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ON A PROBLEM OF COLOURING THE REAL PLANE

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Summary. What is the least number of colours which can be used to colour all points of the real Euclidean plane so that no two points which are unit distance apart have the same colour? This well known problem, open more than 25 years is studied in the paper. Some partial results and open subproblems are presented.

Keywords: Vertex colouring, infinite graph, decomposition of the real plane.

AMS Classification: 05C15.

In this paper we will concentrate on the following question:

Problem 1. *What is the least number $\chi(E_2)$ of colours which can be used to colour all points of the real Euclidean plane E_2 so that no two points which are unit distance apart have the same colour?*

The problem in this sense was published first probably by Hadwiger in 1961 in [6] where he ascribed the authorship to E. Nelson. In any case this problem became well known and popular thanks to Erdös (e.g. [2], [3], [4]). To solve the whole problem seems not to be easy but it was not too difficult to establish the bounds $4 \leqq \chi(E_2) \leqq 7$. The validity of the lower bound was shown by L. Moser and W. Moser [9]

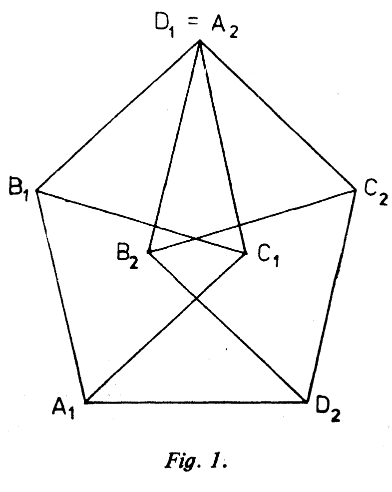

Fig. 1.

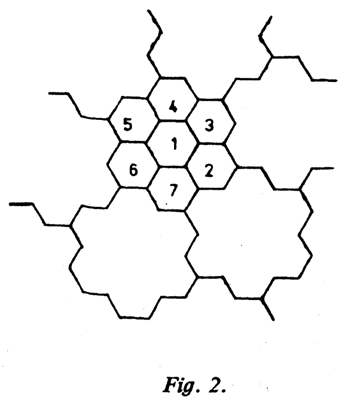

Fig. 2.

who constructed a graph on seven vertices with chromatic number 4 which can be drawn in the plane so that each edge is a straight line segment of unit length (see Fig. 1). The sufficiency of 7 colours was proved by Hadwiger [6] who showed such a decomposition of the real Euclidean plane into 7 sets such that none of them contains two different points unit distance apart. This decomposition (see Fig. 2) consists of monochromatic regular hexagons with the length of the side equal to e.g. 2/5. Since then only some modifications and generalizations of Problem 1 were studied (see e.g. [1], [5], [8], [10]), but no progress has been achieved for more than 25 years in the basic problem.

In our paper we shall use another, a little modified formulation of Problem 1.

Problem 1′ *What is the chromatic number $\chi(G_E)$ of the infinite graph G_E whose vertex set is the set of all points of the real plane and the edges of which exist between all points which are unit distance apart?*

The aim of this paper is to present some partial results on Problem 1′. First we will study the chromatic number of various induced subgraphs of the basic graph G_E.

For the sake of completeness we introduce one trivial assertion from [1].

Theorem 1. *The chromatic number of the real line is* 2.

Now we shall continue with our results.

Theorem 2. *Let P_d be an infinite strip of the real plane with the width $d \in (0, \frac{1}{2}\sqrt{3}\rangle$ Then the chromatic number of the induced subgraph $\langle P_d \rangle$ of the basic graph G_E is $\chi(\langle P_d \rangle) = 3$.*

Proof. The sufficiency of three colours follows from the colouring in Figure 3 because for all $d \leqq \frac{1}{2}\sqrt{3}$ we have $\sqrt{((1/2)^2 + d^2)} \leqq 1$. The necessity of three colours follows from the fact that for $d > 0$ the strip $\langle P_d \rangle$ contains a cycle of odd length

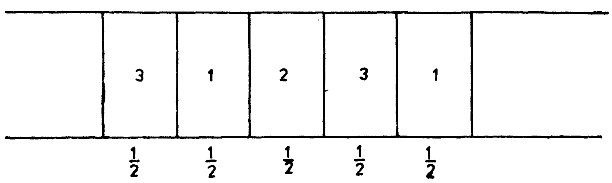

Fig. 3.

(even an infinite number of odd cycles). One example of such a cycle can be constructed in the following way: For any $d > 0$ there must exist a positive real number $c < d$ such that $1 - \sqrt{(1 - c^2)} = 1/2n$ where n is a positive integer. Now let us choose two sequences of points with the following coordinates:

$$X_0 = (0, 0), \quad X_1 = \left(1 - \frac{1}{2n}, c,\right) \quad X_2 = \left(2 - \frac{2}{2n}, 0\right), \ldots,$$

$$\ldots, X_i = \left(i - \frac{i}{2n}, \frac{c}{2} - (-1)^i \frac{c}{2}\right), \ldots, \quad X_{2n} = \left(2n - \frac{2n}{2n}, 0\right),$$

$$Y_0 = X_0 = (0, 0), \quad Y_1 = (1, 0), \quad Y_2 = (2, 0), \ldots, \quad Y_i = (i, 0), \ldots$$

$$\ldots, \quad Y_{2n-1} = X_{2n} = (2n - 1, 0).$$

Then the sequence of points $X_0, X_1, \ldots, X_{2n} = Y_{2n-1}, Y_{2n-2}, \ldots, Y_0 = X_0$ forms a cycle of odd length which can be drawn in P_d.

Theorem 3. *Let P_d be an infinite strip of the real plane with the width $d \in \in (\frac{1}{2}\sqrt{3}, \frac{2}{3}\sqrt{2}\rangle$. Then $\chi(\langle P_d \rangle) = 4$.*

Proof. The sufficiency of 4 colours follows from the example of colouring in Figure 4, because for $d \leqq \frac{2}{3}\sqrt{2}$ we have $\sqrt{((1/3)^2 + d^2)} \leqq 1$.

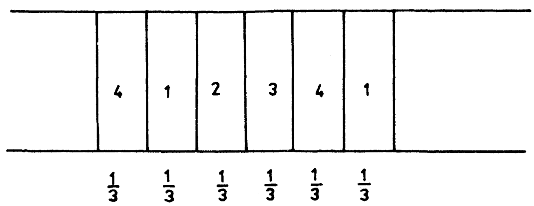

Fig. 4.

Now we prove the necessity of 4 colours. Let us consider a graph $S = (V(S), E(S))$ with $V(S) = \{A, B, C\ D\}$ and $E(S) = \{(A, B), (A, C), (B, C), (B, D), (C, D)\}$. Now for a given $n > 1$ we can construct a graph G_n in the following way: we take n copies $S_1, S_2, \ldots, S_n$ of the graph S and glue together the vertex D_i with the vertex A_{i+1} for all $i = 1, \ldots, n - 1$, and then we join the vertices A_1 and D_n by an edge. An example of the graph G_2 is in Figure 1. It is trivial that this graph G_n on $3n + 1$ vertices can be drawn in the plane with all edges of length 1, because each subgraphS_i can be drawn in the plane in such a way. By a similar method of distributing the points A_i as in the proof of Theorem 2 it can be also easily proved that for a given $d > \frac{1}{2}\sqrt{3}$ there exists a positive integer N_d such that for all $n > N_d$ the graph G_n can be drawn in P_d with all edges of length 1. As $\chi(G_n) = 4$ for all $n > 1$, we have $\chi(\langle P_d \rangle) = 4$.

Theorem 4. *Let P_d be an infinite strip of the real plane with the width $d \in \in (\frac{2}{3}\sqrt{2}, \frac{2}{3}\sqrt{5}\rangle$. Then $4 \leqq \chi(\langle P_d \rangle) \leqq 5$.*

Proof. The necessity of 4 colours follows from Theorem 3 and the sufficiency of 5 colours follows from the example of colouring in Figure 5, because for $d \leqq \frac{2}{3}\sqrt{5}$ we have $\sqrt{((2/3)^2 + (d/2)^2)} \leqq 1$.

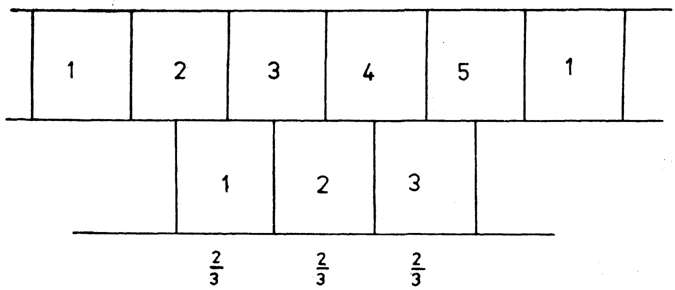

Fig. 5.

Theorem 5. *Let P_d be an infinite strip of the real plane with the width $d \in \in (\frac{2}{3}\sqrt{5}, \sqrt{3}\rangle$. Then $4 \leqq \chi(\langle P_d \rangle) \leqq 6$.*

Proof. The sufficiency of 6 colours follows from the example of colouring in in Figure 6, because for $d \leqq \sqrt{3}$ we have $\sqrt{((1/2)^2 + (d/2)^2)} \leqq 1$.

To prove our next result we need the following lemma.

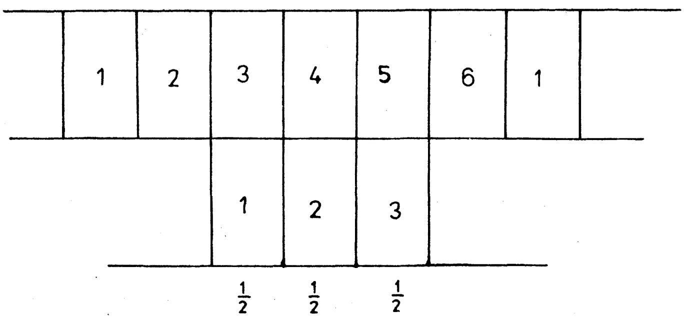

Fig. 6.

Lemma 1. *Let n_1, n_2, n_3 be positive integers and let the greatest common divisor of n_1, n_2, n_3 be 1. Further, let $n_1^2 + n_2^2 = n_3^2$. Then n_3 must be odd and exactly one of the numbers n_1, n_2 must be even.*

Proof. By contradiction. Let n_1, n_2 be both odd, i.e. let $n_1 = 2m_1 + 1$, $n_2 = = 2m_2 + 1$. Then we have $n_1^2 + n_2^2 = 4(m_1^2 + m_2^2 + m_1 + m_2) + 2$ and this is a contradiction because no square of an integer can be divisible by 2 and not divisible by 4.

Using Lemma 1 we can prove a startling result.

Theorem 6. *Let us have an infinite graph $G_R = (V(G_R), E(G_R))$, where $V(G_R)$ is the set of such points in E_2 whose both coordinates are rational numbers and the edges in G_R exist between all points which are unit distance apart. Then the graph G_R contains no cycle of odd length.*

Proof. By contradiction. Let us have an odd cycle $v_1, v_2, \ldots, v_N$, $v_{N+1} = v_1$, $v_i \in V(G_R)$ for all i and N an odd number. Let $({}_xv_i, {}_yv_i)$ be the coordinates of v_i for $i = 1, \ldots, N$. As all these coordinates are rational numbers, their differences must be rational, too. For $i = 1, \ldots, N$ let us denote by ${}_xp_i$, ${}_yp_i$, q_i the integers such that ${}_xp_i/q_i = {}_xv_{i+1} - {}_xv_i$, ${}_yp_i/q_i = {}_yv_{i+1} - {}_yv_i$ and the greatest common divisor of ${}_xp_i$, ${}_yp_i$ and q_i is 1. As the distance between the neighbouring points of the cycle is 1, we necessarily have $({}_xp_i/q_i)^2 + ({}_yp_i/q_i)^2 = 1$. Lemma 1 now implies that q_i is odd for all i and exactly one of the numbers ${}_xp_i$, ${}_yp_i$ is odd. Let ${}_xN$ be the number of odd numbers ${}_xp_i$ and let ${}_yN$ be the number of odd numbers ${}_yp_i$. As the cycle is closed, the equalities

$$\sum_{i=1}^{N} \frac{{}_xp_i}{q_i} = 0 \quad \text{and} \quad \sum_{i=1}^{N} \frac{{}_yp_i}{q_i} = 0$$

must hold. This implies that ${}_xN$ and ${}_yN$ must be even. As $N = {}_xN + {}_yN$ and N is odd, we have obtained a contradiction.

Theorem 6 implies directly the following corollary.

Corollary 1. *Let us have an infinite graph $G_R = (V(G_R), E(G_R))$, where $V(G_R) \subset E_2$ is the set of such points whose both coordinates are rational numbers and the edges in G_R exist between all points which are unit distance apart. Then $\chi(G_R) = 2$, or in other words, the chromatic number of the rational subset of the Euclidean plane is two.*

Another strategy to attack Problem 1 is to suppose that there exists a colouring of E_2 by 4 colours and then to try to get some characteristics of such a colouring. Eventually we should like to get a contradiction and so to improve the lower bound $4 \leqq \chi(E_2)$. The first step of such an attempt is Theorem 7 but before proving it we have to introduce Lemma 2.

Lemma 2. *In E_2 let us have two circles k_1, k_2 of the same radius 1, with centres S_1, S_2, the distance between centres being $d(S_1, S_2) = d$, $0 < d \leqq 1$. Let P be an intersection point of k_1 and k_2 and let $P_1 \in k_1$, $P_2 \in k_2$ be such points that the length of the circular arcs PP_1 and PP_2 is $\frac{2}{3}\pi$ and $d(P_1, P_2) \leqq 1$, $d(S_2, P_1) \leqq 1$. Let M be the set of points of these circular arcs PP_1 and PP_2 including the points P_1 and P_2 but without the point P. Let $\langle M \rangle$ be the subgraph of our basic graph G_E induced on the vertex set M. Then $\chi(\langle M \rangle) = 3$.*

Proof. Let us denote by R_1 the centre of the arc PP_1 and by R_2 the centre of the arc PP_2. Then $d(P, R_1) = d(R_1, P_1) = d(P, R_2) = d(R_2, P_2) = 1$.

1. First we prove that the graph $\langle M \rangle$ can be regularly coloured by 3 colours. We colour the arcs PR_1, PR_2 including the points R_1, R_2 by colour 1, the arc R_1P_1 including P_1 by colour 2 and the arc R_2P_2 including P_2 by colour 3. This is evidently a regular colouring, thus we have $\chi(\langle M \rangle) \leqq 3$.

2. Now we prove that $\langle M \rangle$ contains an odd cycle.

Case A. If $d \in \langle \sqrt{(3)} - 1, 1 \rangle$ then there exists an equilateral triangle with the side 1 such that one vertex of the triangle is on the first arc and two its vertices are on the other. This implies $\chi(\langle M \rangle) = 3$ in this case.

Case B. Let $d \in (0, \sqrt{(3)} - 1)$. Let us choose an arbitrary point Z_1 of the arc PR_1, i.e. $d(Z_1, P) \in (0, 1)$ and such that $d(Z_1, P_2) \geqq 1$. Now for any such point Z_1 we can form a sequence of points $Z_1, Z_2, Z_3, \ldots$ by a construction in four repeating steps (see Fig. 7):

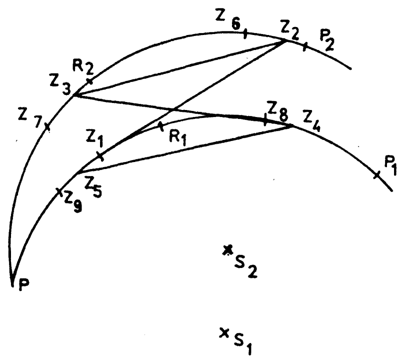

Fig. 7.

Step 1. We denote by Z_2 such point of the arc R_2P_2 that $d(Z_1, Z_2) = 1$.
Step 2. We denote by Z_3 such point of the arc PR_2 that $d(Z_2, Z_3) = 1$.
Step 3. We denote by Z_4 such point of the arc R_1P_1 that $d(Z_3, Z_4) = 1$.
Step 4. We denote by Z_5 such point of the arc PR_1 that $d(Z_4, Z_5) = 1$.

Now we can continue again with Step 1 obtaining Z_6, and so on.

It can be proved by simple geometrical considerations that for all $n \geqq 0$ the following inequalities hold:

$$0 < d(P, Z_{4(n+1)+1}) < d(P, Z_{4n+1}),$$
$$0 < d(P, Z_{4(n+1)+3}) < d(P, Z_{4n+3}) \quad \text{and}$$
$$d^2(Z_{4n+1}, Z_{4n+3}) < d(Z_{4n+1}, Z_{4n+5}) < d(P, Z_{4n+1}).$$

This implies that $\lim_{n \to \infty} d(P, Z_{4n+1}) = 0$ for any possible starting point Z_1 and an arbitrary $d \in (0, \sqrt{(3)} - 1)$.

Now we are prepared to prove the existence of an odd cycle in $\langle M \rangle$ by the following construction. Let us denote by Z_2 the point P_2 and then by Z_1 such point of the arc PR_1 that $d(Z_2, Z_1) = 1$. Let us denote by X_1 such point of the arc R_1P_1 that $d(Z_1, X_1) = 1$. Let us denote by X_2 such point of the arc R_2P_2 that $d(X_1, X_2) = 1$. Let us denote by X_3 such point of the arc PR_1 that $d(X_2, X_3) = 1$. It follows from this construction that all the denoted points are in the set M and that $0 < d(P,X_3) < 1$. Now let us take the points Z_1, Z_2 and begin to construct the sequence of points $Z_1, Z_2, Z_3, \ldots$ in the way we described before. As $\lim_{n \to \infty} d(P, Z_{4n+1}) = 0$ there must exist an integer N such that $d(P, Z_{4N+1}) < d(P, X_3)$ (see Fig. 8, where $N = 1$).

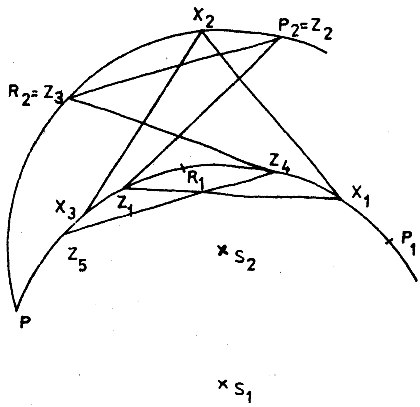

Fig. 8.

The proof of existence of an odd cycle in $\langle M \rangle$ consists now of the following consideration. We move the points Z_1 and X_1 continuously with the same velocity along the circle k_1 towards the point P and we move also all the points $Z_2, \ldots, Z_{4N+1}$, X_2, X_3 of our construction on k_1 and k_2 in such a way that all the distances between the corresponding points remain 1. Then, the points X_3 and Z_{4N+1} on k_1 are moving to P, too. It is evident that there exists a position of X_1 from which we obtain the point X_3 exactly at the point P, but Z_{4N+1} can never reach P because $d(P, Z_{4N+1}) > 0$ always holds. This implies that there must exist some positions of Z_1 and X_1, $d(X_1, Z_1) = 1$, in which the points X_3 and Z_{4N+1} become identical, and in this position we obtain an odd cycle $Z_1, Z_2, \ldots, Z_{4N+1} = X_3, X_2, X_1, Z_1$ on $4N + 3$ vertices in M. We note that for every $d \in (0, \sqrt{(3)} - 1)$ there exists an integer N_d such that for all $n \geqq N_d$ the graph $\langle M \rangle$ contains an odd cycle on $4n + 3$ vertices. Thus we have proved $\chi(\langle M \rangle) = 3$.

Theorem 7. *Let us have a regular colouring α of the real Euclidean plane E_2. Let there exist an open circular disc D with the centre S and the radius $r > 0$ such that every point of the disc D has the same colour 1. Then the colouring α has at least 5 colours.*

Proof. Let us denote by A a point of accumulation of points of colour other than 1 such that for any other point of accumulation A' of points of colour other than 1 we have $d(S, A') \geqq d(S, A) = r_a$. This implies that $r_a \geqq r$ and that for all $r' < r_a$ there is only a finite number of points of colour other than 1 in the circular disc with the centre S and radius r'. This implies that any point X from the circular ring R with the centre S such that $1 - r_a < d(S, X) < 1 + r_a$ must have colour other than 1 because X is adjacent to infinite number of points of colour 1 in the open circular disc with the centre S and radius r_a. Now, let us denote $d = \inf_{x \in M_1} d(X, S)$, where $M_1 = \{X \in E_2, d(S, X) > 1 \text{ and } \alpha(X) = 1\}$. Let us denote by R_d the open circular ring of points X such that $1 - r_a < d(S, X) < d$. Clearly, $X \in R_d$ implies $\alpha(X) \neq 1$.

Case A. Let us assume that $d > 1 + r_a$. Then there exists $\delta_d > 0$ such that for an arbitrary circle with radius 1 and with the centre X such that $d(S, X) < r_a + \delta_d$, the circular arc contained in R_d has a length at least $\frac{4}{3}\pi$. As A is a point of accumulation of points of colour other than 1 then there exist two points X_1, X_2 of the same colour other than 1 with $d(A, X_1) < \delta_d$, $d(A, X_2) < \delta_d$. Let $\alpha(X_1) = \alpha(X_2) = 2$. Let k_1 and k_2 be circles with the centres X_1 and X_2 and with the radius 1. All points of these circles must have colour other than 2. Moreover, these circles have one intersection P in R_d and they have also circular arcs in R_d which begin in P, have a length at least $\frac{4}{3}\pi$ and satisfy the conditions of Lemma 2. This implies that the points of these arcs have at least 3 colours. As no point of these arcs can have colours 1 and 2, we obtain immediately that the colouring α has at least 5 colours.

Case B. Suppose that $d = 1 + r_a$. Then there exists $\delta_d > 0$ such that for an arbitrary circle with radius 1 and with the centre X such that $r_a \leqq d(S, X) < r_a + \delta_d$, the intersection of this circle and R_d consists of two circular arcs of a length at least $\frac{4}{3}\pi$. Now we shall consider 2 subcases.

Subcase B1. Suppose that there exists a point T such that $\alpha(T) = 1$ and $d(S, T) = d$. As any point unit distance apart from T has colour other than 1, there must exist 2 points X_1, X_2 of the same colour other than 1 such that $d(T, X_1) = d(T, X_2) = 1$, $d(S, X_1) < r_a + \delta_d$ and $d(S, X_2) < r_a + \delta_d$. Let $\alpha(X_1) = \alpha(X_2) = 2$. Now as in case A let k_1 and k_2 be circles with centres X_1, X_2 and with the radius 1. These circles intersect at the point T on the boundary of R_d and have circular arcs in R_d that satisfy the conditions of Lemma 2. Using the same considerations as in case A we obtain that α has at least 5 colours in this subcase, too.

Subcase B2. Suppose that there exist no point T such that $\alpha(T) = 1$ and $d(S, T) = d$. This implies that then there must exist a point T which is the point of accumulation of points of colour 1 and $d(T, S) = 1$. Let us denote by T_1 the point of the line segment ST such that $d(S, T_1) = r_a + \delta_d$. Let us denote by T_2 such a point that $d(S, T_2) = d$ and $d(T_1, T_2) = 1$. Further, let us denote by T_3 the point of the line segment ST_2 such that $d(S, T_3) = r_a$ and $d(T_2, T_3) = 1$. Let us denote by k the circle with the centre T_2 and with the radius 1. Then $T_1 \in k$, $T_3 \in k$. As T is a point of accumulation of points of colour 1 then there must exist 4 various points Y_1, Y_2, Y_3, Y_4 of colour 1 in E_2 such that $d(T, Y_1) < \delta_d$, $d(T, Y_2) < \delta_d$, $d(T, Y_3) < \delta_d$, $d(T, Y_4) < \delta_d$. Now let X_1, X_2, X_3, X_4 be such points of the circular arc T_1T_3 of the circle k that $d(X_1, Y_1) = d(X_2, Y_2) = d(X_3, Y_3) = d(X_4, Y_4) = 1$. This implies that the colour of X_1, X_2, X_3 and X_4 cannot be 1. At least two of these four points have the same colour; suppose that $\alpha(X_1) = \alpha(X_2) = 2$. Now we have $d(X_1, T_2) = d(X_2, T_2) = 1$, $d(S, X_1) < r_a + \delta_d$, $d(S, X_2) < r_a + \delta_d$. As in case A let k_1 and k_2 be circles with centres X_1, X_2 and with radius 1. These circles intersect at the point T_2 on the boundary of R_d and have circular arcs in R_d that satisfy the conditions of Lemma 2. Using the same considerations as in case A we obtain that α has at least 5 colours in this subcase, too.

Now it follows from Theorem 7 that if there exists a colouring of E_2 by 4 colours then this colouring contains no small monochromatic area. We believe that this strange result can be much more strengthened and that this is a good way to get finally a contradiction and so to prove at least $\chi(E_2) > 4$.

In the conclusion let us introduce some partial open subproblems of Problem 1.

Problem 2. *Colour regularly by 4 colours an infinite strip of the real Euclidean plane E_2 with the width $d > \frac{2}{3}\sqrt{2}$!*

Problem 3. *Colour regularly by 5 colours an infinite strip of E_2 with the width $d > \frac{2}{3}\sqrt{5}$!*

Problem 4. *Colour regularly by 6 colours an infinite strip of E_2 with the width $d > \sqrt{3}$!*

Problem 5. *Construct a graph G finite or infinite which can be drawn in the plane so that each edge is a straight line segment of unit length and such that it contains two vertices V_1 and V_2 such that in any regular colouring of the graph G by 4 colours the vertices V_1 and V_2 have the same colour!*

References

[1] *R. B. Eggleton, P. Erdös, D. K. Skilton:* Colouring the real line. J. Comb. Theory Ser. B *39* (1985), 86—100.

[2] *P. Erdös:* Some unsolved problems. Publ. Math. Inst. Hung. Acad. Sci. *6* (1961), 221—254.

[3] *P. Erdös, F. Harary, W. T. Tutte:* On dimension of a graph. Mathematika *12* (1965), 118—122.

[4] *P. Erdös:* On combinatorial problems which I would most like to see solved. Combinatorica *1* (1981), 25—42.

[5] *P. Frankl:* Extremal problems and coverings of the space. European J. Comb. *1* (1980), 101—106.

[6] *H. Hadwiger:* Ungelöste Probleme No. 40. Elemente der Math. *16* (1961), 103—104.
[7] *H. Hadwiger, H. Debrunner, V. Klee:* Combinatorial Geometry in the Plane. Holt, Reinehart and Winston, New York (1964).
[8] *D. G. Larman, C. A. Rogers:* The realization of distances within sets in Euclidean space. Mathematika 19 (1972), 1—24.
[9] *L. Moser, W. Moser:* Solution to Problem 10. Canad. Math. Bull. *4* (1961), 187—189.
[10] *N. Wormald:* A 4-chromatic graph with a special plane drawing. J. Austral. Math. Soc. Ser. A 28 (1979), 1—8.

Súhrn

O PROBLÉME FARBENIA REÁLNEJ ROVINY

FILIP GULDAN

Aký je najmenší počet farieb potrebných na ofarbenie všetkých bodov reálnej euklidovskej roviny E_2 tak, aby žiadne dva body vzdialené o jednotku dĺžky nemali rovnakú farbu?

Tento známy problém je nevyriešený už vyše 25 rokov. V článku sú vyriešené niektoré jeho špeciálne prípady a sú uvedené viaceré otvorené podproblémy základného problému.

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