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EDGE-DOMATIC NUMBERS OF CACTI

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Summary. The edge-domatic number of a graph is the maximum number of classes of a partition of its edge set into dominating sets. This number is studied for cacti, i.e. graphs in which each edge belongs to at most one circuit.

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In [1] E. J. Cockayne and S. T. Hedetniemi have introduced the domatic number of a graph. One of its variants is the edge-domatic number of a graph; introduced in [2].

We shall consider finite undirected graphs without loops and multiple edges. Two distinct edges are called adjacent, if they have a common end vertex.

A subset D of the edge set $E(G)$ of a graph G is called dominating, if for each $e \in E(G) - D$ there exists an edge $f \in D$ adjacent to e. An edge-domatic partition of G is a partition of $E(G)$, all of whose classes are dominating edge sets of G. The maximum number of classes of an edge-domatic partition of G is called the edge-domatic number of G and denoted by $ed(G)$.

It is sometimes convenient to consider edge-domatic colourings instead of edge-domatic partitions. A colouring $\mathscr{C}$ of edges of G is called edge-domatic, if each edge of G is adjacent to edges of all colours of $\mathscr{C}$ different from its own. The maximum number of colours of an edge-domatic colouring of G is the edge-domatic number of G. This definition is evidently equivalent to the previous one.

In this paper we shall investigate cacti. A cactus is a connected graph which has the property that each of its edges is contained in at most one circuit.

Thus each block of a cactus is either a circuit, or a complete graph K_2 with two vertices. If a cactus contains only one block, it will be called trivial; otherwise it will be called non-trivial. A cactus in which all blocks are circuits will be called round.

The edge-domatic number of G is evidently equal to the domatic number [1] of the line graph of G. Therefore it easily follows from the results in [1] that $ed(G) \geqq 2$ for each graph G, none of whose connected components is K_2, and $ed(G) \leqq \delta_e(G) + 1$, where $\delta_e(G)$ is the minimum degree of an edge of G. (The degree of an edge is the number of edges adjacent to it.) As any circuit is isomorphic to its line graph, the edge-domatic number of a circuit is equal to its domatic number. Thus we have the following propositions.

Proposition 1. *The edge-domatic number of* K_2 *is* 1.

Proposition 2. *The edge-domatic number of a circuit is* 3 *if and only if its length is divisible by* 3; *otherwise it is* 2.

Thus, in the sequel we shall study only non-trivial cacti. We shall prove a theorem concerning round cacti. Before formulating it, we prove some lemmas.

Lemma 1. *Let G be a round cactus. Then* $ed(G) \leqq 3$.

Proof. For trivial cacti this follows from Proposition 1 and Proposition 2. Let G be a non-trivial cactus. Let C be a terminal block of G, i.e. a block containing only one articulation of G. (Such a block must always exist.) The block C is a circuit and thus it contains two adjacent vertices u, v which are not articulations of G. The vertices u, v have degree 2 and thus also the degree of the edge uv is 2. Thus $\delta_e(G) \leqq 2$ and, according to the above mentioned inequality, $ed(G) \leqq \delta_e(G) + 1 \leqq 3$. □

Now we shall define a certain property of a graph.

A graph G is said to have the property **P**, if $ed(G) = 3$ and there exists an edge-domatic colouring of G with colours such that each vertex of G is incident with edges of at least two colours.

Lemma 2. *Let G be a non-trivial round cactus, let C be its terminal block. Let* G_0 *be the union of all blocks of G except C. Let* $ed(G_0) = 3$ *and let* G_0 *have the property* **P**. *Then* $ed(G) = 3$ *and G has the property* **P**.

Proof. Let $\mathscr{C}_0$ be the colouring of G_0 satisfyi ig the condition of the property **P**. Let a be the articulation of G contained in C. By the assumption the vertex a is incident in G_0 with edges of at least two colours of $\mathscr{C}_0$; without loss of generality we may assume that these colours are 2 and 3. Let c be the length of C and let the vertices of C be $u_1, \ldots, u_c$ and its edges u_iu_{i+1} for $i = 1, \ldots, c - 1$ and u_cu_1. Let $a = u_1$. We shall colour the edges of C in such a way that each edge u_iu_{i+1} for $i = 1, \ldots, c - 1$ obtains the colour congruent with i modulo 3 and the edge u_cu_i obtains the colour congruent with c modulo 3. This colouring together with $\mathscr{C}_0$ gives a colouring $\mathscr{C}$ of G with the property that each vertex of G is incident with edges of at least two colours of $\mathscr{C}$. It remains to prove that $\mathscr{C}$ is edge-domatic. As $\mathscr{C}_0$ is an edge-domatic colouring of G_0, any edge of G_0 is adjacent to edges of all colours different from its own. The edge u_1u_2 has this property, too, because its colour is 1 and it is adjacent to edges of G_0 of the colours 2 and 3 which are incident to $a = u_1$. The edge u_cu_1 is adjacent also to these two edges of G_0 and moreover to u_1u_2 of the colour 1. If $2 \leqq i \leqq c - 2$, then the edge u_iu_{i+1} has the colour congruent with i

modulo 3 and is adjacent to the edge $u_{i-1}u_i$ of the colour congruent with $i - 1$ and to the edge $u_{i+1}u_{i+2}$ of the colour congruent with $i + 1$ modulo 3. This proves the assertion. □

Lemma 3. *Let G be a cactus consisting of two circuits C_1, C_2 of lengths c_1, c_2, respectively, let $c_1 \not\equiv 1 \pmod 3$. Then $ed(G) = 3$ and G has the property $\boldsymbol{P}$.*

Proof. Denote the vertices of C_1 by $u_1, \ldots, u_{c_1}$ and the vertices of C_2 by $v_1, \ldots, v_{c_2}$ in an analogous way as in the proof of Lemma 2. Let the articulation of G be $a = u_1 = v_1$. We colour the edges of C_1 in such a way that $u_i u_{i+1}$ is coloured with the colour congruent with i modulo 3 for each $i = 1, \ldots, c_1 - 1$ and $u_{c_1}u_1$ with the colour congruent with c_1 modulo 3. As $c_1 \not\equiv 1 \pmod 3$, the edges incident with a have different colours. Now let φ be a cyclic permutation of $\{1, 2, 3\}$ such that $\varphi(1)$ is the colour different from the colours of the edges of C_1 incident with a. We colour the edges of C_2 in such a way that $v_i v_{i+1}$ for $i = 1, \ldots, c_2 - 1$ is coloured with the colour $\varphi(j)$, where $j \in \{1, 2, 3\}, j \equiv i \pmod 3$, and $v_{c_2}v_1$ with the colour $\varphi(j)$, where $j \in \{1, 2, 3\}$, $j \equiv c_2 \pmod 3$. Analogously as in the proof of Lemma 2 we prove that this colouring is edge-domatic and satisfies the condition of the property $\boldsymbol{P}$. □

Lemma 4. *Let G be a cactus consisting of two circuits of lengths congruent with 1 modulo 3. Then $ed(G) = 2$.*

Proof. Suppose $ed(G) = 3$. Denote the circuits and their vertices in the same way as in the proof of Lemma 3. Without loss of generality let u_1u_2 be coloured with 1. Then u_2u_3, having the degree 2, must have a colour other than 1; without loss of generality let it be 2. Then the colouring of all edges u_iu_{i+1} for $i = 1, \ldots, c_1 - 1$ is uniquely determined; each edge u_iu_{i+1} must have the colour congruent with i modulo 3. The edge $u_{c_1}u_1$ must have the colour 1. Thus both the edges of C_1 incident with a have the colour 1. Analogously the edges of C_2 must be coloured in such a way that both edges incident with a have the same colour. If this colour is 2 (or 3), then u_1u_2 (or $u_1u_{c_1}$) is not adjacent to an edge of the colour 3 (or 2, respectively). If this colour is 1, then u_1u_2 is not adjacent to an edge of the colour 3, either, and $u_1u_{c_1}$ is not adjacent to an edge of the colour 2. This is a contradiction and therefore $ed(G) = 2$. □

Lemma 5. *Let G be a round cactus with three blocks. Then $ed(G) = 3$ and G has the property $\boldsymbol{P}$.*

Proof. Let C_1, C_2, C_3 be the blocks of G; they are circuits. If some of them has the length not congruent to 1 modulo 3, then the assertion follows from Lemma 3 and Lemma 2. Thus suppose that the lengths c_1, c_2, c_3 of C_1, C_2, C_3 are all congruent with 1 modulo 3. The graph G can have either one or two articulations. Consider the first case. Let φ_1 be the identity permutation of $\{1, 2, 3\}$, let φ_2, φ_3 be the cyclic permutations of $\{1, 2, 3\}$ such that $\varphi_2(1) = 2$, $\varphi_3(1) = 3$. Let the vertices of C_j for $j = 1, 2, 3$ be $u_1^{(j)}, \ldots, u_{c_j}^{(j)}$, and let the edges be $u_i^{(j)}u_{j+1}^{(j)}$ for $i = 1, \ldots, c_j - 1$ and $u_{c_j}^{(j)}u_1^{(j)}$. Let the articulation of G be $a = u_1^{(1)} = u_1^{(2)} = u_1^{(3)}$. We colour any edge $u_i^{(j)}u_{i+1}^{(j)}$ with the colour congruent with $\varphi_j(i)$ modulo 3 and any edge $u_{c_j}^{(j)}u_1^{(j)}$ by j. The reader may verify that G has the property $\boldsymbol{P}$. Now consider the second case. The vertices and edges of C_1 and C_3 will be the same as in the preceding case. The articulations will be $a_1 = u_1^{(1)}$ and $a_2 = u_1^{(3)}$. Both a_1, a_2 will be contained in C_2. Then C_2 is the union of two edge-disjoint paths P_1, P_2 connecting a_1 with a_2. Let p_1, p_2 be their lengths. We have $p_1 + p_2 \equiv c_2 \equiv 1 \pmod 3$; therefore (without loss of generality) either $p_1 \equiv 1 \pmod 3$ and $p_2 \equiv 0 \pmod 3$, or $p_1 \equiv p_2 \equiv 2 \pmod 3$. Let the vertices of P_1 (or P_2) be $v_0, \ldots, v_{p_1}$ (or $w_0, \ldots, w_{p_2}$) and let the edges be v_iv_{i+1} (or w_iw_{i+1}) for $i = 0, \ldots, p_1 - 1$ (or $i = 0, \ldots, p_2 - 1$, respectively). The notation will be chosen so that $v_1 = w_1 = a_1$, $v_{p_1} = w_{p_2} = a_2$. If $p_1 \equiv 1 \pmod 3$ and $p_2 \equiv 0 \pmod 3$, we colour each edge v_iv_{i+1} with the colour congruent with $\varphi_3(i)$ modulo 3 and each edge w_iw_{i+1} with the colour congruent with i modulo 3. Then the edges of C_2 incident with a_1 have the colours 2 and 3 and the edges of C_2 incident with a_2 have the colours 1 and 2. Now we colour the edges of C_1 and C_3 in the same way as in the preceding case. The graph G has the property $\boldsymbol{P}$, as the reader may verify himself. If $p_1 \equiv p_2 \equiv 2 \pmod 3$, then we colour the edges of C_2 in the same way. The edges of C_2 incident with a_1 have again the colours 2 and 3, and the edges of C_2 incident with a_2 have the colours 1 and 3. The edges of C_1 will be coloured as in the preceding case and the edges of C_3 in such a way as the edges of C_2 in the case of the articulation. Again G has the property $\boldsymbol{P}$. □

Now we can prove a theorem.

Theorem. *Let G be a non-trivial round cactus. Then $ed(G) = 2$ if and only if G consists of two circuits of lengths congruent with* 1 *modulo* 3; *otherwise $ed(G) = 3$.*

Proof. According to Lemma 1 we have $ed(G) \leqq 3$. If G consists of two circuits of lengths congruent with 1 modulo 3, then $ed(G) = 2$ according to Lemma 4. Otherwise G contains a subcactus G_0 consisting either of two circuits, at least one of which has a length non-congruent with 1 modulo 3, or of three circuits. Then from Lemma 3 or Lemma 5 by using iteratively Lemma 2 we obtain the assertion. □

For cacti which are not round the theorem does not hold. For trees (which are a particular case of cacti) it was proved in [2] that $ed(G) = \delta_e(G) + 1$.

References

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Souhrn

HRANOVĚ DOMATICKÁ ČÍSLA KAKTUSŮ

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Hranově domatické číslo grafu je maximální počet tříd rozkladu množiny jeho hran na dominantní množiny. Toto číslo je v článku studováno pro kaktusy, tj. grafy, v nichž každá hrana patří do nejvýše jednoho cyklu.

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