear on the intervals $[\vartheta_i^k, \beta_i^k]$, $[\beta_i^k, \tau_i^k]$ for $i = 1, 2, \ldots, m_k$; $\lambda_n(t) = t$ for $t \in [0, \vartheta_1^k) \cup \bigcap_{i=2}^{m_k} (\tau_{i=1}^k, \vartheta_i^k) \cup (\tau_{m_k}^k, 1]$. The function λ_n is increasing and continuous, $\lambda_n(0) = 0$, $\lambda_n(1) = 1$. Since $t_i^k \leqq \gamma_{i,n} < \tau_i^k$ and $\tau_i^k - t_i^k < 1/2k$, we have $|\lambda_n(\beta_i^k) - \beta_i^k| < 1/2k$. Consequently

$\|\lambda_n - \mathrm{id}\| < 1/2k$ for $n = n_k, n_k + 1, \ldots, n_{k+1} - 1$.

Now we aim at proving that $q_n \circ \lambda_n \rightrightarrows q_0$. Assume that $n_k \leqq n < n_{k+1}$. Let $t \in [\vartheta_i^k, \beta_i^k]$, $i = 1, 2, \ldots, m_k$. Then $\lambda_n(t) \in [\vartheta_i^k, \gamma_{i,n}]$; let us notice that q_0 and q_n are lipschitzian with the constant 1 on $[\vartheta_i^k, \beta_i^k]$ and $[\vartheta_i^k, \gamma_{i,n}]$, respectively. We have

$$\begin{aligned}
&|q_n(\lambda_n(t)) - q_0(t)| \leqq [q_n(\lambda_n(t)) - q_n(\vartheta_i^k)] + \\
&+ |q_n(\vartheta_i^k) - q_0(\vartheta_i^k)| + |q_0(\vartheta_i^k) - q_0(t)| < \\
&< [\lambda_n(t - \vartheta_i^k] + 1/4k + \vartheta_i^k - t \leqq \\
&\leqq 2 \,.\, [\tau_i^k - \vartheta_i^k] + 1/4k \leqq 2 \,.\, 3/4k + 1/4k = 7/4k \,.
\end{aligned}$$

Assume that $t \in (\beta_i^k, \tau_i^k]$ $i = 1, 2, \ldots, m_k$; then $\lambda_n(t) \in (\gamma_{i,n}, \tau_i^k]$. We have

$$\begin{aligned}
&q_n(\lambda_n(t)) - q_0(t) \leqq q_n(\tau_i^k)) - q_0(t) = [q_n(\tau_i^k) - q_0(\tau_i^k)] + \\
&+ [q_0(\tau_i^k) - q_0(t)] < 1/4k + [q_0(\tau_i^k) - q_0(\beta_i^k+)] = \\
&= 1/4k + [q_0(\tau_i^k) - \beta_i^k] \leqq 1/4k + [\tau_i^k - \beta_i^k] \leqq 1/4k + 1/4k = 1/2k \,; \\
&q_n(\lambda_n(t)) - q_0(t) \geqq q_n(\gamma_{i,n}+) - q_0(t) = \gamma_{i,n} - q_0(t) \geqq \gamma_{i,n} - t \geqq \\
&\geqq \gamma_{i,n} - \tau_i^k \geqq -1/2k \,.
\end{aligned}$$

Let $t \in (\sigma_{ij-1}^k, \sigma_{ij}^k]$, $i = 1, 2, \ldots, m_k + 1$, $j = 0, 1, \ldots, r_i^k - 1$. Then

$$\begin{aligned}
&q_n(\lambda_n(t)) - q_0(t) = q_n(t) - q_n(t) \leqq q_n(\sigma_{ij}^k) - q_0(\sigma_{ij-1}^k) = \\
&= [q_n(\sigma_{ij}^k) - q_0(\sigma_{ij}^k)] + [\sigma_{ij}^k - \sigma_{ij-1}^k] + [\sigma_{ij-1}^k - q_0(\sigma_{ij-1}^k)] < \\
&< 1/4k + 1/4k + 1/k = 3/2k \,; \\
&q_n(\lambda_n(t)) - q_0(t) = q_n(t) - q_0(t) \geqq q_n(\sigma_{ij-1}^k) - q_0(\sigma_{ij}^k) = \\
&= [q_n(\sigma_{ij-1}^k) - q_0(\sigma_{ij-1}^k)] + [q_0(\sigma_{ij-1}^k) - q_0(\sigma_{ij}^k)] > -1/4k + \\
&+ [q_0(\sigma_{ij-1}^k) - \sigma_{ij-1}^k] + [\sigma_{ij-1}^k - \sigma_{ij}^k] > -1/4k - 1/k - 1/4k = \\
&= -3/2k \,.
\end{aligned}$$

Taking into account (1.23), we can conclude that

$$|q_n(\lambda_n(t)) - q_0(t)| < 2/k \quad \text{for every} \quad t \in [0, 1] \quad \text{and} \quad n \geqq n_k \,.$$

1.20. Theorem. *Assume that a sequence of increasing functions $(f_n)_{n=1}^{\infty} \subset \mathscr{R}_1^-$ is given such that*

(1.25) $f_n(0) = 0$, $f_n(1) = 1$, *the continuous part of f_n is increasing for every* $n \in \mathbb{N}$;

(1.26) *for every* $\varepsilon > 0$ *there is* $\delta_\varepsilon \in (0, \varepsilon]$ *such that the following holds: If* $t \in [0, 1)$ *and* $f_n(1) - f_n(t+) < \delta_\varepsilon$, *then* $f_n(t+) - f_n(t) < \varepsilon$.

Then there is a sequence of increasing continuous functions $(\varphi_n)_{n=1}^{\infty} \subset \Lambda$ *such that*

$$\|(f_n)_{-1} - \varphi_n^{-1}\| \to 0 \quad \textit{with} \quad n \to \infty ,$$

and the set $\{f_n \circ \varphi_n^{-1};\ n = 1, 2, \ldots\}$ *is relatively compact in the metric space* $(\mathscr{R}_1^-; \varrho)$.

Proof. For a fixed integer n let us find an increasing function $g_n \in \mathscr{R}_1^-$ such that $g_n(0) = 0$, $g_n(1) = 1$, g_n has finitely many points of discontinuity and

$$(1.27) \qquad \|f_n - g_n\| < 1/n \quad \text{and} \quad \|(f_n)_{-1} - (g_n)_{-1}\| < 1/n .$$

The function g_n will be constructed in the following way:

There is a division $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$ such that $t_i - t_{i-1} < 1/n$ for $i = 1, 2, \ldots, k + 1$ and

$$[f^J(1) - f^J(0)] - \sum [f(t_i+) - f(t_i)] = \sum [f^J(t_i) - f^J(t_{i-1}+)] < 1/n$$

where we denote by f^J, f^C the jump part and the continuous part of f.

Let us define

$$g_n(0) = 0 , \quad g_n(t) = f_n(t_i) + \frac{f_n(t_i) - f_n(t_{i-1}+)}{f_n^C(t_i) - f_n^C(t_{i-1})} \cdot$$

$$\cdot\, [f_n^C(t) - f_n^C(t_i)] \quad \text{for} \quad t \in (t_{i-1}, t_i] , \quad i = 1, 2, \ldots, k + 1 .$$

We have

$$(1.28) \qquad g_n(t_i) = f_n(t_i) \quad \text{and} \quad g_n(t_i+) = f_n(t_i+) \quad \text{for} \quad i = 1, 2, \ldots, k .$$

For $t \in (t_{i-1}, t_i]$ we have

$$|g_n(t) - f_n(t)| =$$

$$= |[f_n^J(t_i) - f_n^J(t_{i-1}+)]\,[f_n^C(t) - f_n^C(t_{i-1}+)]| < 1/n .$$

Hence $\|g_n - f_n\| < 1/n$.

By virtue of (1.28), for every $s \in (f_n(t_{i-1}+), f_n(t_i)] = (g_n(t_{i-1}+), g_n(t_i)]$ both $(f_n)_{-1}(s)$ and $(g_n)_{-1}(s)$ belong to $(t_{i-1}, t_i]$. From the assumption $t_i - t_{i-1} < 1/n$ we conclude that

$$|(f_n)_{-1}(s) - (g_n)_{-1}(s)| < 1/n .$$

If $s \in (f_n(t_i), f_n(t_i+)] = (g_n(t_i), g_n(t_i+)]$, then $(f_n)_{-1}(s) = (g_n)_{-1}(s) = t_i$. We have found that $|(f_n)_{-1}(s) - (g_n)_{-1}(s)| < 1/n$ for every $s \in [0, 1]$.

For every $i = 1, 2, \ldots, k$ let us find a point s_i satisfying

$$(1.29) \qquad t_{i-1} < s_i < t_i , \quad t_i - 1/n < s_i \quad \text{and} \quad g_n(t_i) - g_n(s_i) < 1/n .$$

Denote $g_n(s_i) = \sigma_i$, $g_n(t_i) = \tau_i$, $g_n(t_i+) = \vartheta_i$.

Let us define a function $\varphi_n \in \Lambda$ as follows: For $t \in [0, s_1] \cup \bigcup_{i=2}^{k} (t_{i-1}, s_i] \cup (t_k, 1]$ we define $\varphi_n(t) = g_n(t)$. For $t \in (s_i, t_i]$, $i = 1, 2, \ldots, k$ let us define

$$\varphi_n(t) = g_n(s_i) + \frac{g_n(t_i+) - g_n(s_i)}{g_n(t_i) - g_n(s_i)} \cdot [g_n(t) - g_n(s_i)] = \sigma_i + \frac{\vartheta_i - \sigma_i}{\tau_i - \sigma_i} \cdot [g_n(t) - \sigma_i] .$$

Further let us define $q_n = g_n \circ \varphi_n^{-1}$. The function q_n has the form

$$q_n(s) = \sigma_i + \frac{\vartheta_i - \sigma_i}{\tau_i - \sigma_i} \cdot [s - \sigma_i] \quad \text{for} \quad s \in (\sigma_i, \vartheta_i] ,$$

$$i = 1, 2, \ldots, k ,$$

$$q_n(s) = s \quad \text{for} \quad s \in [0, \sigma_1] \cup \bigcup_{i=2}^{k} (\vartheta_{i-1}, \sigma_1] \cup (\vartheta_k, 1] .$$

It is evident that $q_n \in Q$.

Let $i = 1, 2, \ldots, k$. Since $g_n(s_i) = \sigma_i = \varphi_n(s_i)$, $g_n(\tau_i+) = \vartheta_i = \varphi_n(\tau_i)$ and the functions g_n, φ_n are increasing, we have $(g_n)_{-1}(\sigma_i) = s_i = \varphi_n^{-1}(\sigma_i)$, $(g_n)_{-1}(\vartheta_i) = t_i = \varphi_n^{-1}(\vartheta_i)$. Hence $s_i \leqq (g_n)_{-1}(s) \leqq t_i$ and $s_i \leqq \varphi_n^{-1}(s) \leqq t_i$ for every $s \in (\sigma_i, \vartheta_i]$. Since $t_i - s_i < 1/n$ by (1.29), we obtain the estimate

$$|(g_n)_{-1}(s) - \varphi_n^{-1}(s)| < 1/n \quad \text{for every} \quad s \in (\sigma_i, \vartheta_i] .$$

If $s \in [0, \sigma_1] \cup \bigcup_{i=2}^{k} (\vartheta_{i-1}, \sigma_i] \cup (\vartheta_k, 1]$ then $(g_n)_{-1}(s) = \varphi_n^{-1}(s)$. Consequently $\|(g_n)_{-1} - \varphi_u^{-1}\| < 1/n$. By (1.27) we have

(1.30) $$\|(f_n)_{-1} - \varphi_n^{-1}\| < 2/n .$$

Let us prove

(1.31) If for $\varepsilon > 0$ the value of δ_ε is taken from (1.26), then $q_n(s) \in (1 - 2\varepsilon, 1)$ for every $s \in (1 - \delta_\varepsilon, 1)$.

Let $s \in (1 - \delta_\varepsilon, 1)$. Either $s \in [0, \sigma_1] \cup \bigcup_{i=2}^{k} (\vartheta_{i-1}, \sigma_i] \cup (\vartheta_k, 1]$, then $q_n(s) = s$, and $q_n(s) \in (1 - \delta_\varepsilon, 1)$. Or $s \in (\sigma_i, \vartheta_i]$ for some $i \in \{1, 2, \ldots, k\}$; then

$$1 - q_n(s) = [1 - s] + [s - \sigma_i] \cdot \frac{\vartheta_i - \tau_i}{\vartheta_i - \sigma_i} \leqq [1 - s] + [\vartheta_i - \tau_i] =$$

$$= [1 - s] + [f_n(t_i+) - f_n(t_i)] < \delta_\varepsilon + \varepsilon \leqq 2\varepsilon .$$

If we define for every integer n functions g_n, φ_n, q_n in this way, by (1.30) it is clear that $\|(f_n)_{-1} - \varphi_n^{-1}\| \to 0$. Let us prove that the set $\{f_n \circ \varphi_n^{-1};\ n \in \mathbb{N}\}$ is relatively compact in the metric space $(\mathscr{R}_1^-; \varrho)$. Let $(f_{n_i} \circ \varphi_{n_i}^{-1})_{i=1}^{\infty}$ be an arbitrary subsequence. By Helly's Choice Theorem the sequence $(q_{n_i})_{i=1}^{\infty}$ contains a pointwise convergent subsequence $q_{n_{i_l}}(s) \to q(s)$ for every $s \in [0, 1]$. Let us define $q_0(0) = 0$, $q_0(s) = q(s-)$ for $s \in (0, 1]$. Let us prove that $q_0(1) = 1$.

For a given $\varepsilon > 0$ let us find δ_ε by (1.26). Let $s \in (1 - \delta_\varepsilon, 1)$. Since $q_{n_{i_l}}(s) \to q(s)$, there is $l_0 \in \mathbb{N}$ such that $|q_{n_{i_l}}(s) - q(s)| < \varepsilon$ for every $l \geqq l_0$. Let us fix $l \geqq l_0$ and denote $n = n_{i_l}$. Then $0 \leqq 1 - q(s) = [1 - q_n(s)] + [q_n(s) - q(s)] < [1 - q_n(s)] + \varepsilon < 3\varepsilon$ according to (1.31). Consequently $q_0(1) = q(1-) = 1$.

By Lemma 1.18 the function q_0 belongs to Q and by Lemma 1.19 there is a sequence $(\lambda_l)_{l=1}^{\infty}$ such that $q_{n_{i_l}} \circ \lambda_l \rightrightarrows q_0$. Then $\|(f_{n_{i_l}} \circ \varphi_{n_{i_l}}^{-1}) \circ \lambda_l - q_0\| \leqq \|f_{n_{i_l}} - g_{n_{i_l}}\| + \|(g_{n_{i_l}} \circ \varphi_{n_{i_l}}^{-1}) \circ \lambda_l - q_0\| < 1/n_{i_l} + \|q_{n_{i_l}} \circ \lambda_l - q_0\| \to 0$ with $l \to \infty$. Hence the sequence $(f_{n_{i_l}} \circ \varphi_{n_{i_l}}^{-1})_{l=1}^{\infty}$ is convergent in the metric space $(\mathscr{R}_1^-; \varrho)$.

1.21. Theorem. *Let a sequence of nondecreasing functions $(h_n)_{n=1}^{\infty} \subset \mathscr{R}_1^-$ be given such that $h_n(0) = 0$. Assume that there is a nondecreasing continuous function $\eta: [0, \infty) \to [0, \infty)$, $\eta(0) = 0$ such that*

$$\limsup_{n\to\infty} [h_n(t'') - h_n(t')] \leqq \eta(h_0(t'') - h_0(t')) \tag{1.32}$$

provided h_0 is continuous at t', t''; $0 \leqq t' < t'' \leqq 1$.

Then there is a subsequence $(h_{n_k})_{k=1}^{\infty}$, a sequence of increasing continuous functions $(v_k)_{k=1}^{\infty} \subset \Lambda$ and a function $v \in \mathscr{R}_-^1$ so that

(1.33) *the functions $h_{n_k} \circ v_{-1}^k$ are uniformly convergent;*

(1.34) *$v_k(t) \to v(t)$ for every $t \in [0, 1]$ at which v is continuous;*

(1.35) *$v(t'') - v(t') \leqq t'' - t' + \eta(h_0(t'') - h_0(t'))$ for every $t' < t''$.*

Proof. Let us define

$$f_n(t) = \frac{t + h_n(t)}{1 + h_n(1)} \quad \text{for} \quad n = 1, 2, \ldots.$$

Then $f_n \in \mathscr{R}_1^-$, $f_n(0) = 0$, $f_n(1) = 1$ and the continuous part of f_n is increasing.

The assumption (1.32) implies that there is K such that $1 + h_n(1) \leqq K$ for every $n \in \mathbb{N}$.

If h_0 is continuous at $t' < t''$, then

$$\limsup_{n\to\infty} [f_n(t'') - f_n(t')] = \limsup_{n\to\infty} \frac{t'' - t' + h_n(t'') - h_n(t')}{1 + h_n(1)} \leqq$$
$$\leqq \limsup_{n\to\infty} [t'' - t' + h_n(t'') - h_n(t')] \leqq t'' - t' + \eta(h_0(t'') - h_0(t')). \tag{1.36}$$

Let us verify the assumption (1.26) of Theorem 1.20. Since the function h_0 is left-continuous at 1 and η is right-continuous at 0, for a given $\varepsilon > 0$ there is $\lambda \in (0, \varepsilon/2)$ such that

$$\eta(h_0(1) - h_0(t)) < \varepsilon/2 \quad \text{for every} \quad t \in (1 - \lambda, 1).$$

Let $\tau \in (1 - \lambda, 1 - \lambda/2]$ be fixed so that h_0 is continuous at τ. By (1.36) we have

$$\limsup_{n\to\infty} [f_n(1) - f_n(\tau)] \leqq [1 - \tau] + \eta(h_0(1) - h_0(\tau)) < \varepsilon.$$

There is $n_0 \in N$ such that

$$(1.37) \qquad f_n(1) - f_n(\tau) < \varepsilon \quad \text{for every} \quad n \geqq n_0.$$

Let $n = 1, 2, \ldots, n_0 - 1$. There is $s_n \in (0, 1)$ such that $f_n(1) - f_n(s_n) < \varepsilon$. Denote $\delta_n = f_n(1) - f_n(s_n)$. If $f_n(1) - f_n(t+) < \delta_n$ then $f_n(t+) > f_n(s_n)$, which implies $t \geqq s_n$. Consequently $f_n(t+) - f_n(t) \leqq f_n(1) - f_n(s_n) < \varepsilon$.

Denote $\delta_\varepsilon = \min\{\delta_1, \delta_2, \ldots, \delta_{n_0-1}, \lambda/2K, \varepsilon\}$. Assume that $f_n(1) - f_n(t+) < \delta_\varepsilon$, $n \geqq n_0$. Then

$$1 - t \leqq 1 - t + h_n(1) - h_n(t+) = [f_n(1) - f_n(t+)](1 + h_n(1)) \leqq$$
$$\leqq [f_n(1) - f_n(t+)] K < \delta_\varepsilon K \leqq \lambda/2.$$

Then $t \in (1 - \tau, 1)$. By (1.37) we have $f_n(t+) - f_n(t) \leqq f_n(1) - f_n(\tau) < \varepsilon$.

By Helly's Choice Theorem there is a function v_0 and a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $f_{n_k} \to v_0(t)$ for every $t \in [0, 1]$. Define $v(0) = 0$, $v(1) = 1$, $v(t) = v_0(t-)$ for $t \in (0, 1)$. From (1.36) we get (1.35); hence $v \in \mathscr{R}_1^-$.

Since the assumptions of Theorem 1.20 are satisfied, there is a sequence $(\varphi_k)_{k=1}^{\infty} \subset \subset \Lambda$ such that $\|(f_{n_k})_{-1} - \varphi_k^{-1}\| \to 0$ with $k \to \infty$ and the set $\{f_{n_k} \circ \varphi_k^{-1};\ k \in N\}$ is relatively compact in $(\mathscr{R}_1^-; \varrho)$. Consequently there is a subsequence which for simplicity will be denoted again by $(f_{n_k} \circ \varphi_k^{-1})$, a function $q \in \mathscr{R}_1^-$ and a sequence $(\lambda_k)_{k=1}^{\infty} \subset \Lambda$ such that

$$(f_{n_k} \circ \varphi_k^{-1}) \circ \lambda_k \rightrightarrows q \quad \text{and} \quad \lambda_k \rightrightarrows \mathrm{id}.$$

Since $f_n(t'') - f_n(t') \geqq (t'' - t')/K$ for every $t' < t''$, $n \in N$, we get $v(t'') - v(t') \geqq \geqq (t'' - t')/K$ for $t' < t''$. Then the function v is increasing. Proposition 1.13 implies that $(f_{n_k})_{-1} \rightrightarrows v_{-1}$. Then also $\varphi_{-1}^k \rightrightarrows v_{-1}$. By Proposition 1.13 we obtain that

$$(1.38) \qquad \varphi_k(t) \to v(t) \quad \text{provided } v \text{ is continuous at} \quad t \in [0, 1].$$

Let us denote $v_k = \lambda_k^{-1} \circ \varphi_k$ for every $k \in N$. Then $v_k \in \Lambda$; (1.38) implies (1.34).

Since the functions $f_{n_k} \circ v_k^{-1}$ are uniformly convergent, the functions $h_{n_k} \circ v_{k-1}$ are also uniformly convergent.

1.22. Proposition. *For every nondecreasing function $\varkappa$: $[0, 1] \to [0, \infty)$ such that $\lim_{r\to 0+} \varkappa(r) = 0 = \varkappa(0)$, there is a continuous concave increasing function η: $[0, 1] \to \to [0, \infty)$ such that $\eta(0) = 0$ and $\varkappa(r) \leqq \eta(r)$ for every $r \in [0, 1]$.*

Proof. Denote $\mu(0) = 0$ and define $\zeta_0 = \sup\{\zeta \in R;\ \varkappa(r) \leqq \varkappa(1) - \zeta(1 - r)$

for every $r \in [0, 1]\}$. Since the function $\varkappa$ is nondecreasing, we have $\zeta_0 \geqq 0$. Define

$$\mu(r) = \varkappa(1) - \zeta_0(1 - r) \quad \text{for} \quad r \in [\tfrac{1}{2}, 1].$$

For $k = 1, 2, \ldots$ let us assume that the function μ has been defined on $[2^{-k}, 1]$. Denote

$$\zeta_k = \sup\{\zeta \in \mathbb{R};\ \varkappa(r) \leqq \mu(2^{-k}) - \zeta(2^{-k} - r) \quad \text{for every} \quad r \in [0, 2^{-k}]\}$$

and define

$$\mu(r) = \mu(2^{-k}) - \zeta_k(2^{-k} - r) \quad \text{for} \quad r \in [2^{-k-1}, 2^{-k}).$$

Obviously $0 \leqq \varkappa(r) \leqq \mu(r)$ for every $r \in [0, 1]$, and the function μ is continuous on $(0, 1]$ (it is piecewise linear). Since the function $\varkappa$ is nondecreasing, we have $\zeta_k \geqq 0$ for every $k = 0, 1, 2, \ldots$ and consequently the function μ is nondecreasing on $[0, 1]$.

Let us show that the function μ is concave. For $k = 0, 1, 2, \ldots$, $r \in [0, 2^{-k-1}]$ we have the inequality

$$\begin{aligned}\varkappa(r) &\leqq \mu(r) = \mu(2^{-k}) - \zeta_k(2^{-k} - r) = \\ &= [\mu(2^{-k}) - \zeta_k(2^{-k} - 2^{-k-1})] - \zeta_k(2^{-k-1} - r) = \\ &= \mu(2^{k-1}) - \zeta_k(2^{-k-1} - r).\end{aligned}$$

Consequently $\zeta_{k+1} \geqq \zeta_k$, hence the function μ is concave on $(0, 1]$. Since $\mu(r) \geqq 0$ on $(0, 1]$ and $\mu(0) = 0$, this function is concave on the whole interval $[0,1]$.

Let us prove that $\mu(0+) = \lim\limits_{r \to 0+} \mu(r) = 0$. Let us denote $\beta = \mu(0+)$. Assume that $\beta > 0$. Since $\varkappa(0+) = 0$, there is $\delta > 0$ such that $\varkappa(r) < \beta/2$ for every $r \in (0, \delta)$. There is an integer k_0 such that $2^{-k_0} < \delta$. Then for any $k \geqq k_0$ and $r \in (0, 2^{-k}]$ we have

$$\begin{aligned}\varkappa(r) &\leqq \varkappa(2^{-k}) < \beta \,.\, \tfrac{1}{2} = \mu(0+) \,.\, \tfrac{1}{2} \leqq \\ &\leqq \mu(2^{-k}) \,.\, \tfrac{1}{2} < \mu(2^{-k}) (\tfrac{1}{2} + 2^{k-1} r) = \mu(2^{-k}) (1 - 2^{k-1}(2^{-k} - r)) = \\ &= \mu(2^{-k}) - [\mu(2^{-k}) \,.\, 2^{k-1}] (2^{-k} - r).\end{aligned}$$

Taking into account the definition of ζ_k, we find that

$$\zeta_k \geqq \mu(2^{-k}) \,.\, 2^{k-1}.$$

Then

$$\begin{aligned}\beta &= \mu(0+) \leqq \mu(2^{-k-1}) = \mu(2^{-k}) - \zeta_k(2^{-k} - 2^{-k-1}) \leqq \\ &\leqq \mu(2^{-k}) - [\mu(2^{-k}) \,.\, 2^{k-1}] (2^{-k} - 2^{-k-1}) = \\ &= \mu(2^{-k}) (1 - 2^{k-1}(2^{-k} - 2^{-k-1})) = \mu(2^{-k}) \,.\, 3/4\end{aligned}$$

holds for any integer $k \geqq k_0$. Passing to infinity with k, we obtain

$$\mu(0+) \leqq \mu(0+) \,.\, 3/4\,,$$

which is a contradiction with $\beta > 0$.

We have proved that μ is a continuous, concave and nondecreasing function. Then the function $\eta(r) = \mu(r) + r$, $r \in [0, 1]$ satisfies the requirements of Proposition 1.22.

2. CHARACTERIZATIONS OF COMPACT SETS IN $\mathscr{R}_N[a, b]$

2.1. Lemma. *Assume that the set $\mathscr{A} \subset \mathscr{R}_N[a, b]$ is equiregulated. Then for every $\varepsilon > 0$ there is a division $a = t_0 < t_1 < \ldots < t_k = b$ such that*

$$(2.1) \qquad |x(t) - x(t')| \leqq \varepsilon \quad \textit{for every} \quad x \in \mathscr{A} \quad \textit{and} \quad [t, t'] \subset (t_{j-1}, t_j)\,,$$
$$j = 1, 2, \ldots, k\,.$$

Proof. By D let us denote the set of all $d \in (a, b]$ such that there is a division $a = t_0 < t_1 \ldots < t_k = d$ for which (2.1) holds.

There is $\delta_1 \in (0, b - a]$ such that $|x(t) - x(a+)| \leqq \varepsilon/2$ for every $x \in \mathscr{A}$, $t \in \in (a, a + \delta_1)$; denote $d_1 = a + \delta_1$, $a = t_0 < t_1 = d_1$. For $[t, t'] \subset (a, d_1)$ and $x \in \mathscr{A}$ we have the inequality $|x(t) - x(t')| \leqq |x(t) - x(a+)| + |x(t') - x(a+)| \leqq \varepsilon$. Hence $d_1 \in D$. Denote $\tilde{d} = \sup D$. There is $\delta > 0$ such that $|x(\tilde{d}-) - x(t)| \leqq \varepsilon/2$ for every $x \in \mathscr{A}$, $t \in (\tilde{d} - \delta, \tilde{d}) \cap [a, b]$. Find $d \in D \cap (\tilde{d} - \delta, \tilde{d})$ and a division $a = t_0 < t_1 \ldots < t_k = d$ such that (2.1) holds. Denote $t_{k+1} = \tilde{d}$. For $[t, t'] \subset \subset (t_k, t_{k+1})$ and $x \in \mathscr{A}$ we have the inequality $|x(t) - x(t')| \leqq |x(t) - x(\tilde{d}-)| + + |x(t') - x(\tilde{d}-)| \leqq \varepsilon$. Hence $\tilde{d} \in D$. If $\tilde{d} < b$ then it would be possible to find $d_2 \in (d, b]$ such that $d_2 \in D$ in similar way as d_1 was defined. But this contradicts $\tilde{d} = \sup D$ and consequently $\tilde{d} = b$.

2.2. Lemma. *Assume that a set $\mathscr{A} \subset \mathscr{R}_N[a, b]$ is equiregulated and for any $t \in [a, b]$ there is a number γ_t such that*

$$(2.2) \qquad |x(t) - x(t-)| \leqq \gamma_t \quad \textit{holds for} \quad t \in (a, b]\,;$$
$$|x(t+) - x(t)| \leqq \gamma_t \quad \textit{holds for} \quad t \in [a, b)\,.$$

Then there is $K > 0$ such that $|x(t) - x(a)| \leqq K$ for every $x \in \mathscr{A}$, $t \in [a, b]$.

Proof. Denote by B the set of all $\tau \in (a, b]$ for which there is $K_\tau > 0$ such that $|x(t) - x(a)| \leqq K_\tau$ for any $x \in \mathscr{A}$, $t \in [a, \tau]$. Since the set $\mathscr{A}$ is equiregulated, there is $\delta > 0$ such that $|x(t) - x(a+)| \leqq 1$ for every $x \in \mathscr{A}$, $t \in (a, a + \delta]$. For every $t \in (a, a + \delta]$ and $x \in \mathscr{A}$ we have the estimate

$$|x(t) - x(a)| \leqq |x(t) - x(a+)| + |x(a+) - x(a)| \leqq 1 + \gamma_a = K_{(a+\delta)}\,.$$

Hence $(a, a + \delta] \subset B$. Denote $\tau_0 = \sup B$.

There is $\delta' > 0$ such that $|x(t) - x(\tau_0-)| \leqq 1$ for every $x \in \mathscr{A}$, $t \in [\tau_0 - \delta', \tau_0)$. Let us fix a point $\tau \in B \cap [\tau_0 - \delta', \tau_0)$. For $x \in \mathscr{A}$, $t \in (\tau, \tau_0)$ we have

$$|x(t) - x(a)| \leqq |x(t) - x(\tau_0-)| + |x(\tau_0-) - x(\tau)| + |x(\tau) - x(a)| \leqq$$
$$\leqq 1 + 1 + K_\tau ;$$

then also $|x(\tau_0-) - x(a)| \leqq 2 + K_\tau$ and

$$|x(\tau_0) - x(a)| \leqq |x(\tau_0) - x(\tau_0-)| + (x(\tau_0-) - x(a)| \leqq \gamma_{\tau_0} + 2 + K_\tau .$$

Hence $\tau_0 \in B$ with $K_{\tau_0} = \gamma_{\tau_0} + 2 + K_\tau$.

For $\tau_0 < b$ we can find $\delta'' > 0$ such that

$$|x(t) - x(\tau_0+)| \leqq 1 \quad \text{for any} \quad x \in \mathscr{A} , \quad t \in (\tau_0, \tau_0 + \delta''] .$$

Then $|x(t) - x(a)| \leqq |x(t) - x(\tau_0+)| + |x(\tau_0+) - x(\tau_0)| + |x(\tau_0) - x(a)| \leqq 1 + \gamma_{\tau_0} + K_{\tau_0} = K_{(\tau_0+\delta'')}$. Hence $\tau_0 + \delta'' \in B$ and we get a contradiction with the definition of B. Consequently $\tau_0 = b \in B$.

2.3. Proposition. *A set $\mathscr{A} \subset \mathscr{R}_N[a, b]$ is relatively compact in the sup-norm topology if and only if it is equiregulated, satisfies* (2.2) *and there is $\alpha > 0$ such that $|x(a)| \leqq \alpha$ for any $x \in \mathscr{A}$.*

Proof. It is well-known that a subset A of a Banach space X is relatively compact if and only if it is totally bounded, i.e. for every $\varepsilon > 0$ there is a finite ε-net F for A – i.e. such a subset $F = \{x_1, x_2, \ldots, x_k\}$ of X that for every $x \in A$ there is $x_n \in F$ satisfying $\|x - x_n\| \leqq \varepsilon$.

(i) Assume that $\mathscr{A}$ is relatively compact. Then it is bounded by a constant C; evidently (2.2) is satisfied with $\gamma_t = 2C$ for every $t \in [a, b]$.

Let $t_0 \in [a, b]$ and $\varepsilon > 0$ be given. Let $\{x_1, x_2, \ldots, x_k\} \subset \mathscr{R}_N[a, b]$ be a finite $\varepsilon/3$-net for $\mathscr{A}$. For every $n = 1, 2, \ldots, k$ there is $\delta_n > 0$ such that

$$|x_n(t) - x_n(t_0+)| < \varepsilon/3 \quad \text{for} \quad t \in (t_0, t_0 + \delta_n) \cap [a, b] \quad \text{and}$$
$$|x_0(t_0-) - x_n(t)| < \varepsilon/3 \quad \text{for} \quad t \in (t_0 - \delta_n, t_0) \cap [a, b] .$$

Denote $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_k\}$.

For arbitrary $x \in \mathscr{A}$ let us find x_n such that $\|x - x_n\| \leqq \varepsilon/3$. For every $t \in$ $t \in (t_0, t_0 + \delta) \cap [a, b]$ we have the inequality

$$|x(t) - x(t_0+)| \leqq |x(t) - x_n(t)| + |x_n(t) - x_n(t_0+)| +$$
$$+ |x_n(t_0+) - x(t_0+)| \leqq 2\|x - x_n\| + |x_n(t) - x_n(t_0+)| < \varepsilon ,$$

and similarly $|x(t_0-) - x(t)| < \varepsilon$ for $t \in (t_0 - \delta, t_0) \cap [a, b]$.

(ii) Assume that $\mathscr{A}$ is equiregulated, (2.2) holds and $|x(a)| \leqq \alpha$ for every $x \in \mathscr{A}$.

By Lemma 2.2 there is $K > 0$ such that $|x(t) - x(a)| \leqq K$ for any $x \in \mathscr{A}$ and $t \in [a, b]$. Hence $|x(t)| \leqq |x(t) - x(a)| + |x(a)| \leqq K + \alpha$. If we denote $\gamma = K + \alpha$

then $\|x\| \leqq \gamma$ for any $x \in \mathscr{A}$.

Let $\varepsilon > 0$ be given. By Lemma 2.1 there is a division $a = t_0 < t_1 < \ldots < t_k = b$ such that (2.1) holds, when ε is replaced by $\varepsilon/2$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a finite $\varepsilon/2$-net of the compact set $\{\alpha \in \mathbb{R}^N; |\alpha| \leqq \gamma\}$. Define $F = \{x: [a, b] \to \mathbb{R}^N;$ x is constant on (t_{j-1}, t_j) for every $j = 1, 2, \ldots, k$ and $x(t) \in \in \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ for every $t \in [a, b]\}$. The set $F \subset \mathscr{R}_N[a, b]$ is evidently finite.

Let us verify that F is an ε-net for $\mathscr{A}$. Let $x \in \mathscr{A}$ be given. For every $n = 0, 1, \ldots, k$ there is $i_n \in \{1, 2, \ldots, m\}$ such that $|x(t_n) - \alpha_{i_n}| \leqq \varepsilon$, for every $n = 1, 2, \ldots, k$ there is $j_n \in \{1, 2, \ldots, m\}$ such that $|x(t_n-) - \alpha_{j_n}| \leqq \varepsilon/2$.

Let us define $z(t_n) = \alpha_{i_n}$, $n = 0, 1, \ldots, k$, $z(t) = \alpha_{j_n}$ for $t \in (t_{n-1}, t_n)$ and any $n = 1, 2, \ldots, k$. Then $z \in F$ and $|z(t_n) - x(t_n)| \leqq \varepsilon$, $|z(t) - x(t)| = |\alpha_{j_n} - x(t)| \leqq \leqq |\alpha_{j_n} - x(t_n-)| + |x(t_n-) - x(t)| \leqq \varepsilon$ for $t \in (t_{n-1}, t_n)$; hence $\|z - x\| \leqq \varepsilon$. We have proved that $\mathscr{A}$ is totally bounded.

2.4. Corollary. *A set $\mathscr{A} \subset \mathscr{R}_N[a, b]$ si relatively compact if and only if it is equiregulated and for every $t \in [a, b]$ the set $\{x(t); x \in \mathscr{A}\}$ is bounded in $\mathbb{R}^N$.*

Proof. If $\mathscr{A}$ is relatively compact then it is equiregulated by Proposition 2.3 and evidently it is bounded.

Assume that $\mathscr{A}$ is equiregulated and $|x(t)| \leqq \beta_t$ for $x \in \mathscr{A}$, $t \in [a, b]$.

Let $t \in (a, b)$ be given. There is $\delta > 0$ such that $|x(\tau) - x(t-)| \leqq 1$ for $x \in \mathscr{A}$, $\tau \in (t - \delta, t)$ and $|x(\tau) - x(t+)| \leqq 1$ for $\tau \in (t, t + \delta)$. Let $\tau_1 \in (t - \delta, t)$, $\tau_2 \in \in (t, t + \delta)$ be fixed. Then

$$|x(t) - x(t-)| \leqq |x(t)| + |x(\tau_1)| + |x(t-) - x(\tau_1)| \leqq \beta_t + \beta_{\tau_1} + 1 ;$$
$$|x(t+) - x(t)| \leqq |x(t+) - x(\tau_2)| + |x(\tau_2)| + |x(t)| \leqq 1 + \beta_{\tau_2} + \beta_t .$$

Let us denote $\gamma_t = 1 + \beta_t + \max\{\beta_{\tau_1}, \beta_{\tau_2}\}$. Analogously γ_a, γ_b can be defined.

Hence the condition (2.2) is fulfilled and $\mathscr{A}$ is relatively compact by Proposition 2.3.

Remark. This result can be found also e.g. in [3].

2.5. By the symbol V let us denote the set of all increasing functions $v: [0, 1] \to \to [0, 1]$ such that $v(0) = 0$, $v(1) = 1$. Any function $v \in V$ transforms the interval $[0, 1]$ onto a subset of $[0, 1]$ having the form $[0, 1] \setminus \{ \bigcap_{\substack{t \in (0,1] \\ v(t-) < v(t)}} [v(t-), v(t)) \cup \cup \bigcap_{\substack{t \in [0,1) \\ v(t) < v(t+)}} (v(t), v(t+)]\}$.

2.6. Definition. *Let $v \in V$ be given. By the symbol L_v let us denote the set of all functions $y \in \mathscr{R}_N$ for which the following conditions hold:*

(2.3) *If $t \in (0, 1]$ is a point such that $v(t-) < v(t)$ then the function y is left-continuous at the point $v(t-)$ and linear on the interval $[v(t-), v(t)]$.*

(2.4) *If $t \in [0, 1)$ is a point such that $v(t) < v(t+)$ then the function y is right-continuous at $v(t+)$ and linear on $[v(t), v(t+)]$.*

2.7. Definitiin. *Let an increasing function $v \in V$ and a regulated function $x \in \mathscr{R}_N$ be given. A regulated function $y \in \mathscr{R}_N$ is called the linear prolongation of the function x along the function v, if $y \in L_v$ and $x(t) = y(v(t))$ for every $t \in [0, 1]$.*

2.8. Proposition. *Let $v \in V$. If $y_1, y_2 \in L_v$ are functions such that $y_1(v(t)) = y_2(v(t))$ for every $t \in [a, b]$ where $[a, b] \subset [0, 1]$, then $y_1(\tau) = y_2(\tau)$ for every $\tau \in [v(a), v(b)]$.*

Proof. Denote $y = y_1 - y_2$, then $y \in L_v$ and $y(v(t)) = 0$ for every $t \in [a, b]$. If $t \in (a, b]$ is such that $v(t-) < v(t)$ then $y(v(t-)) = 0$ by the assumption (2.3). Since the function y is linear on the interval $[v(t-), v(t)]$, it vanishes on all this interval. Similarly for every $t \in [a, b)$ such that $v(t) < v(t+)$ we have $y(v(t)) = y(v(t+)) = 0$ and consequently $y(\tau) = 0$ for every $\tau \in [v(t), v(t+)]$. Then $y_1(\tau) - y_2(\tau) = y(\tau) = 0$ for every $\tau \in [v(a), v(b)]$.

2.9. Proposition. *Let $v \in V$. Any function $x \in \mathscr{R}_N$ has exactly one linear prolongation along v.*

Proof. For a given function $x \in \mathscr{R}_N$ let us define a function $y: [0, 1] \to R^N$ in the following way:

(2.5) $y(\tau) = x(t)$ provided $\tau = v(t)$, $t \in [0, 1]$;
if $v(t-) \neq v(t)$ then $y(\tau) = x(t-)$ for $\tau = v(t-)$ and y is linear on $[v(t-), v(t)]$;
if $v(t) \neq v(t+)$ then $y(\tau) = x(t+)$ for $\tau = v(t+)$ and y is linear on $[v(t), v(t+)]$.

To prove that y is regulated, it is sufficient to verify that

$$(2.6) \qquad \begin{aligned} &\lim_{\tau \to \tau_0 -} y(\tau) = x(t_0-) \quad \text{for every} \quad \tau_0 = v(t_0-), \quad t_0 \in (0, 1];\\ &\lim_{\tau \to \tau_0 +} y(\tau) = x(t_0+) \quad \text{for every} \quad \tau_0 = v(t_0+), \quad t_0 \in [0, 1). \end{aligned}$$

Let $t_0 \in (0, 1]$, denote $\tau_0 = v(t_0-)$. For a given $\varepsilon > 0$ there is $\delta > 0$ such that $|x(t) - x(t_0-)| < \varepsilon$ for every $t \in (t_0 - \delta, t_0)$. For arbitrary $\tau \in (v(t_0 - \delta), v(t_0-))$ we can find $t \in (t_0 - \delta, t_0)$ such that $\tau \in [v(t-), v(t+)]$ (this interval contains only one point when v is continuous at t). If $\tau \in [v(t-), v(t)]$, there is $\lambda \in [0, 1]$ such that $\tau = \lambda\, v(t-) + (1 - \lambda)\, v(t)$; since y is linear on $[v(t-), v(t)]$, it has the form $y(\tau) = \lambda\, x(t-) + (1 - \lambda)\, x(t)$. We get the inequality $|y(\tau) - x(t_0-)| \leqq \lambda |x(t-) - x(t_0-)| + (1 - \lambda)\, |x(t) - x(t_0-)| < \varepsilon$. In the latter case $\tau \in [v(t), v(t+)]$ we can find $\mu \in [0, 1]$ such that $\tau = \mu\, v(t) + (1 - \mu)\, v(t+)$, and again we get $|y(\tau) - x(t_0-)| < \varepsilon$. Consequently $\lim y(\tau) = x(t_0-)$. The other equality in (2.6) can be verified analogously.

It is evident that $y \in L_v$. It follows from Proposition 2.8 that the linear prolongation is unique.

2.10. Proposition. *Let $v \in V$. The linear prolongation of a function $x \in \mathscr{R}_N$ along v is continuous if and only if the condition*

$$(2.7) \qquad \text{if } t \in (0, 1], \ x(t-) \neq x(t) \text{ then } v(t-) \neq v(t); \\ \text{if } t \in [0, 1), \ x(t) \neq x(t+) \text{ then } v(t) \neq v(t+)$$

holds.

Proof. Let us denote by y the linear prolongation of x along v. Assume that y is continuous. If $v(t-) = v(t)$ for some $t \in (0, 1]$ then $x(t-) = \lim_{\tau \to t-} y(v(\tau)) = y(v(t)) = x(t)$; if $v(t+) = v(t)$ for some $t \in [0, 1)$ then $x(t+) = x(t)$. Hence (2.7) is satisfied.

Assume that the condition (2.7) holds. In order to verify that y is continuous, it is sufficient to whow that

$$\lim_{\tau \to \tau_0 -} y(\tau) = y(\tau_0) \quad \text{for every} \quad \tau_0 = v(t_0 -), \quad t_0 \in (0, 1] \quad \text{and}$$
$$\lim_{\tau \to \tau_0 +} y(\tau) = y(\tau_0) \quad \text{for every} \quad \tau_0 = v(t_0 +), \quad t_0 \in [0, 1).$$

Let $t_0 \in (0, 1]$, denote $\tau_0 = v(t_0-)$. If $v(t_0-) \neq v(t_0)$ then $y(\tau_0) = x(t_0-)$ by (1.6); from (1.7) we get $\lim_{\tau \to \tau_0 -} y(\tau) = x(t_0-) = y(\tau_0)$. If $v(t_0-) = v(t_0)$ then $x(t_0-) = x(t_0)$ by virtue of (2.7) and from (2.6) we get the equality $\lim_{\tau \to \tau_0 \tau -} y(\tau) = x(t_0) = y(\tau_0)$.

The equality $\lim_{\tau \to \tau_0 +} y(\tau) = y(\tau_0)$ for $\tau_0 = v(t_0+)$ can be verified analogously.

2.11. Proposition. *Assume that $v \in V$. For every two functions $y_1, y_2 \in L_v$ we have the equality*

$$\|y_1 - y_2\| = \|y_1 \circ v - y_2 \circ v\|.$$

Proof. Let us denote $y = y_1 - y_2$. Evidently $\|y \circ v\| \leqq \|y\|$. If $\sigma = v(t)$, $t \in [0, 1]$, then

$$(2.8) \qquad |y(\sigma)| = |y(v(t))| \leqq \|y \circ v\|.$$

If $\vartheta = v(t-)$ and $v(t-) \neq v(t)$ then the function y is continuous at $v(t-)$ due to (2.3); from (2.8) we get

$$(2.9) \qquad |y(\vartheta)| = \lim_{s \to t-} |y(v(s))| \leqq \|y \circ v\|.$$

Since y is linear on $[v(t-), v(t)]$ and we have estimates (2.8), (2.9) for $\vartheta = v(t-)$, $\sigma = v(t)$, for every $\tau \in [v(t-), v(t)]$ the inequality $|y(\tau)| \leqq \|y \circ v\|$ holds.

Similarly $|y(\tau)| \leqq \|y \circ v\|$ for every $\tau \in [v(t), v(t+)]$ where $t \in [0, 1)$ is such that $v(t) \neq v(t+)$. Hence $\|y\| \leqq \|y \circ v\|$.

It has been proved that $\|y_1 - y_2\| = \|y\| = \|y \circ v\| = \|y_1 \circ v - y_2 \circ v\|$.

2.12. Proposition. *Let functions $x \in \mathscr{R}_N$ and $v \in V$ be given, assume that there is a continuous increasing concave function η: $[0, 1] \to [0, \infty)$, $\eta(0) = 0$ such that*

$$(2.10) \qquad |x(t_2) - x(t_1)| \leqq \eta(v(t_2) - v(t_1)) \quad \textit{for every} \quad 0 \leqq t_1 < t_2 \leqq 1 .$$

Let the function $y \in L_v$ be the linear prolongation of the function x along v. Then

$$|y(\tau_2) - y(\tau_1)| \leqq \eta(\tau_2 - \tau_1) \quad \textit{for every} \quad 0 \leqq \tau_1 < \tau_2 \leqq 1 .$$

Proof. Let us denote by Z the closure of the set $\{\tau \in [0, 1];\ \tau = v(t)$ for some $t \in [0, 1]\}$. If $\tau_1, \tau_2 \in [0, 1]$ are points such that $\tau_1 = v(t_1)$, $\tau_2 = v(t_2)$ and $t_1 < t_2$, then (2.1) implies that

$$|y(\tau_2) - y(\tau_1)| = |x(t_2) - x(t_1)| \leqq \eta(v(t_2) - v(t_1)) = \eta(\tau_2 - \tau_1) .$$

Since the functions y, η are continuous, the inequality

$$(2.11) \qquad |y(\tau_2) - y(\tau_1)| \leqq \eta(\tau_2 - \tau_1)$$

holds for every $\tau_1, \tau_2 \in Z$ such that $\tau_1 < \tau_2$.

(a) Assume that (a, b) is a component of the open set $(0, 1) \setminus Z$, let $a \leqq \tau_1 < \tau_2 \leqq \leqq b$. Since $a, b \in Z$, the inequality $|y(b) - y(a)| \leqq \eta(b - a)$ holds.

Either $(a, b) = (v(t-), v(t))$ or $(a, b) = (v(t), v(t+))$ for some $t \in [0, 1]$. Since $y \in L_v$, the function y is linear on $[a, b]$; hence

$$y(\tau_2) - y(\tau_1) = \frac{\tau_2 - \tau_1}{b - a} \cdot [y(b) - y(a)] .$$

Owing to the fact that η is a concave function and $\eta(0) = 0$, we get the inequality

$$|y(\tau_2) - y(\tau_1)| \leqq \frac{\tau_2 - \tau_1}{b - a} \cdot \eta(b - a) \leqq \eta \left(\frac{\tau_2 - \tau_1}{b - a} \cdot (b - a) \right) =$$
$$= \eta(\tau_2 - \tau_1) .$$

(b) It remains to consider the case when $\tau_1, \tau_2 \in [0, 1]$ are points such that $a \leqq \leqq \tau_1 \leqq b \leqq c \leqq \tau_2 \leqq d$, where $a, b, c, d \in Z$ and the following holds: If $a < b$ then y is linear on $[a, b]$; if $c < d$ then y is linear on $[c, d]$. Let $\lambda_1, \lambda_2 \in [0, 1]$ be such that $\tau_1 = (1 - \lambda_1) a + \lambda_1 b$ and $\tau_2 = (1 - \lambda_2) c + \lambda_2 d$.

Since the function η is concave, (2.2) yields the estimate

$$\begin{aligned}
&|y(\tau_2) - y(\tau_1)| = \\
&= |[(1 - \lambda_2)\, y(c) + \lambda_2\, y(d)] - [(1 - \lambda_1)\, y(a) + \lambda_1\, y(b)]| = \\
&= |(1 - \lambda_2)\, [(1 - \lambda_1)\, (y(c) - y(a)) + \lambda_1(y(c) - y(b))] + \\
&+ \lambda_2[(1 - \lambda_1)\, (y(d) - y(a)) + \lambda_1(y(d) - y(b))]| \leqq \\
&\leqq (1 - \lambda_2)\, [(1 - \lambda_1)\, \eta(c - a) + \lambda_1 \eta(c - b)] + \\
&+ \lambda_2[(1 - \lambda_1)\, \eta(d - a) + \lambda_1 \eta(d - b)] \leqq \eta(\tau_2 - \tau_1) .
\end{aligned}$$

2.13. Lemma. *If two sets $M^- \subset (0, 1]$ and $M^+ \subset [0, 1)$ are at most countable, there exists an increasing function $v \in V$ such that*

$$(2.12) \qquad M^- = \{t \in (0, 1];\ v(t-) \neq v(t)\} \quad \text{and}$$
$$M^+ = \{t \in [0, 1);\ v(t) \neq v(t+)\}\,.$$

Proof. Let us order the sets M^-, M^+ into sequences $M^- = \{s_1, s_2, \ldots\}$, $M^+ = \{\sigma_1, \sigma_2, \ldots\}$ (finite sequences if the sets are finite). Let us take any sequences of positive numbers $\{r_1, r_2, \ldots\}$ and $\{\varrho_1, \varrho_2, \ldots\}$ such that $\sum r_j < \infty$, $\sum \varrho_j < \infty$. Let us define $w(t) = t + \sum_{0 < s_j \leqq t} r_j + \sum_{0 \leqq \sigma_j < t} \varrho_j$ for every $t \in [0, 1]$. Then the function w is increasing, $w(0) = 0$, $0 < w(1) < \infty$. The function $v(t) = w(1)^{-1}\, w(t)$ belongs to V and satisfies (2.12).

2.14. Theorem. *For an arbitrary function $x\colon [0, 1] \to \mathbb{R}^N$ the following conditions are equivalent:*

(i) *The function x is regulated.*

(ii) *There is a continuous function $y\colon [0, 1] \to \mathbb{R}^N$ and an increasing function $v \in V$ such that $x(t) = y(v(t))$ for every $t \in [0, 1]$.*

(iii) *There is an increasing function $v\colon [0, 1] \to [0, 1]$ and a continuous increasing function $\eta\colon [0, 1] \to [0, \infty)$ such that $\eta(0) = 0$ and*

$$(2.13) \qquad |x(t_2) - x(t_1)| \leqq \eta(v(t_2) - v(t_1)) \quad \textit{provided} \quad 0 \leqq t_1 < t_2 \leqq 1\,.$$

Proof. (i) $\Rightarrow$ (ii). Let us denote

$$(2.14) \qquad M^- = \{t \in (0, 1];\ x(t-) \neq x(t)\} \quad \text{and}$$
$$M^+ = \{t \in [0, 1);\ x(t) \neq x(t+)\}\,.$$

By virtue of the property 1.6 the sets M^-, M^+ are at most countable. By Lemma 2.13 there is a function $v \in V$ such that (2.12) holds. If $y \in L_v$ is the linear prolongation of x along v, it is continuous according to Proposition 2.10.

(ii) $\Rightarrow$ (iii) The function η is a modulus of continuity of the function y.

(iii) $\Rightarrow$ (i) Let $t_0 \in (0, 1]$. For an arbitrary $\varepsilon > 0$ there is $\lambda > 0$ such that $\eta(\lambda) < \varepsilon$ and there is $\delta > 0$ such that $v(t_0-) - v(t_0 - \delta) \leqq \lambda$. If $t_0 - \delta \leqq t' < t'' < t_0$, then $v(t'') - v(t') \leqq v(t_0-) - v(t_0 - \delta) \leqq \lambda$, hence $|x(t'') - x(t')| \leqq \eta(v(t'') - v(t')) \leqq \eta(\lambda) < \varepsilon$. It is well-known that this implies the existence of the limit $\lim_{t \to t_0-} x(t) = x(t_0 -)$. Similarly for every $t_0 \in [0, 1)$ the limit $x(t_0+)$ exists.

2.15. Remark. If the function x belongs to $\mathscr{R}_N^-$, the set M^- is empty and the set M^+ does not contain the point 0. Hence the function v in Theorem 2.14 is also left-continuous on $(0, 1]$ and right-continuous at 0.

2.16. Lemma. *If a set $\mathscr{A} \subset \mathscr{R}_N$ is equiregulated then the sets*

(2.15) $M^- = \{t \in (0,1]$; *there is* $x \in \mathscr{A}$ *such that* $x(t-) \neq x(t)\}$ *and*
$M^+ = \{t \in [0,1)$; *there is* $x \in \mathscr{A}$ *such that* $x(t) \neq x(t+)\}$

are at most countable.

Proof. Only the set M^- will be dealt with – the proof for M^+ is quite analogous. For every $j \in \mathbb{N}$ let us denote

$$M_j = \{t \in (0,1];\ \text{there is } x \in \mathscr{A} \text{ such that } |x(t) - x(t-)| \geqq 1/j\}\,.$$

Since $M^- = \bigcup_{j=1}^{\infty} M_j$, it is sufficient to prove that the set M_j is finite for every $j \in \mathbb{N}$.

Assume that there is j such that the set M_j is infinite. Let us choose a strictly monotone sequence $(t_n)_{n=1}^{\infty} \subset M_j$ and denote its limit by t_0. For instance, assume that the sequence (t_n) is decreasing.

For every $n \in \mathbb{N}$ there is $x_n \in \mathscr{A}$ such that $|x_n(t_n) - x_n(t_n-)| \geqq 1/j$. Since the set $\mathscr{A}$ is equiregulated, there is $\delta > 0$ such that

$$|x(t) - x(t_0+)| \leqq 1/3j \quad \text{for every} \quad x \in \mathscr{A}\,, \quad t \in (t_0, t_0 + \delta)\,.$$

There is $n_0 \in \mathbb{N}$ such that $t_n \in (t_0, t_0 + \delta)$ for every $n \geqq n_0$. If $n \geqq n_0$ then

$$\begin{aligned} 1/j &\leqq |x_n(t_n) - x_n(t_n-)| \leqq \\ &\leqq |x_n(t_n) - x_n(t_0+)| + |x_n(t_n-) - x_n(t_0+)| \leqq 2/3j\,, \end{aligned}$$

which is a contradiction; hence M_j is finite.

2.17. Theorem. *For any set of regulated functions* $\mathscr{A} \subset \mathscr{R}_N$ *the following properties are equivalent:*

(i) $\mathscr{A}$ *is equiregulated and satisfies* (2.2).

(ii) *There is an increasing function* $v \in V$ *and an increasing continuous function* $\eta: [0, \infty) \to [0, \infty)$, $\eta(0) = 0$ *such that*

(2.16) $|x(t'') - x(t')| \leqq \eta(v(t'') - v(t'))$ *for every* $x \in \mathscr{A}$,
$0 \leqq t' < t'' \leqq 1$.

(iii) *There is* $v \in V$ *and an equicontinuous set* $\mathscr{B} \subset \mathscr{C}_N$ *such that* $\mathscr{A} \subset \mathscr{B} \circ v$, *i.e. for every* $x \in \mathscr{A}$ *there is a continuous function* $y \in \mathscr{B}$ *such that* $x = y \circ v$.

Proof. (i) $\Rightarrow$ (ii) By Lemma 2.16 the sets M^-, M^+ defined in (2.15) are at most countable. By Lemma 2.13 we can construct a function $v \in V$ such that (2.12) holds. This function is defined so that

(2.17) $v(t'') - v(t') \geqq c(t'' - t')\,, \quad 0 \leqq t' < t'' < 1$

for some $c > 0$. For every $r > 0$ let us define

$$\varkappa(r) = \sup \{|x(t'') - x(t')|;\ x \in \mathscr{A},\ 0 \leqq t' < t'' \leqq 1,\ v(t'') - v(t') \leqq r\}\,.$$

For $t' < t''$ let us denote $r = v(t'') - v(t')$. Then

$$(2.18)\qquad |x(t'') - x(t')| \leqq \varkappa(r) = \varkappa(v(t'') - v(t')) \quad \text{for any} \quad x \in \mathscr{A}\,.$$

Lemma 2.2 implies that $\varkappa(r) < \infty$ for every $r > 0$. The function $\varkappa$ is evidently nondecreasing on $(0, \infty)$.

Let us prove that $\varkappa(0+) = 0$. For every $r > 0$ there is $x_r \in \mathscr{A}$ and $t'_r < t''_r$ such that

$$v(t''_r) - v(t'_r) \leqq r \quad \text{and} \quad |x_r(t''_r) - x_r(t'_r)| \geqq \tfrac{1}{2}\varkappa(r)\,.$$

By (2.17) we have

$$t''_r - t'_r \leqq \frac{1}{c}\,[v(t''_r) - v(t'_r)] \leqq \frac{1}{c}\,r\,;$$

hence $t''_r - t'_r \to 0$ with $r \to 0$.

Since the nets $(t'_r)_{r>0}$ and $(t''_r)_{r>0}$ are contained in the compact interval $[0, 1]$, there are convergent subsequences

$$(2.19)\qquad t'_{r_n} \to t_0 \quad \text{and} \quad t''_{r_n} \to t_0 \quad \text{with} \quad r_n \to 0\,.$$

Denote $x_{r_n} = x_n$, $t'_{r_n} = t'_n$, $t''_{r_n} = t''_n$ for $n \in \mathbb{N}$.

Since the set $\mathscr{A}$ is equiregulated, for every $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that we have for every $x \in \mathscr{A}$, $t \in [0, 1]$:

$$(2.20)\qquad \begin{aligned} &\text{if} \quad t_0 - \delta_\varepsilon < t < t_0 \quad \text{then} \quad |x(t_0-) - x(t)| < \varepsilon\,;\\ &\text{if} \quad t_0 < t < t_0 + \delta_\varepsilon \quad \text{then} \quad |x(t) - x(t_0+)| < \varepsilon\,. \end{aligned}$$

(a) Assume that the sequence (r_n) can be found so that $t'_n = t_0$ for every $n \in \mathbb{N}$. Then

$$v(t''_n) - v(t_0) \leqq r_n \to 0\,; \quad \text{consequently} \quad v(t_0+) - v(t_0) = 0\,.$$

Then $t_0 \notin M^+$ and $x(t_0+) = x(t_0)$ holds for every $x \in \mathscr{A}$. If for a given $\varepsilon > 0$ the integer n is big enough so that $t''_n < t_0 + \delta_\varepsilon$, then (2.20) yields

$$\varkappa(r_n) \leqq 2|x_n(t''_n) - x_n(t_0)| < 2\varepsilon\,.$$

(b) Similarly, if $t''_n = t_0$ for every $n \in \mathbb{N}$, then $v(t_0) - v(t'_n) \leqq r_n$, hence $v(t_0) - v(t_0-) = 0$, and $x(t_0) = x(t_0-)$ for every $x \in \mathscr{A}$. Then $\varkappa(r_n) \leqq 2|x_n(t_0) - x_n(t'_n)| < 2\varepsilon$ for every n such that $t_0 - \delta_\varepsilon < t'_n$.

(c) If we can find sequences (t'_n), (t''_n) such that $t'_n < t_0 < t''_n$, the inequality $v(t''_n) - v(t'_n) \leqq r_n \to 0$ implies $v(t_0+) - v(t_0-) = 0$. Hence $t_0 \notin M^- \cup M^+$ and $x(t_0-) = x(t_0) = x(t_0+)$ for any $x \in \mathscr{A}$.

If for $\varepsilon > 0$ an integer n satisfies $t_0 - \delta_\varepsilon < t'_n < t_0 < t''_n < t_0 + \delta_\varepsilon$, then

$$\varkappa(r_n) \leqq 2[|x_n(t''_n) - x_n(t_0)| + |x_n(t_0) - x_n(t'_n)|] < 4\varepsilon\,.$$

(d) Assume that $t_n' < t_n'' < t_0$ for every $n \in \mathbb{N}$. If for a given $\varepsilon > 0$ the inequality $t_0 - \delta_\varepsilon < t_n'$ holds, then

$$\varkappa(r_n) \leqq 2[|x_n(t_n'') - x_n(t_0-)| + |x_n(t_0-) - x_n(t_n')|] < 4\varepsilon .$$

(e) Similarly in the case of $t_0 < t_n' < t_n''$ we get:

$$\text{if} \quad t_n < t_0 + \delta_\varepsilon \quad \text{then} \quad \varkappa(r_n) < 4\varepsilon .$$

We conclude that $\varkappa(r_n) \to 0$ with $n \to \infty$ in each of the cases mentioned. Consequently $\varkappa(0+) = 0$.

By Proposition 1.22 there is an increasing continuous function $\eta: [0, \infty) \to [0, \infty)$ such that $\eta(0) = 0$ and $\varkappa(r) \leqq \eta(r)$ for every $r > 0$. Then from (2.18) we obtain (2.16).

(ii) $\Rightarrow$ (iii) According to Proposition 1.22 the function η in (2.16) can be replaced by a concave increasing function $\tilde{\eta}$ such that $\eta(r) \leqq \tilde{\eta}(r)$ for $r \in [0, 1]$. From (2.16) we get

$$|x(t+) - x(t)| \leqq \eta(v(t+) - v(t)) \quad \text{for any} \quad x \in \mathscr{A}, \quad t \in [0, 1) ;$$
$$|x(t) - x(t-)| \leqq \eta(v(t) - v(t-)) \quad \text{for any} \quad x \in \mathscr{A}, \quad t \in (0, 1] .$$

Consequently (2.7) is satisfied for any $x \in \mathscr{A}$.

Let us denote by $\mathscr{B}$ the set of the linear prolongations of all functions from $\mathscr{A}$ along v. Then $\mathscr{A} = \mathscr{B} \circ v$ holds. According to Proposition 2.11 all functions from $\mathscr{B}$ are continuous. Moreover, by Proposition 2.12 every $y \in \mathscr{B}$ satisfies

$$|y(\tau'') - y(\tau')| \leqq \tilde{\eta}(\tau'' - \tau') \quad \text{for} \quad 0 \leqq \tau' < \tau'' \leqq 1 .$$

The function $\tilde{\eta}$ is a uniform modulus of continuity of the set $\mathscr{B}$.

(iii) $\Rightarrow$ (i) If $\mathscr{A} \subset \mathscr{B} \circ v$ where $\mathscr{B} \subset \mathscr{C}_N$ is an equicontinuous set, it is well-known that there is such $K > 0$ that $|y(\tau) - y(0)| \leqq K$ for every $\tau \in [0, 1]$, $y \in \mathscr{B}$. Then (2.2) is satisfied.

Let us prove that the set $\mathscr{A}$ is equiregulated. Let $\varepsilon > 0$ be given. There is $\lambda > 0$ such that the following holds: If $|\tau'' - \tau'| \leqq \lambda$ then $|y(\tau'') - y(\tau')| < \varepsilon$ for any $y \in \mathscr{B}$.

Let $t_0 \in (0, 1]$ be given, denote $\tau_0 = v(t_0-)$. There is $\delta > 0$ such that $v(t_0-) - - v(t_0 - \delta) \leqq \lambda$. For any $t \in (t_0 - \delta, t_0)$ denote $\tau = v(t)$. Then $\tau_0 - \tau < \lambda$. If $x = y \circ v$ then $|x(t_0-) - x(t)| = |y(v(t_0-)) - y(v(t)))| = |y(\tau_0) - y(\tau)| < \varepsilon$. Similarly for every $t_0 \in [0, 1)$ there is $\delta > 0$ such that $|x(t) - x(t_0+)| < \varepsilon$ for any $x \in \mathscr{A}$, $t \in (t_0, t_0 + \delta)$. Hence the set $\mathscr{A}$ is equiregulated.

Now we will formulate an important theorem about various characterizations of relatively compact sets in $\mathscr{R}_N$.

2.18. Theorem. *For any set of regulated functions $\mathscr{A} \subset \mathscr{R}_N$ the following properties are equivalent:*

(i) *$\mathscr{A}$ is relatively compact in the sup-norm topology in $\mathscr{R}_N$.*

(ii) *$\mathscr{A}$ is equiregulated, satisfies* (2.2) *and*

(2.21) *there is* $\alpha > 0$ *such that* $|x(0)| \leqq \alpha$ *for any* $x \in \mathscr{A}$.

(iii) *The set $\mathscr{A}$ satisfies* (2.16) *and* (2.21).

(iv) *There is $v \in V$ and a compact set of continuous functions $\mathscr{B} \subset \mathscr{C}_N$ such that $\mathscr{A} \subset \mathscr{B} \circ v$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) was established in Proposition 2.3. Here we will give another proof of (ii) $\Rightarrow$ (i), proving successively the implications (ii) $\Rightarrow$ $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Now let us use only the fact that (i) $\Rightarrow$ (ii) was proved in Proposition 2.3.

(ii) $\Rightarrow$ (iii) is the same as (i) $\Rightarrow$ (ii) in Theorem 2.17, together with the assumption (2.21).

(iii) $\Rightarrow$ (iv): By (ii) $\Rightarrow$ (iii) in Theorem 2.17 there is $v \in V$ and an equicontinuous set $\mathscr{B}_1 \subset \mathscr{C}_N$ such that $\mathscr{A} = \mathscr{B}_1 \circ v$. By (2.21) the inequality $|y(0)| \leqq \alpha$ holds for every $y \in \mathscr{B}_1$. By the Arzelà-Ascoli Theorem the set $\mathscr{B}_1$ is relatively compact in $\mathscr{C}_N$. Then there is a compact set $\mathscr{B} \subset \mathscr{C}_N$ such that $\mathscr{B}_1 \subset \mathscr{B}$; hence $\mathscr{A} \subset \mathscr{B} \circ v$.

(iv) $\Rightarrow$ (i) Let $(x_n)_{n=1}^{\infty} \subset \mathscr{A}$ be an arbitrary sequence; for any $n \in \mathbb{N}$ there is $y_n \in \mathscr{B}$ such that $x_n = y_n \circ v$. Since the set $\mathscr{B}$ is compact, there is a convergent subsequence $y_{n_k} \rightrightarrows y_0$. Then $x_{n_k} = y_{n_k} \circ v \rightrightarrows y_0 \circ v$; hence (x_{n_k}) is a Cauchy subsequence. Consequently $\mathscr{A}$ is relatively compact.

3. POINTWISE CONVERGENCE OF REGULATED FUNCTIONS

3.1. It is well-known that functions of bounded variation have a nice property expressed in *Helly's Choice Theorem*:

Assume that for a sequence $(z_n)_{n=1}^{\infty} \subset BV_N[a, b]$ there are positive numbers γ, K such that $|z_n(a)| \leqq \gamma$ and $\operatorname{var}_a^b z_n \leqq K$ holds for every $n \in \mathbb{N}$. Then there is a function z_0 and a subsequence $(z_{n_k})_{k=1}^{\infty}$ such that $z_{n_k}(t) \to z_0(t)$ holds for every $t \in [a, b]$. The function z_0 is of bounded variation and

$$\operatorname{var}_a^b z_0 \leqq \liminf_{n \to \infty} \operatorname{var}_a^b z_n .$$

In order to extend this result to the space $\mathscr{R}_N[a, b]$, it is possible to reason in this way: Let a sequence of regulated functions $(x_n)_{n=1}^{\infty} \subset \mathscr{R}_N[a, b]$ be given such that $|x_n(a)| \leqq \gamma$ for any $n \in \mathbb{N}$. Assume that in an arbitrary close "neighbourhood" (in the sup-norm) of the sequence (x_n) we can find a sequence (z_n) the members of which have uniformly bounded variations. Then we can find a pointwise convergent subsequence (z_{n_k}), using Helly's Choice Theorem. Since the functions z_{n_k}, $k \in \mathbb{N}$ are "near" to the functions x_{n_k}, $k \in \mathbb{N}$, we can expect that the subsequence (x_{n_k}) is "almost" pointwise convergent. More precisely:

Assume that for every $\varepsilon > 0$ there is a sequence $(z_n^\varepsilon)_{n=1}^\infty \subset BV_N[a, b]$ and a number $K_\varepsilon > 0$ such that

$$\|x_n - z_n^\varepsilon\|_{[a,b]} \leqq \varepsilon \quad \text{and} \quad \operatorname{var}_a^b z_n^\varepsilon \leqq K_\varepsilon \quad \text{holds for any} \quad n \in \mathbb{N} .$$

Let $(\varepsilon_m)_{m=1}^\infty$ be an arbitrary sequence of positive numbers such that $\varepsilon_m \to 0$. For every $m \in \mathbb{N}$ the sequence $(z_n^{\varepsilon_m})_{n=1}^\infty$ contains a pointwise convergent subsequence (by Helly's Choice Theorem). Using diagonalization process, we can find an increasing sequence of indices $(n_k)_{k=1}^\infty$ such that

$$z_{n_k}^{\varepsilon_m}(t) \to z_0^{\varepsilon_m}(t) \quad \text{holds for every} \quad t \in [a, b] \quad \text{and} \quad m \in \mathbb{N} .$$

Let us show that $(z_0^{\varepsilon_m})_{m=1}^\infty$ is a Cauchy sequence in the sup-norm topology. Let $\eta > 0$ be given. There is $m_0 \in \mathbb{N}$ such that $\varepsilon_m < \eta/4$ for any $m \geqq m_0$. Let $m, p \geqq m_0$ and $t \in [a, b]$ be fixed. There is $k \in \mathbb{N}$ such that

$$|z_{n_k}^{\varepsilon_m}(t) - z_0^{\varepsilon_m}(t)| < \eta/4 \quad \text{and} \quad |z_{n_k}^{\varepsilon_p}(t) - z_0^{\varepsilon_p}(t)| < \eta/4 . \quad \text{Then}$$

$$|z_0^{\varepsilon_m}(t) - z_0^{\varepsilon_p}(t)| \leqq |z_0^{\varepsilon_m}(t) - z_{n_k}^{\varepsilon_m}(t)| + |z_0^{\varepsilon_p}(t) - z_{n_k}^{\varepsilon_p}(t)| +$$

$$+ |z_{n_k}^{\varepsilon_m}(t) - x_{n_k}(t)| + |z_{n_k}^{\varepsilon_p}(t) - x_{n_k}(t)| < \eta/4 + \eta/4 + \varepsilon_m + \varepsilon_p < \eta .$$

We find that $\|z_0^{\varepsilon_m} - z_0^{\varepsilon_p}\| < \eta$ holds for any $m, p \geqq m_0$. Hence $(z_0^{\varepsilon_m})_{m=1}^\infty$ is a Cauchy sequence and it has a uniform limit x_0. It is easy to verify that $x_{n_k}(t) \to x_0(t)$ for every $t \in [a, b]$. In this way we have found a subsequence of (x_n) which is pointwise convergent.

3.2. Definition. *For an arbitrary function* $x: [a, b] \to \mathbb{R}^N$ *and a positive number* $\varepsilon > 0$ *let us define*

$$\varepsilon\text{-}\operatorname{var}_a^b x = \inf \{\operatorname{var}_a^b z;\ z \in BV_N[a, b],\ \|x - z\|_{[a,b]} \leqq \varepsilon\} .$$

We set $\inf \emptyset = \infty$.

3.3. Definition. *We say that a set* $\mathcal{A} \subset \mathcal{R}_N[a, b]$ *has uniformly bounded* ε*-variations, when for every* $\varepsilon > 0$ *there is a number* $K_\varepsilon > 0$ *such that* $\varepsilon\text{-}\operatorname{var}_a^b x \leqq K_\varepsilon$ *for every* $x \in \mathcal{A}$.

3.4. Proposition. *A function* $x: [a, b] \to \mathbb{R}^N$ *is regulated if and only if* $\varepsilon\text{-}\operatorname{var}_a^b x < \infty$ *for every* $\varepsilon > 0$.

Proof. If the function x is regulated, then the property 1.5 implies that for every $\varepsilon > 0$ there is a piecewise constant function $z: [a, b] \to \mathbb{R}^N$ such that $\|x - z\| \leqq \varepsilon$. Of course, the function z has bounded variation.

Now let us assume that $1/n - \operatorname{var}_a^b x < \infty$ fot every $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there is $z_n \in BV[a, b]$ such that $\|x - z_n\| \leqq 1/n$. Since the functions z_n are regulated, it follows from 1.8 that $x \in \mathcal{R}_N[a, b]$.

3.5. Proposition. *For every function* $x \in \mathscr{R}[a, b]$ *and positive number* ε *there is a function* $z \in BV_N[a, b]$ *such that* $\|x - z\| \leqq \varepsilon$ *and* $\operatorname{var}_a^b z = \varepsilon\text{-}\operatorname{var}_a^b x$.

Proof. For every $k \in \mathbb{N}$ there is a function $z_k \in BV_N[a, b]$ such that $\|x - z_k\| \leqq \varepsilon$ and

$$\varepsilon\text{-}\operatorname{var}_a^b x \leqq \operatorname{var}_a^b z_k < \varepsilon\text{-}\operatorname{var}_a^b x + 1/k .$$

Hence $\varepsilon\text{-}\operatorname{var}_a^b x = \lim_{k \to \infty} \operatorname{var}_a^b z_k$.

Since the sequence $(z_k)_{k=1}^{\infty}$ is bounded and its members have uniformly bounded variations, by Helly's Choice Theorem there is a subsequence $(z_{k_j})_{j=1}^{\infty}$ and a function z such that

$$z_{k_j}(t) \to z(t) \quad \text{for any} \quad t \in [a, b], \quad \text{and} \quad \operatorname{var}_a^b z \leqq \liminf_{j \to \infty} \operatorname{var}_a^b z_{k_j} = \\ = \varepsilon\text{-}\operatorname{var}_a^b x .$$

On the other hand, since obviously $\|x - z\| \leqq \varepsilon$, it follows from Definition 3.2 that $\varepsilon\text{-}\operatorname{var}_a^b x \leqq \operatorname{var}_a^b z$. This completes the proof of the equality $\varepsilon\text{-}\operatorname{var}_a^b x = \operatorname{var}_a^b z$.

3.6. Proposition. *Assume that the members of a sequence* $(x_n)_{n=1}^{\infty} \subset \mathscr{R}_N[a, b]$ *have uniformly bounded* ε*-variations. If* $x_n(t) \to x_0(t)$ *for every* $t \in [a, b]$, *then the function* x_0 *is regulated and*

$$\text{(3.1)} \qquad \varepsilon\text{-}\operatorname{var}_a^b x_0 \leqq \liminf_{n \to \infty} \varepsilon\text{-}\operatorname{var}_a^b x_n \quad \textit{for every} \quad \varepsilon > 0 .$$

Proof. For every $\varepsilon > 0$ there is $K_\varepsilon > 0$ such that $\varepsilon\text{-}\operatorname{var}_a^b x_n \leqq K_\varepsilon$ holds for any $n \in \mathbb{N}$. Let $\varepsilon > 0$ be fixed. There is a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that

$$\liminf_{n \to \infty} \varepsilon\text{-}\operatorname{var}_a^b x_n = \lim_{k \to \infty} \varepsilon\text{-}\operatorname{var}_a^b x_{n_k} .$$

By Proposition 3.5 for any $k \in \mathbb{N}$ there is $z_k^\varepsilon \in BV_N[a, b]$ such that $\|x_{n_k} - z_k^\varepsilon\| \leqq \varepsilon$ and $\varepsilon\text{-}\operatorname{var}_a^b x_{n_k} = \operatorname{var}_a^b z_k^\varepsilon$. By Helly's Choice Theorem there is a subsequence $(z_{k_j}^\varepsilon)_{j=1}^{\infty}$ and a function z_0^ε such that $z_{k_j}^\varepsilon(t) \to z_0^\varepsilon(t)$ for every $t \in [a, b]$, and $\operatorname{var}_a^b z_0^\varepsilon \leqq \liminf_{j \to \infty} \operatorname{var}_a^b z_{k_j}^\varepsilon$. Let $t \in [a, b]$ and $\eta > 0$ be given. There is an integer j such that $|x_{n_{k_j}}(t) - x_0(t)| < \eta/2$ and $|z_{k_j}^\varepsilon(t) - z_0^\varepsilon(t)| < \eta/2$. Then $|x_0(t) - z_0^\varepsilon(t)| \leqq$

$$\leqq |x_0(t) - x_{n_{k_j}}(t)| + |x_{n_{k_j}}(t) - z_{k_j}^\varepsilon(t)| + |z_{k_j}^\varepsilon(t) - z_0^\varepsilon(t)| < \eta/2 + \varepsilon + \eta/2 = \varepsilon + \eta.$$

Since this estimate holds for any t and η, we conclude that $\|x_0 - z_0^\varepsilon\| \leqq \varepsilon$. Definition 3.2 yields $\varepsilon\text{-}\operatorname{var}_a^b x_0 \leqq \operatorname{var}_a^b z_0^\varepsilon$. Further $\operatorname{var}_a^b z_0^\varepsilon \leqq \liminf_{j \to \infty} \operatorname{var}_a^b z_{k_j}^\varepsilon = \liminf_{j \to \infty} \varepsilon\text{-}\operatorname{var}_a^b x_{n_{k_j}} = \lim_{k \to \infty} \varepsilon\text{-}\operatorname{var}_a^b x_{n_k} = \liminf_{n \to \infty} \varepsilon\text{-}\operatorname{var}_a^b x_n$. Hence (3.1) holds. Moreover, it is evident that $\liminf_{n \to \infty} \varepsilon\text{-}\operatorname{var}_a^b x_n \leqq K_\varepsilon$; then $\varepsilon\text{-}\operatorname{var}_a^b x_0$ is finite for every $\varepsilon > 0$. By Proposition 3.4 the function x_0 is regulated.

3.7. Proposition. *If a set* $\mathscr{A} \subset \mathscr{R}_N[a, b]$ *has uniformly bounded* ε*-variations, then there is* $\alpha > 0$ *such that* $|x(t_2) - x(t_1)| \leqq \alpha$ *for any* $x \in \mathscr{A}$, $a \leqq t_1 < t_2 \leqq b$.

Moreover, if the set $\{x(a);\ x \in \mathscr{A}\}$ *is bounded, then there is* $\beta > 0$ *such that* $\|x\| \leqq \beta$ *for any* $x \in \mathscr{A}$.

Proof. There is $K > 0$ such that $1\text{-var}_a^b\, x \leqq K$ for any $x \in \mathscr{A}$. For arbitrary $x \in \mathscr{A}$ there is $z \in BV_N[a, b]$ such that $\|x - z\| \leqq 1$ and $\text{var}_a^b\, z \leqq K$. If $a \leqq t_1 < t_2 \leqq b$ then

$$|x(t_2) - x(t_1)| \leqq |x(t_2) - z(t_2)| + |z(t_2) - z(t_1)| + |z(t_1) - x(t_1)| \leqq$$
$$\leqq 2\|x - z\| + \text{var}_{t_1}^{t_2}\, z \leqq 2 + K = \alpha\,.$$

If there is $\gamma > 0$ such that $|x(a)| \leqq \gamma$ for any $x \in \mathscr{A}$, then

$$|x(t)| \leqq |x(a)| + |x(t) - x(a)| \leqq \gamma + \alpha = \beta \quad \text{for every} \quad x \in \mathscr{A}\,,$$

$t \in [a, b]$. Consequently $\|x\| \leqq \beta$.

Using the notion of ε-variation, let us formulate the main theorem of this section, which is an analogue of Helly's Choice Theorem in the space of regulated functions.

3.8. Theorem. *Assume that the sequence* $(x_n)_{n=1}^{\infty} \subset \mathscr{R}_N[a, b]$ *has uniformly bounded* ε*-variations and that there is* $\gamma > 0$ *such that* $|x_n(a)| \leqq \gamma$ *for every* $n \in \mathbb{N}$. *Then there is a subsequence* $(x_{n_k})_{k=1}^{\infty}$ *and a function* $x_0 \in \mathscr{R}_N[a, b]$ *such that* $x_{n_k}(t) \to x_0(t)$ *for every* $t \in [a, b]$.

An outline of the proof is given in 3.1. However, this proof will not be presented in detail at this moment, because Theorem 3.8 will be proved later in another way.

In the following we will work on the interval $[0, 1]$, because the notion of linear prolongation will be used, which was defined for the interval $[0, 1]$. Of course, all results can be simply transferred to an arbitrary compact interval $[a, b]$.

3.9. Lemma. *Assume that an equicontinuous set* $\mathscr{B} \subset \mathscr{C}_N$ *is given. Then for any* $\varepsilon > 0$ *there is* $K_\varepsilon > 0$ *such that for every* $y \in \mathscr{B}$ *there is a function* $\zeta\colon [0, 1] \to \mathbb{R}^N$ *which is lipschitzian with the constant* K_ε *and such that* $\|y - \zeta\| < \varepsilon$.

Proof. For a given $\varepsilon > 0$ let us find $\delta > 0$ such that

$$\text{if} \quad |\tau'' - \tau'| < \delta \quad \text{then} \quad |y(\tau'') - y(\tau')| < \varepsilon/2$$

holds for every $y \in \mathscr{B}$.

Let $0 = \tau_1 < \tau_2 < \ldots < \tau_k = 1$ be a division such that

$$\delta/2 \leqq \tau_i - \tau_{i-1} < \delta \quad \text{for} \quad i = 1, 2, \ldots, k\,.$$

For any $y \in \beta$ let us define a function $\zeta\colon [0, 1] \to \mathbb{R}^N$ such that $\zeta(\tau_i) = y(\tau_i)$ for $i = 0, 1, \ldots, k$ and ζ is linear on each of the intervals $[\tau_{i-1}, \tau_i]$, $i = 1, 2, \ldots, k$; i.e.

$$\zeta(\tau) = y(\tau_{i-1}) + \frac{y(\tau_i) - y(\tau_{i-1})}{\tau_i - \tau_{i-1}} . (\tau - \tau_{i-1}) \quad \text{for} \quad \tau \in [\tau_{i-1}, \tau_i] .$$

For $i = 1, 2, \ldots, k$ we have

$$\left|\frac{y(\tau_i) - y(\tau_{i-1})}{\tau_i - \tau_{i-1}}\right| \leqq \frac{2}{\delta} . |y(\tau_i) - y(\tau_{i-1})| < \frac{2}{\delta} . \frac{\varepsilon}{2} = \frac{\varepsilon}{\delta} .$$

Hence ζ is lipschitzian with the constant $K_\varepsilon = \varepsilon/\delta$. If $\tau \in [\tau_{i-1}, \tau_i]$ then

$$|\zeta(\tau) - y(\tau)| = \left|y(\tau_{i-1}) + \frac{y(\tau_i) - y(\tau_{i-1})}{\tau_i - \tau_{i-1}} . (\tau - \tau_{i-1}) - y(\tau)\right| \leqq$$

$$\leqq |y(\tau_{i-1}) - y(\tau)| + |y(\tau_i) - y(\tau_{i-1})| < \varepsilon .$$

Consequently $\|\zeta - y\| < \varepsilon$.

3.10. Theorem. *For an arbitrary set of regulated functions $\mathscr{A} \subset \mathscr{R}_N$ the following conditions are equivalent:*

(i) *The set $\mathscr{A}$ has uniformly bounded ε-variations.*

(ii) *There is an increasing continuous function $\eta: [0, 1] \to [0, \infty)$, $\eta(0) = 0$ such that for every $x \in \mathscr{A}$ there is an increasing function $v_x \in V$ satisfying*

(3.2) $\quad |x(t'') - x(t')| \leqq \eta(v_x(t'') - v_x(t'))$ *for* $0 \leqq t' < t'' \leqq 1$;

(3.3) $\quad v_x(t'') - v_x(t') \geqq \frac{1}{2}(t'' - t')$ *for* $0 \leqq t' < t'' \leqq 1$;

(3.4) *if x is continuous at* 0 *or* 1, *then v_x is continuous at* 0 *or* 1, *respectively*;

and

(3.5) *if the set $\mathscr{A}$ has uniform one-sided limits at* 0 *and* 1, *then also the set $\{v_x, x \in \mathscr{A}\}$ has uniform one-sided limits at* 0 *and* 1.

(iii) *There is an equicontinuous set $\mathscr{B} \subset \mathscr{C}_N$ such that for any $x \in \mathscr{A}$ there are $y_x \in \mathscr{B}$ and $v_x \in V$ satisfying $x = y_x \circ v_x$ (this can be written as $\mathscr{A} \subset \mathscr{B} \circ V$).*

Proof. (i) ⇒ (ii) By Proposition 3.7 there is $\alpha > 0$ such that

(3.6) $\quad |x(t'') - x(t')| \leqq \alpha$ holds for any $x \in \mathscr{A}$, $0 \leqq t' < t'' \leqq 1$.

For any $j \in \mathbb{N}$ there is $K_j > 0$ such that $1/j\text{-var}_0^1 x \leqq K_j$ for every $x \in \mathscr{A}$.
Let $x \in \mathscr{A}$ be given. For any integer j there is $z_{x,j} \in BV_N$ such that

(3.7) $\quad \|x - z_{x,j}\| \leqq 1/j$ and $\text{var}_0^1 z_{x,j} \leqq K_j$.

Let us define

(3.8) $\quad \tau_{x,j} = \sup \{\tau \in (0, \frac{1}{4}]; |x(t) - x(0+)| \leqq 1/2j$ for every $t \in (0, \tau]\}$,

$$\sigma_{x,j} = \inf \{\sigma \in [\tfrac{3}{4}, 1);\ |x(1-) - x(t)| \leqq 1/2j \text{ for every } t \in [\sigma, 1)\}.$$

Evidently $\tau_{x,j} > 0$ and $\sigma_{x,j} < 1$.

Let us define

(3.9) $$\zeta_{x,j}(0) = x(0)\,;\quad \zeta_{x,j}(t) = x(0+) + \frac{x(\tau_{x,j}-) - x(0+)}{\tau_{x,j}}\,.\,t \text{ for } t \in (0, \tau_{x,j})\,;$$

$$\zeta_{x,j}(t) = z_{x,j}(t) \quad \text{for} \quad t \in [\tau_{x,j}, \sigma_{x,j}]\,;$$

$$\zeta_{x,j}(t) = x(1-) + \frac{x(1-) - x(\sigma_{x,j}+)}{1 - \sigma_{x,j}}\,(t - 1) \quad \text{for } t \in (\sigma_{x,j}, 1)\,,$$

$$\zeta_{x,j}(1) = x(1)\,.$$

For $t \in (0, \tau_{x,j})$ we have

$$|\zeta_{x,j}(t) - x(t)| \leqq |x(\tau_{x,j}-) - x(0+)| + |x(t) - x(0+)| \leqq 1/j\,.$$

Similarly

$$|\zeta_{x,j}(t) - x(t)| \leqq 1/j \quad \text{for any} \quad t \in (\sigma_{x,j}, 1)\,.$$

Hence

(3.10) $$\|\zeta_{x,j} - x\| \leqq 1/j\,.$$

By (3.6), (3.7) and (3.10) we have an estimate

$$\begin{aligned} \operatorname{var}_0^1 \zeta_{x,j} &= \operatorname{var}_0^{\tau_{x,j}} \zeta_{x,j} + \operatorname{var}_{\tau_{x,j}}^{\sigma_{x,j}} z_{x,j} + \operatorname{var}_{\sigma_{x,j}}^1 \zeta_{x,j} \leqq \\ &\leqq |x(0+) - x(0)| + |x(\tau_{x,j}-) - x(0+)| + |z_{x,j}(\tau_{x,j}) - x(\tau_{x,j}-)| + \\ &+ \operatorname{var}_0^1 z_{x,j} + |x(\sigma_{x,j}+) - z_{x,j}(\sigma_{x,j})| + |x(1-) - x|\sigma_{x,j}+)| + \\ &+ |x(1) - x(1-)| \leqq 6\alpha + 2\|z_{x,j} - x\| + \operatorname{var}_0^1 z_{x,j} \leqq 6\alpha + 2j + K_j\,. \end{aligned}$$

If we denote

(3.11) $$M_j = 6\alpha + 2/j + K_j\,,$$

then

(3.12) $$\operatorname{var}_0^1 \zeta_{x,j} \leqq M_j\,.$$

Using (3.10), (3.11) we find that $\zeta_{x,j}$ has similar properties as $z_{x,j}$ in (3.7), but moreover it has a special form near the endpoints of the interval $[0, 1]$.

Let us define

(3.13) $$v_{x,j}(t) = \operatorname{var}_0^t \zeta_{x,j} \quad \text{for} \quad t \in [0, 1]\,.$$

From (3.12) it follows that

(3.14) $$0 \leqq v_{x,j}(t) \leqq M_j \quad \text{holds for any} \quad t \in [0, 1]\,.$$

Let us define

(3.15) $$v_x(t) = a_x t + \sum_{j=1}^{\infty} 2^{-j-1} \,.\, (1/M_j)\, v_{x,j}(t) \quad \text{for} \quad t \in [0, 1]\,,$$

where the number $a_x \in [1/2, 1]$ is chosen so that $v_x(1) = 1$. We have the inequality

(3.16) $$v_{x,j}(t'') - v_{x,j}(t') \leqq 2^{j+1} M_j [v_x(t'') - v_x(t')] \quad \text{for} \quad t' < t''\,.$$

From (3.14) it follows that the series in (3.15) is uniformly absolutely convergent.

Since $a_x \geqq 1/2$, the property (3.3) is evident.

Assume that x is continuous from the right at 0. Since $\zeta_{x,j}$ is linear on $(0, \tau_{x,j})$ and $\zeta_{x,j}(0) = x(0)$, $\zeta_{x,j}(0+) = x(0+)$, it is evident that $\zeta_{x,j}$ are, as well as $v_{x,j}$, continuous at 0 for every $j \in \mathbb{N}$.

For a given $\varepsilon \in (0, 1)$ there is an integer j_0 such that $2^{-j_0-1} < \varepsilon/4$. For $j = 1, 2, \ldots, j_0$ denote

$$\delta_j = \varepsilon \,.\, \tau_{x,j}\,.$$

Further, denote

(3.17) $$\delta = \min\left\{\frac{\varepsilon}{4a_x}, \delta_1, \delta_2, \ldots, \delta_{j_0}\right\}.$$

By (3.11) we have $\alpha > M_j$. If $t \in (0, \delta)$, then

(3.18) $$v_{x,j}(t) = |x(\tau_{x,j}-) - x(0)| \,.\, \frac{t}{\tau_{x,j}} \leqq \alpha \,.\, \frac{\delta}{\tau_{x,j}} < M_j \,.\, \frac{\delta_j}{\tau_{x,j}} \leqq M_j \varepsilon\,.$$

By (3.14), (3.17) and (3.18) we get an estimate

$$|v_x(t) - v_x(0)| = v_x(t) \leqq$$

$$\leqq a_x t + \sum_{j=1}^{j_0} 2^{-j-1} \,.\, \frac{1}{M_j} \,.\, v_{x,j}(t) + \sum_{j=j_0+1}^{\infty} 2^{-j-1} \leqq$$

$$\leqq a_x \delta + \sum_{j=1}^{j_0} 2^{-j-1} \,.\, \frac{1}{M_j} \,.\, M_j \varepsilon + 2 < a_x \,.\, \frac{\varepsilon}{4a_x} + 2^{-1}\varepsilon + \varepsilon/4 = \varepsilon\,.$$

Consequently v_x is right-continuous at the point 0. Similarly it can be proved that if x is left-continuous at 1, then v_x is left-continuous at 1. Hence (3.4) holds.

For $r > 0$ let us define

(3.19) $$\varkappa(r) = \sup\{|x(t'') - x(t')|\,, \quad \text{where} \quad x \in \mathscr{A}\,,$$
$$0 \leqq t' < t'' \leqq 1,\ v_x(t'') - v_x(t') \leqq r\}\,.$$

Evidently the inequality

(3.20) $$|x(t'') - x(t')| \leqq \varkappa(v_x(t'') - v_x(t'))$$

holds for every $x \in \mathscr{A}$, $0 \leqq t' < t'' \leqq 1$.

It is obvious that the function $\varkappa$ is nondecreasing. Let us prove that $\varkappa(0+) = 0$.

On the contrary, assume that $\varkappa(0+) = \varkappa > 0$. Let us find $j \in \mathbb{N}$ such that $2/j < \varkappa/4$. Denote

$$(3.21) \qquad r = \varkappa/4 \,.\, 2^{-j-1} \,.\, \frac{1}{M_j} \,.$$

Since $\varkappa(r) \geqq \varkappa(0+) = \varkappa$, there are $x \in \mathscr{A}$ and $t' < t''$ such that

$$|x(t'') - x(t')| > \varkappa/2 \quad \text{and} \quad v_x(t'') - v_x(t') \leqq r \,.$$

By (3.10), (3.13), (3.16) and (3.21) we have

$$\tfrac{1}{2}\varkappa < |x(t'') - x(t')| \leqq 2\|x - \zeta_{x,j}\| + |\zeta_{x,j}(t'') - \zeta_{x,j}(t')| \leqq$$
$$\leqq 2/j + [v_{x,j}(t'') - v_{x,j}(t')] \leqq 2/j + 2^{j+1} \,.\, M_j(v_x(t'') - v_x(t')) <$$
$$< \varkappa/4 + 2^{j+1} \,.\, M_j \,.\, r = \varkappa/2 \,,$$

which is a contradiction with $\varkappa > 0$. By Proposition 1.22 there is a continuous increasing function $\eta\colon [0, 1] \to [0, \infty)$ such that $\eta(0) = 0$, $\varkappa(r) \leqq \eta(r)$ for any $r \in (0, 1]$. Now we can get (3.2) from (3.20).

In this part of the proof it remains to prove (3.5). Assume that the set $\mathscr{A}$ has uniform one-sided limits at the points 0,1. Let $\lambda \in (0, 1)$ be given. There is $j' \in \mathbb{N}$ such that

$$\frac{1}{2j'} < \lambda \leqq \frac{1}{2(j' - 1)} \,.$$

Then also $2^{-j'} < \lambda$. For any $j = 1, 2, \ldots, j' - 1$ there is $\Delta_j > 0$ such that

$$(3.22) \qquad |x(t) - x(0+)| < \lambda \quad \text{for any} \quad t \in (0, \Delta_j) \,, \quad x \in \mathscr{A} \,,$$
$$|x(1-) - x(t)| < \lambda \quad \text{for any} \quad t \in (1 - \Delta_j, 1) \,, \quad x \in \mathscr{A} \,.$$

Denote $\Delta_0 = \min\{1/4, \Delta_1, \Delta_2, \ldots, \Delta_{j'-1}\}$. Let $x \in \mathscr{A}$ and $j \in \{1, 2, \ldots, j' - 1\}$ be given. Since

$$\Delta_0 \leqq \Delta_j \,, \quad \Delta_0 \leqq \tfrac{1}{4} \quad \text{and} \quad \lambda \leqq \frac{1}{2(j' - 1)} \leqq \frac{1}{2j} \,,$$

(3.22) together with (3.8) imply that $\tau_{x,j} \geqq \Delta_0$ and $\sigma_{x,j} \leqq 1 - \Delta_0$. Denote $\Delta = \Delta_0 \,.\, \lambda$; then $\Delta \leqq \lambda/4$.

Let $x \in \mathscr{A}$ and $t \in (0, \Delta)$ be given. Since $t \in (0, \tau_{x,j})$ for any $j = 1, 2, \ldots, j' - 1$, by the definitions of $\zeta_{x,j}$ and $v_{x,j}$ we have an estimate

$$(3.23) \qquad |v_{x,j}(t) - v_{x,j}(0+)| = |x(\tau_{x,j}-) - x(0+)| \,.\, \frac{t}{\tau_{x,j}} \leqq \frac{1}{j} \,.\, \frac{\Delta}{\Delta_0} \leqq \lambda \,.$$

Since $M_j > 6\alpha$ by (3.11), we get by (3.14) and (3.23)

$$|v_x(t) - v_x(0+)| = a_x t + \sum_{j=1}^{\infty} 2^{-j-1} \cdot \frac{1}{M_j} [v_{x,j}(t) - v_{x,j}(0+)] \leqq$$

$$\leqq a_x \Delta + \sum_{j=1}^{j'-1} 2^{-j-1} \cdot \frac{1}{M_j} \cdot \lambda + \sum_{j=j'}^{\infty} 2^{-j-1} \cdot \frac{1}{M_j} \cdot v_{x,j}(t) \leqq$$

$$\leqq \Delta + \sum_{j=1}^{j'-1} 2^{-j-1} \cdot \frac{\lambda}{6\alpha} + \sum_{j=j'}^{\infty} 2^{-j-1} < \frac{\lambda}{4} + \frac{\lambda}{12\alpha} + 2^{-j'} < \lambda \cdot \left(\frac{5}{4} + \frac{1}{12\alpha}\right).$$

Consequently the set $\{v_x;\ x \in \mathscr{A}\}$ has uniform right-sided limits at 0. Similarly we can prove that it has uniform left-sided limits at 1; hence (3.5) holds.

(ii) ⇒ (iii) By Proposition 1.22 there is a continuous increasing concave function $\tilde{\eta}: [0, 1] \to [0, \infty)$ such that $\tilde{\eta}(0) = 0$ and $\eta(r) \leqq \tilde{\eta}(r)$, $r \in [0, 1]$. Then the inequality

$$|x(t_2) - x(t_1)| \leqq \tilde{\eta}(v_x(t_2) - v_x(t_1)), \quad 0 \leqq t_1 < t_2 \leqq 1$$

holds for every $x \in \mathscr{A}$.

For $x \in \mathscr{A}$ let us denote by y_x the linear prolongation of the function x along v_x. Denote $\mathscr{B} = \{y_x;\ x \in \mathscr{A}\}$. It follows from Proposition 2.12 that

$$|y_x(\tau_2) - y_x(\tau_1)| \leqq \tilde{\eta}(\tau_2 - \tau_1), \quad 0 \leqq \tau_1 < \tau_2 \leqq 1 .$$

This means that the set $\mathscr{B}$ is equicontinuous.Evidently $\mathscr{A} = \{y_x \circ v_x;\ x \in \mathscr{A}\} \subset \subset \mathscr{B} \circ V$.

(iii) ⇒ (i) For a given $\varepsilon > 0$ let us find the number K_ε by Lemma 3.9. For any $x \in \mathscr{A}$ there are $y \in \mathscr{B}$ and $v \in V$ such that $x = y \circ v$. By Lemma 3.9 there is $\zeta \in \mathscr{C}_N$ which is K_ε-lipschitzian and such that $\|\zeta - y\| < \varepsilon$. Denote $z = \zeta \circ v$. Then

$$\|z - x\| = \|\zeta \circ v - y \circ v\| \leqq \|\zeta - y\| < \varepsilon ,$$

and $\mathrm{var}_0^1\, z \leqq \mathrm{var}_0^1\, \zeta \leqq K_\varepsilon$. Consequently $\varepsilon\text{-}\mathrm{var}_0^1\, x \leqq K_\varepsilon$.

Using Theorem 3.10 and the well-known Arzelà-Ascoli Theorem, we obtain an important theorem which is an analogue of Theorem 2.18.

3.11. Theorem. *For an arbitrary set of regulated functions $\mathscr{A} \subset \mathscr{R}_N$ the following conditions are equivalent:*

(i) *The set $\mathscr{A}$ has uniformly bounded ε-variations and there is $\gamma > 0$ such that $|x(0)| \leqq \gamma$ holds for any $x \in \mathscr{A}$.*

(ii) *There is an increasing continuous function $\eta: [0, 1] \to [0, \infty)$, $\eta(0) = 0$ such that for every $x \in \mathscr{A}$ there is an increasing function $v_x \in V$ satisfying* (3.3), (3.4) *and*

$$|x(t'') - x(t')| \leqq \eta(v_x(t'') - v_x(t')) \quad for \quad 0 \leqq t' < t'' \leqq 1 ,$$

and

(3.24) *there is such $\beta > 0$ that $\|x\| \leqq \beta$ holds for any $x \in \mathscr{A}$.*

(iii) *There is a set* $\mathscr{B} \subset \mathscr{C}_N$ *which is compact in the sup-norm topology so that for every* $x \in \mathscr{A}$ *there are* $y_x \in \mathscr{B}$ *and* $v_x \in V$ *satisfying* $x = y_x \circ v_x$ *(i.e.* $\mathscr{A} \subset B \circ V$*).*

Proof. (i) ⇒ (ii) The property (3.24) follows from Proposition 3.7, the remaining part follows from Theorem 3.10.

(ii) ⇒ (iii) Let us denote by $\mathscr{B}_0$ the set of the linear prolongations y_x along v_x of all functions x from $\mathscr{A}$. By Theorem 3.10 the set $\mathscr{B}_0$ is equicontinuous. By Proposition 2.11 and (3.24) we have

$$\|y_x\| \leqq \beta \quad \text{for any} \quad y_x \in \mathscr{B}_0 .$$

Since $\mathscr{B}_0$ is equicontinuous and bounded, by the Arzelà-Ascoli Theorem the set $\mathscr{B}_0$ is relatively compact in the sup-norm topology on $\mathscr{C}_N$. If we denote by $\mathscr{B}$ the closure of $\mathscr{B}_0$, then $\mathscr{B}$ is compact and $\mathscr{A} \subset \mathscr{B} \circ V$.

(iii) ⇒ (i) follows immediately from Theorem 3.10.

At this moment we have an effective tool for proving a theorem formulated earlier.

3.8. Theorem. *Assume that the sequence* $(x_n)_{n=1}^{\infty} \subset \mathscr{R}_N[a, b]$ *has uniformly bounded ε-variations, and that there is* $\gamma > 0$ *such that* $|x_n(a)| \leqq \gamma$ *for every* $n \in \mathbb{N}$*. Then there is a subsequence* $(x_{n_k})_{k=1}^{\infty}$ *and a function* $x_0 \in \mathscr{R}_N[a, b]$ *such that* $x_{n_k}(t) \to x_0(t)$ *for every* $t \in [a, b]$*.*

Proof. Let us define

$$x_n'(t) = x_n(a + (b - a)\, t) \quad \text{for any} \quad t \in [0, 1]\,, \quad n \in \mathbb{N}\,.$$

Evidently the set $\{x_n';\ n \in \mathbb{N}\}$ has uniformly bounded ε-variations and $|x_n'(0)| \leqq \gamma$ for $n \in \mathbb{N}$. By Theorem 3.11 there is a compact set $\mathscr{B} \subset \mathscr{C}_N$ such that for every $n \in \mathbb{N}$ there are $y_n \in \mathscr{B}$ and $v_n \in V$ satisfying $x_n' = y_n \circ v_n$. Since $\mathscr{B}$ is compact, there is $y_0 \in \mathscr{C}_N$ and a uniformly convergent subsequence $(y_{n_k})_{k=1}^{\infty}$ such that $y_{n_k} \rightrightarrows y_0$. By Helly's Choice Theorem there is a nondecreasing function v_0 and a subsequence of $(v_{n_k})_{k=1}^{\infty}$ which will be denoted again by (v_{n_k}), such that $v_{n_k}(t) \to v_0(t)$ for any $t \in [0, 1]$.

If we define

$$x_0' = y_0 \circ v_0 \quad \text{and} \quad x_0(t) = x_0'\left(\frac{t - a}{b - a}\right) \quad \text{for} \quad t \in [a, b]\,,$$

then

$$x_{n_k}'(t) \to x_0'(t) \quad \text{for any} \quad t \in [0, 1]\,, \quad \text{and} \quad x_{n_k}(t) \to x_0(t) \quad \text{for any}$$

$t \in [a, b]$.

3.12. If we compare the results of the second and third sections, we can feel some relationship between the uniform convergence of regulated functions and the pointwise convergence of such regulated functions which have uniformly bounded ε-variations.

It would be an interesting result if an arbitrary sequence of pointwise convergent functions having uniformly bounded variations could be transformed to another sequence of regulated functions which is uniformly convergent, and if this transformation could be made by compositions with continuous increasing functions. More formally, if $x_n(t) \to x_0(t)$ for $t \in [0, 1]$ and the functions x_n, $n \in \mathbb{N}$ have uniformly bounded ε-variations, we would like to find continuous increasing functions $w_n \in \Lambda$, $n \in \mathbb{N}$ such that the functions $\xi_n = x_n \circ w_n^{-1}$ were uniformly convergent, or at least equiregulated. Such result would be useful in the theory of ordinary differential and integral equations.

Regrettably, this is not true; but a result like this takes place for some subsequence of (x_n). This result will be formulated now for the space $\mathscr{R}_N^-$.

3.13. Theorem. *Assume that a sequence $(x_n)_{n=0}^\infty \subset \mathscr{R}_N^-$ has uniformly bounded ε-variations and that it has uniform one-sided limits at the points* 0, 1. *Assume that*

$$x_n(t) \to x_0(t) \quad \text{for any} \quad t \in [0, 1] \quad \text{at which } x_0 \text{ is continuous}.$$

Then there is a subsequence $(x_n^k)_{k=1}^\infty$, a sequence of regulated functions $(\xi_k)_{k=0}^\infty \subset \subset \mathscr{R}_N^-$, a sequence of increasing continuous functions $(w_k)_{k=1}^\infty \subset \Lambda$ and an increasing function $w_0 \in V \cap \mathscr{R}_1^-$ such that

$$x_n^k = \xi_k \circ w_k \quad \text{for any} \quad k \in \mathbb{N}, \quad x_0 = \xi_0 \circ w_0 \quad \text{and} \tag{3.25}$$

$$\xi_k \rightrightarrows \xi_0, \quad w_k(t) \to w_0(t) \quad \text{for every} \quad t \in [0, 1] \quad \text{at which } w_0 \text{ is continuous.} \tag{3.26}$$

Proof. By Theorem 3.11 there is a compact set $\mathscr{B} \subset \mathscr{C}_N$ and for any $n \in \mathbb{N}$ there are $y_n \in \mathscr{B}$ and $v_n \in V$ such that $x_n = y_n \circ v_n$, and (3.3) (3.4), (3.5) hold.

For any $n \in \mathbb{N}$ let us denote $v_n'(0) = 0$, $v_n'(t) = v_n(t-)$ for $t \in (0, 1]$. Since $v_n(0+) = v_n(0) = 0$ and $v_n(1-) = v_n(1) = 1$ by (3.4), we have $v_n \in V \cap \mathscr{R}_1^-$. Since $x_n \in \mathscr{R}_N^-$ and y_n is continuous, we find that

$$x_n(t) = \lim_{\tau \to t-} x_n(\tau) = \lim_{\tau \to t-} y_n(v_n(\tau)) = y_n(v_n(t-)) = y_n(v_n'(t)) \quad \text{for} \quad t \in (0, 1].$$

Hence $x_n = y_n \circ v_n'$ where $v_n' \in V \cap \mathscr{R}_1^-$.

By Helly's Choice Theorem there is a subsequence $(v_{n_k}')_{k=1}^\infty$ and a function v_0' such that $v_{n_k}'(t) \to v_0'(t)$ for any $t \in [0, 1]$. From (3.3) it follows that v_0' is increasing.

By (3.5) the functions v_n, $n \in \mathbb{N}$ have uniform one-sided limits at 0 and 1. Hence for a given $\lambda > 0$ there is $\delta > 0$ such that $|v_n(t) - v_n(0+)| = v_n(t) < \lambda/2$ holds for any $t \in (0, \delta)$, $n \in \mathbb{N}$. Let $t \in (0, \delta)$ be given. There is an integer k such that $|v'_{n_k}(t) - v'_0(t)| < \lambda/2$. Let us find $\tau \in [t, \delta)$ such that v_{n_k} is continuous at τ. Then

$$|v'_0(t) - v'_0(0)| = v'_0(t) \leqq |v'_{n_k}(t) - v'_0(t)| + v_{n_k}(\tau) < \lambda .$$

Hence v'_0 is continuous at 0, and similarly v'_0 is also continuous at 1. If we define $w_0(0) = 0$, $w_0(t) = v'_0(t-)$ for $t \in (0, 1]$, then $w_0 \in V \cap \mathscr{R}_1^-$ and

$$v'_{n_k}(t) \to w_0(t) \quad \text{for any} \quad t \in [0, 1] \quad \text{at which } w_0 \text{ is continuous} . \tag{3.27}$$

If we replace f_n by v'_{n_k}, then the assumption (1.25) of Theorem 1.20 is satisfied.

By (3.27) the assumption (1.32) of Theorem 1.21 is satisfied when h_n, h_0, η are replaced by v'_{n_k}, w_0, id. As is shown in the proof of Theorem 1.21, the assumption (1.26) of Theorem 1.20 is satisfied. By Theorem 1.20 there is a sequence $(v_k)_{k=1}^{\infty} \subset \Lambda$ such that $\|(v'_{n_k})_{-1} - v_k^{-1}\| \to 0$ and the set $\{v'_{n_k} \circ v_k^{-1};\ k \in \mathbb{N}\}$ is relatively compact in the metric space $(\mathscr{R}_1^-; \varrho)$. Then

$$v_k(t) \to w_0(t) \quad \text{for every} \quad t \in [0, 1] \quad \text{at which } w_0 \text{ is continuous} .$$

Let us denote $q_k = v'_{n_k} \circ v_k^{-1}$, $k \in \mathbb{N}$.

There is a subsequence of (q_k) which for simplicity will be denoted again by (q_k), and a sequence $(\lambda_k)_{k=1}^{\infty} \subset \Lambda$ such that $\lambda_k \rightrightarrows \text{id}$ and $q_k \circ \lambda_k \rightrightarrows q_0 \in \mathscr{R}_1^-$.

Since the sequence (y_{n_k}) is contained in a compact set $\mathscr{B} \subset \mathscr{C}_N$, there is $y_0 \in \mathscr{C}_N$ and a subsequence which will be denoted again by y_{n_k}, such that $y_{n_k} \rightrightarrows y_0$.

Let us denote $\xi_k = y_{n_k} \circ q_k \circ \lambda_k$ for any $k \in \mathbb{N}$, $\xi_0 = y_0 \circ q_0$; $w_k = \lambda_k^{-1} \circ v_k$ for $k \in \mathbb{N}$. Then (3.25), (3.26) hold.

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Souhrn

REGULOVANÉ FUNKCE

DANA FRAŇKOVÁ

První kapitola sestává z pomocných výsledků o neklesajících reálných funkcích. Druhá kapitola přináší novou charakterizaci relativně kompaktních množin regulovaných funkcí v supremální topologii, třetí kapitola obsahuje mimo jiné analogii Hellyovy věty o výběru v prostoru regulovaných funkcí.

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