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# TOTAL EDGE-DOMATIC NUMBER OF A GRAPH

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*Summary.* The total edge-domatic number of a graph is introduced as an edge analogue of the total domatic number. Its values are studied for some special classes of graphs. The concept of totally edge-domatically full graph is introduced and investigated.

*Keywords:* total dominating edge set, total edge-domatic number, totally edge-domatically full graph.

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We consider finite undirected graphs without loops and multiple edges. The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi in [1], the total domatic number by the same authors together with R. M. Dawes in [2].

A subset $D$ of the vertex set $V(G)$ of a graph $G$ is called dominating (or total dominating), if for each vertex $x \in V(G) - D$ (or each $x \in V(G)$, respectively) there exists a vertex $y \in D$ adjacent to $x$. A partition of $V(G)$, all of whose classes are dominating (or total dominating) sets in $G$, is called a domatic (or total domatic, respectively) partition of $G$. The maximum number of classes of a domatic (or total domatic) partition of $G$ is called the domatic (or total domatic, respectively) number of $G$. The domatic number of a graph $G$ is denoted by $d(G)$, the total domatic number by $d_t(G)$.

Analogous concepts can be defined for edges instead of vertices. Two edges will be called adjacent, if they have one end vertex in common. The number of edges which are adjacent to an edge $e$ in a graph $G$ will be called the degree of $e$ in $G$. The graph in which the degree of each edge is equal to the same number $r$ will be called an edge-regular graph of degree $r$.

A subset $D$ of the edge set $E(G)$ of a graph $G$ is called a dominating (or total dominating) edge set, if for each edge $e \in E(G) - D$ (or each $e \in E(G)$, respectively) there exists an edge $f \in D$ adjacent to $e$. A partition of $E(G)$, all of whose classes are dominating (or total dominating) edge sets, is called an edge-domatic (or total edge-domatic, respectively) partition of $G$. The maximum number of classes of an edge-domatic (or total edge-domatic) partition of $G$ is called the edge-domatic (or total edge-domatic, respectively) number of $G$. The edge-domatic number of $G$ is denoted by $ed(G)$, the total edge-domatic number by $ed_t(G)$. The edge-domatic number of a graph was investigated in [3].

Note that the edge-domatic number is well-defined for any graph with at least one edge, while the total edge-domatic number only for graphs having no connected component with exactly one edge. If $G$ contains at least one edge, then the whole set $E(G)$ is a dominating edge set in $G$ and there exists at least one edge-domatic partition of $G$, namely $|E(G)|$. But if a graph $G$ contains a connected component with exactly one edge $e$, then no subset of $E(G)$ is total dominating, because there exists no edge adjacent to $e$.

The following inequality is analogous to an inequality for the domatic number of a graph proved by E. J. Cockayne and S. T. Hedetniemi.

**Proposition.** *Let $G$ be a finite graph, let $\delta_e(G)$ be the minimum degree of an edge in $G$. Then*

$$ed_t(G) \leqq \delta_e(G).$$

Proof. Consider a total edge-domatic partition $\mathscr{D}$ of $G$ with $ed_t(G)$ classes. Any edge of $G$ must be adjacent to edges from all classes of $D$, i.e. to at least $ed_t(G)$ edges, which implies the assertion. □

If the equality $ed_t(G) = \delta_e(G)$ holds, the graph $G$ is called totally edge-domatically full.

We shall now look for the values of $ed_t(G)$ for certain special graphs.

**Theorem 1.** *Let $C_n$ be the circuit of length $n$. If $n$ is divisible by 4, then $ed_t(C_n) = 2$; otherwise $ed_t(C_n) = 1$.*

Proof. Evidently the total edge-domatic number of a graph is equal to the total domatic number of its line graph. Every circuit is isomorphic to its own line graph and therefore its total edge-domatic number is equal to its total domatic number. The assertion follows from the corresponding assertion on the total domatic number in [2]. □

**Theorem 2.** *Let $K_{m,n}$ be the complete bipartite graph with the bipartition classes of cardinalities $m$, $n$. Then*

$$ed_t(K_{m,n}) = [n/2] \quad for \quad m = 1, \quad n \geqq 2,$$
$$ed_t(K_{m,n}) = \max(m, n) \quad for \quad m \geqq 2, \quad n \geqq 2.$$

Proof. Let $A$, $B$ be the bipartition classes of $K_{m,n}$, let $|A| = m$, $|B| = n$. If $m = 1$, $n \geqq 2$, then $K_{m,n}$ is a star with $n$ edges. Any subset of the edge set of a star which contains at least two edges is total dominating. Hence an edge-domatic partition of $K_{m,n}$ with the maximum number of classes consists of classes of cardinality 2, except at most one class of cardinality 3; the number of classes is $[n/2]$. Now suppose $m \geqq 2$, $n \geqq 2$ and let $D$ be a subset of $E(K_{m,n})$. If each vertex of $A$ is incident with an edge of $D$, then each edge of $K_{m,n}$, being incident with a vertex of $A$, is adjacent to an edge of $D$; the set $D$ is dominating. Analogously, if each vertex of $B$ is incident with an edge of $D$, then the set $D$ is dominating. If there exist $a \in A$ and $b \in B$ such that none of these vertices is an end vertex of an edge of $D$, then the edge joining a and b is adjacent to no edge of $D$ and $D$ is not dominating. Thus if $D$ is dominating edge set, we must have $|D| \geqq \min(m, n)$. As $K_{m,n}$ has $mn$ edges, we have $ed_t(G) \leqq ed(G) \leqq mn/\min(m, n) = \max(m, n)$. Without loss of generality suppose that $m \leqq n$. For each $x \in B$ let $D(x)$ be the set of all edges joining $x$ with vertices of $A$. Evidently the sets $D(x)$ for $x \in B$ form a total edge-domatic partition of $K_{m,n}$ with $|B| = n = \max(m, n)$ classes and $ed_t(K_{m,n}) = \max(m, n)$. □

The following theorem will concern totally edge-domatically full graphs.

**Theorem 3.** *Let G be a finite connected edge-regular graph of degree r. Then G is totally edge-domatically full if and only if it is obtained from a regular graph G′ of degree r having a decomposition into r linear factors by inserting one vertex onto each edge.*

Proof. If $r = 2$, then the assertion follows from Theorem 1. Thus suppose $r \geqq 3$. Suppose that $G$ is totally edge-domatically full. Then there exists a total edge-domatic partition $\mathscr{D} = \{D_1, \ldots, D_r\}$ of $G$. Each edge $e$ of $G$ is adjacent to exactly one edge from each class of $D$. Let the end vertices of $e$ be $u$ and $v$. Let $i$ be such a number that $e \in D_i$. The edge $e$ is adjacent to exactly one edge $f \in D_i$; without loss of generality let $u$ be the common end vertex of $e$ and $f$. Suppose that there exists an edge $e'$ incident with $u$ and different from $e$ and $f$. Then $e'$ is adjacent to at least two edges of $D_i$ and thus to at most $r - 2$ edges of other classes of $\mathscr{D}$; this implies that there exists a class of $\mathscr{D}$ such that $e'$ is adjacent to no edge from it, which is a contradiction. Hence $u$ has degree 2 in $G$. All edges which are adjacent to $e$ and different from $f$ are incident with $v$ and thus $v$ has degree $r$. As $e$ was chosen arbitrarily, we have proved that each edge of $G$ joins a vertex of degree 2 with a vertex of degree $r$. Let $V'$ be the set of all vertices of degree $r$ in $G$. Let $G'$ be the graph whose vertex set is $V'$ and in which two vertices $x, y$ are adjacent if and only if there exists a vertex $z$ of degree 2 in $G$ adjacent to both $x$ and $y$. The graph $G'$ is evidently regular of degree $r$. In $G$ any two edges adjacent to the same vertex of degree 2 belong to the same class of $\mathscr{D}$. Let $\mathscr{F} = \{F_1, \ldots, F_r\}$ be the partition of $E(G')$ such that an edge of $G'$ belongs to $F_i$ if and only if the two edges of the path of length 2 connecting its end vertices in $G$ belong to $D_i$ (for all $i \in \{1, \ldots, r\}$). As each vertex of $V'$ is incident in $G$ with exactly one edge from each class of $\mathscr{D}$, it is incident in $G'$ with exactly one edge from each class of $\mathscr{F}$; the edges of any class of $F$ form a linear factor of $G'$. The graph $G$ is obtained from $G'$ by inserting one vertex onto each edge.

Now suppose that $G$ has the described structure. As $G'$ has a decomposition into $r$ linear factors, there exists a partition $\mathscr{F} = \{F_1, \ldots, F_r\}$ of $E(G')$ such that each class of $\mathscr{F}$ is the edge set of one of these factors. Now we may define the partition $\mathscr{D} = \{D_1, \ldots, D_r\}$ of $G$ in such a way that if an edge of $G'$ belongs to $F_i$, then both edges of $G$ obtained from it by inserting one vertex belong to $D_i$ (for all $i \in \{1, \ldots, r\}$). The partition $D$ is evidently a total edge-domatic partition of $G$ and $G$ is totally edge-domatically full. □

A wheel $W_n$, where $n$ is an integer, $n \geqq 3$, is the graph obtained from a circuit $C_n$ of length $n$ by adding a new vertex and joining it by edges with all vertices of $C_n$.

**Theorem 4.** *The total edge-domatic number of every wheel is* 3.

Proof. Let the vertices of $C_n$ be $v_1, \ldots, v_n$ and let its edges be $v_i v_{i+1}$, where the sum $i + 1$ is taken modulo $n$, for all $i \in \{1, \ldots, n\}$. Let the added vertex of $W_n$ be $w$. We shall construct a total edge-domatic partition $\{D_1, D_2, D_3\}$ of $W_n$. If $n \equiv 0 \pmod 3$, then each $D_j$ for $j \in \{1, 2, 3\}$ consists of all edges $v_i v_{i+1}$ and $v_i w$ for $i \equiv j \pmod 3$. If $n \equiv 1 \pmod 3$, then each $D_j$ for $j \in \{1, 2, 3\}$ contains all edges $v_i v_{i+1}$ and $v_i w$ for $i \equiv j \pmod 3$ and $i \leqq n - 3$; further, $D_1$ contains $v_{n-1} w$, $D_2$ contains $v_{n-2} v_{n-1}$ and $v_{n-1} v_n$ and $D_3$ contains $v_{n-2} w$, $v_n w$ and $v_n v_1$. If $n \equiv 2 \pmod 3$, then $D_j$ for $j \in \{1, 2, 3\}$ contains $v_i v_{i+1}$ and $v_i w$ for $i \equiv j \pmod 3$ and $i \leqq n - 2$; further, $D_1$ contains $v_{n-1} w$, $D_2$ contains $v_n w$ and $D_3$ contains $v_{n-1} v_n$ and $v_n v_1$. The reader may verify himself that in all these cases $\{D_1, D_2, D_3\}$ is a total edge-domatic partition of $W_n$. We have $\delta_e(W_n) = 4$ and according to Theorem 3 the wheel $W_n$ is not totally edge-domatically full; hence $ed_t(W_n) = 3$. □

A graph $G$ for which $d(G) = \delta(G) + 1$, where $d(G)$ is the domatic number and $\delta(G)$ is the minimum degree of a vertex of $G$, is called domatically full.

**Theorem 5.** *Let $G$ be a regular graph of degree* 3 *and let $G$ be domatically full. Then $ed_t(G) = 3$.*

Proof. As $G$ is a regular graph of degree 3, it is edge-regular of degree 4 and $ed_t(G) \leqq 4$ according to Proposition. According to Theorem 3 the graph $G$ is not totally edge-domatically full and therefore $ed_t(G) \leqq 3$. As $G$ is domatically full, $d(G) = 4$. Let $\mathscr{D} = \{D_1, D_2, D_3, D_4\}$ be a domatic partition of $G$. Evidently each vertex of $G$ is adjacent to exactly one vertex of any class of $\mathscr{D}$ different from its own and to no vertex of its own class. For each $i \in \{1, 2, 3\}$ let $D'_i$ be the set of all edges of $G$ which join a vertex of $D_i$ with a vertex of $D_{i-1} \cup D_4$, where the subscript $i - 1$ is taken modulo 3. Let $e$ be an edge of $G$; then $e \in D'_i$ for some $i \in \in \{1, 2, 3\}$. If $e$ joins a vertex $v_i \in D_i$ with a vertex $v_{i-1} \in D_i$, then the vertex $v_i$ is its common end vertex with an edge of $D'_i$ (distinct from $e$) and with an edge of $D'_{i+1}$, and the vertex $v_{i-1}$ is its common end vertex with two edges of $D'_{i-1}$. If $e$ joins a vertex $v_i \in D_i$ with a vertex $v_4 \in D_4$, then the vertex $v_i$ is its common end vertex with an edge of $D'_i$ distinct from $e$ and with an edge of $D'_{i+1}$, and the vertex $v_4$ is its common end vertex with an edge of $D'_{i-1}$ and an edge of $D'_{i+1}$. Therefore $e$ is adjacent to edges from all classes $D'_1, D'_2, D'_3$ and these classes form a total edge-domatic partition of $G$ end $ed_t(G) = 3$.

Domatically full graphs which are regular of degree 3 are studied in [4].

In the end we shall prove an inequality for the total edge-domatic number of a complete graph.

**Theorem 6.** *Let $K_n$ be a complete graph with $n$ vertices. Then*

$$ed_t(K_n) \leqq 3n/4 .$$

Proof. Let $D$ be a total dominating edge set of minimum cardinality in $K_n$. There exists at most one vertex of $K_n$ which is not an end vertex of an edge of $D$; otherwise an edge joining two such vertices would not be adjacent to an edge of $D$. Further, each edge of $D$ must be adjacent to another edge of $D$. Therefore the subgraph of $K_n$ formed by the edges of $D$ and their end vertices contains all vertices of $K_n$ except at most one, and each of its connected components has at least two edges. If $n \equiv 1 \pmod 3$, then such a graph consists of $(n - 1)/3$ vertex-disjoint paths of length 2 and $|D| = 2(n - 1)/3$. In the case $n \equiv 2 \pmod 3$ we have $|D| = 2(n - 2)/3 + 1 = (2n - 1)/3 > 2(n - 1)/3$; in the case $n \equiv 0 \pmod 3$ we have $|D| = 2n/3 > 2(n - 1)/3$. In any case $|D| \geqq 2(n - 1)/3$. Thus $ed_t(K_n)$ is less than or equal to the number $n(n - 1)/2$ of edges of $K_n$ divided by $2(n - 1)/3$, i.e. $ed_t(K_n) \leqq 3n/4$.

## *References*


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## Souhrn

## TOTÁLNÍ HRANOVĚ DOMATICKÉ ČÍSLO GRAFU

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Podmnožina $D$ množiny hran $E(G)$ grafu $G$ se nazývá totální dominantní, jestliže ke každé hraně $e \in E(G)$ existuje hrana $f \in D$ mající společný koncový uzel s hranou $e$. Totální hranově domatické číslo $ed_t(G)$ grafu $G$ je maximální počet tříd rozkladu množiny $E(G)$ na totální dominantní množiny. Zkoumají se základní vlastnosti totálního hranově domatického čísla grafu.



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