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APPROXIMATION AND ENTROPY NUMBERS OF EMBEDDINGS IN WEIGHTED ORLICZ SPACES

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Summary. Upper estimates are obtained for approximation and entropy numbers of the embeddings of weighted Sobolev spaces into appropriate weighted Orlicz spaces. Results are given when the underlying space domain is bounded and for certain unbounded domains.

Keywords: Sobolev spaces, Orlicz spaces, weights, approximation and entropy numbers.

AMS Classification: 41A46, 46E35.

1. INTRODUCTION

Much work has been done on the estimation of entropy and approximation numbers of embeddings of Sobolev spaces in L^p spaces or Orlicz spaces (see [1], [2], [4], [9], [12]). In particular, when Ω is an open subset of $\boldsymbol{R}^n$, it is known that in the critical case in which $rp = n$, $W^{r,p}(\Omega)$ can be compactly embedded in appropriate Orlicz spaces (see [3], [13]); estimates for the entropy and approximation numbers of this embedding were obtained in [2]. In recent years a great deal of effort has been devoted to the study of embeddings between *weighted* function spaces (see[6], [7], [10], [11]), particular attention having been paid to the situation in which the target space is a weighted Lebesgue space. This paper is devoted to the study of embeddings of weighted Sobolev spaces $W_\varrho^{r,p}(\Omega)$ in weighted Orlicz spaces $L_\sigma^\Phi(\Omega)$ when $rp \geqq n$; we give estimates for the approximation and entropy numbers of these embeddings first when Ω is bounded and then when it is unbounded. Here ϱ and σ are nonnegative weight functions which are related by a condition of A_{pq} type (see [10]).

Our results extend those of [2], in which $\varrho = \sigma = 1$.

2. PRELIMINARIES

Denote points in n-dimensional Euclidean space $\boldsymbol{R}^n$ by $x = (x_1, \ldots, x_n)$, and for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}_0^n$ (where $\boldsymbol{N}_0 \boldsymbol{N} \cup \{0\}$) put

$$|\alpha| = \sum_{j=1}^{n} \alpha_j , \quad \alpha! = \prod_{j=1}^{n} \alpha_j ,$$

$$x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}, \quad D^\alpha = \prod_{j=1}^{n} (\partial/\partial x_j)^{\alpha_j}.$$

Let $p \in (1, \infty)$, $r \in N$, $t \in (1, \infty]$ and suppose that

$$(1) \qquad \frac{1}{p} = \left(\frac{r}{n} - \frac{1}{t}\right) t'$$

where $1/t + 1/t' = 1$, and

$$(2) \qquad 1 < \frac{n}{r} < t.$$

Let ϱ and σ be measurable functions on R^n which are positive almost everywhere and which satisfy the following condition for all $q \in [p, \infty)$:

$$(\text{A}) \qquad \sup_Q \left\{ \left(\frac{1}{|Q|} \int_Q (\sigma(x))^t \, dx \right)^{1/qt} \left(\frac{1}{|Q|} \int_Q (\varrho(x))^{-tp'/p} \, dx \right)^{1/tp'} \right\} \leqq K q^a ,$$

where the supremum is taken over all cubes Q^n in R (with sides parallel to the coordinate axes), $|Q|$ is the volume of Q, K and a are constants and $0 \leqq a < r/n$.

Let Ω be an open subset of R^n. The weighted Sobolev space $W_\varrho^{r,p}(\Omega)$ is defined to be

$$\{u \colon D^\alpha u \in L_\varrho^p(\Omega) \quad \text{for all} \quad \alpha \in N_0^n \quad \text{with} \quad |\alpha| \leqq r\},$$

endowed with the norm

$$\|u\|_{r,p,\varrho,\Omega} := \Big(\sum_{|\alpha| \leqq r} \|D^\alpha u\|_{p,\varrho,\Omega}^p \Big)^{1/p},$$

where the derivatives are taken in the sense of distributions, functions equal almost everywhere are identified, and

$$\|v\|_{p,\varrho,\Omega} := \left(\int_\Omega |v(x)|^p \varrho(x) \, dx \right)^{1/p}$$

is the norm on $L_\varrho^p(\Omega)$. Here the functions may be real- or complex-valued. Note that since ϱ and σ satisfy condition (A),

$$\varrho^{-1/(p-1)} \in L_{\text{loc}}^t(\Omega) \subset L_{\text{loc}}^1(\Omega);$$

hence by Theorem 1.2 of [8], $W_\varrho^{r,p}(\Omega)$ is a Banach space.

By $W_{0,\varrho}^{r,p}(\Omega)$ we shall denote the completion (whenever this is meaningful) of the set $C_0^\infty(\Omega)$ (of all infinitely differentiable functions with compact support in Ω) in $W_\varrho^{r,p}(\Omega)$. As shown in Theorem 1.2 in [8], $W_{0,\varrho}^{r,p}(\Omega)$ *is* meaningful if, in addition, we require that

$$(3) \qquad \varrho \in L_{\text{loc}}^1(\Omega).$$

An Orlicz function is a map $\phi\colon [0, \infty) \to R$ which is continuous, convex and such that $\lim_{t \to 0} \phi(t)/t = 0$ and $\lim_{t \to \infty} \phi(t)/t = \infty$. Given such a ϕ, the weighted Orlicz space

$L^{\phi}_{\sigma}(\Omega)$ is defined to be the linear hull of the set of all real- or complex-valued functions u on Ω (with the convention that functions equal almost everywhere are identified) such that

$$\int_{\Omega} \phi(|u(x)|)\, \sigma(x)\, dx < \infty ,$$

furnished with the Luxemburg norm given by

$$\|u\|_{\phi,\sigma,\Omega} = \inf\{\lambda > 0 : \int_{\Omega} \phi(|u(x)|/\lambda)\, \sigma(x)\, dx \leqq 1\} .$$

With this norm, $L^{\phi}_{\sigma}(\Omega)$ is a Banach space.

Let X, Y be Banach spaces and let $T \in \mathscr{B}(X, Y)$, the set of all bounded linear maps from X to Y. Given any $s \in N$, the s-th approximation number $a_s(T)$ of T is defined by

$$a_s(T) = \inf\{\|T - F\| : F \in \mathscr{B}(X, Y),\ \dim F(X) < s\} ,$$

and the s-th entropy number $e_s(T)$ of T is given by

$$e_s(T) = \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{s-1} \text{ closed balls of radius } \varepsilon\} ,$$

where B_X is the closed unit ball in X.

Much information about weighted Sobolev spaces is contained in [7]; details of the main properties of approximation and entropy numbers are given in Chapter II of [4].

3. THE CASE WHEN Ω IS BOUNDED

To obtain our results concerning the entropy and approximation numbers the following lemmas are required.

Lemma 1. *Suppose that $\mu_1 > 0$ and $0 < \mu_2 < 1$. Then the series*

$$S(z) := \sum_{j=0}^{\infty} z^j(\mu_1 + \mu_2 j)^{\mu_1 + \mu_2 j}/j!$$

converges for all $z > 0$ and there is a constant K_1, depending only on μ_1 and μ_2, such that for all $z > 0$,

$$S(z) \leqq K_1 \exp\{(ze)^{\mu_2/(1-\mu_2)} (e^{\mu_1} z)^{1/(1-\mu_2)}\} .$$

Proof. Let k be the integer part of $\mu_1/(1 - \mu_2)$. For $j \geqq k + 1$ we have $\mu_1 + \mu_2 j \leqq j$, and for $j \leqq k$ the inequality $\mu_1 + \mu_2 j \leqq \mu_1/(1 - \mu_2)$ holds. Thus

$$S(z) \leqq \sum_{j=0}^{k} \frac{z^j}{j!}\left(\frac{\mu_1}{1 - \mu_2}\right)^{\mu_1/(1-\mu_2)} + \sum_{j=k+1}^{\infty} \frac{z^j}{j!} j^{\mu_1 + \mu_2 j} = S_1(z) + S_2(z), \text{ say} .$$

Since $j^{\mu_1} \leqq e^{\mu_1 j}$ we have

$$S_2(z) \leqq \sum_{j=k+1}^{\infty} \frac{j^{\mu_2 j}}{j!} (e^{\mu_1} z)^j ,$$

and by the proof of Theorem V.6.6 of [4], there is an absolute constant c_0 such that for all $z > 0$, this last series is majorised by

$$c_0 \exp\left\{(ze)^{\mu_2/(1-\mu_2)} \left(e^{\mu_1} z\right)^{1/(1-\mu_2)}\right\}.$$

Moreover,

$$S_1(z) \leqq \left(\frac{\mu_1}{1-\mu_2}\right)^{\mu_1/(1-\mu_2)} e^z \leqq$$

$$\leqq \left(\frac{\mu_1}{1-\mu_2}\right)^{\mu_1/(1-\mu_2)} \exp\left\{(ze)^{\mu_2/(1-\mu_2)} \left(e^{\mu_1} z\right)^{1/(1-\mu_2)}\right\}.$$

The result follows.

Lemma 2 *For $i = 1, \ldots, n$ let $a_i, b_i \in \boldsymbol{R}$ be such that $a_i < b_i$; let $Q = \{x \in \boldsymbol{R}^n : : a_i < x_i < b_i$ for $i = 1, \ldots, n\}$, $p \in (1, \infty)$, $r \in \boldsymbol{N}$, $t \in (1, \infty]$, and suppose (1) and (2) hold and that $q \geqq p$. Suppose that ϱ and σ are weight functions which satisfy (A). For all $u \in C^r(\bar{Q}) \cap W_\varrho^{r,p}(Q)$ and all $x \in \boldsymbol{R}^n$ put*

$$(P_{r,Q}u)(x) = \frac{\chi_Q(x)}{|Q|} \sum_{|\alpha| \leqq r-1} \int_{\boldsymbol{R}^n} \chi_Q(y) \frac{(x-y)^\alpha}{\alpha!} D^\alpha u(y)\, \mathrm{d}y$$

where χ_Q is the characteristic function of Q. Let Q be subdivided into 2^{nN} congruent boxes Q_j and set

$$(P_N u)(x) = \sum_{j=1}^{2^{nN}} \chi_{Q_j}(x) (P_{r,Q_j}u)(x) \quad (x \in \boldsymbol{R}^n).$$

Then for all $u \in C^r(\bar{Q})$ with $\|u\|_{r,p,\varrho,Q} = 1$,

$$\|u - P_N u\|_{q,\sigma,Q} \leqq C_1 K q^a (2^{-nN} |Q|^{1/q+1/p't}) (1 + q/p')^{(1/q+1/p')t'},$$

where C_1 is a constant which depends only on n and r.

Proof. For $u \in C^r(\bar{Q}) \cap W_\varrho^{r,p}(Q)$ we have for any $x \in \boldsymbol{R}^n$, by Taylor's formula and setting $u = 0$ outside $\bar{Q}$,

$$u(x) - (P_{r,Q}u)(x) =$$

$$= \chi_Q(x) |Q|^{-1} \sum_{|\alpha|=r} \frac{r}{\alpha!} \int_{\boldsymbol{R}^n} \chi_Q(y) \int_0^1 (1-\tau)^{r-1} (x-y)^\alpha D^\alpha u(\tau x + y - \tau y)$$

$$\mathrm{d}\tau\, \mathrm{d}y = \sum_{|\alpha|=r} \frac{1}{\alpha!} F_\alpha(x), \quad \text{say}.$$

Then

$$|F_\alpha(x)| \leqq |Q|^{-1} (g_\alpha * \chi_Q |D^\alpha u|)(x),$$

where

$$g_\alpha(x) = \textstyle\int_0^1 |x^\alpha| \tau^{-n-1} \chi_{2Q_0}(x/\tau)\, \mathrm{d}\tau,$$

and Q_0 is the box centred at 0 and obtained by translation of Q. A routine calculation shows that

(4) $$\|g_\alpha\|_{m,\mathbf{R}^n} \leqq A\, 2^{r+n/m} |Q|^{1/m + r/n} \{(r-n)\, m + n\}^{-1/m}$$

provided that $(r-n)\, m + n > 0$ and $m > 1$; here A is a constant which depends only on n and r. (See Lemma V.6.1 of [4].)

Let $1/m = 1 - 1/p + 1/q$, $1/s = 1/p - 1/q$ and suppose that $q > p$; note that

$$p'(1 - m/q) = m\,, \quad s(1 - p/q) = p \quad \text{and} \quad 1/p' + 1/q + 1/s = 1\,.$$

Use of Hölder's inequality now shows that

$$\begin{aligned}
|(g_\alpha * \chi_Q |D^\alpha u|)(x)| &\leqq \left(\int_{\mathbf{R}^n} |g_\alpha(x-z)|^{p'(1-m/q)} \chi_Q(z)\, \varrho(z)^{-p'/p}\, dz\right)^{1/p'} \times \\
&\times \left(\int_{\mathbf{R}^n} |g_\alpha(x-z)|^m \chi_Q(z)\, |D^\alpha u(z)|^p\, \varrho(z)\, dz\right)^{1/q} \times \\
&\times \left(\int_{\mathbf{R}^n} \chi_Q(z)\, |D^\alpha u(z)|^{s(1-p/q)}\, \varrho(z)\, dz\right)^{1/s} \leqq \\
&\leqq \|D^\alpha u\|_{p,\varrho,Q}^{p/s} \|g_\alpha\|_{mt',\mathbf{R}^n}^{m/p'} \|\varrho^{-p'/p}\|_{t,\mathbf{R}^n}^{1/p'} \times \\
&\times \left(\int_{\mathbf{R}^n} |g_\alpha(x-z)|^m \chi_Q(z)\, |D^\alpha u|^p\, \varrho(z)\, dz\right)^{1/q}.
\end{aligned}$$

It follows that

(5) $$\begin{aligned}
&\left(\int_{\mathbf{R}^n} \chi_Q(x)\, |(g_\alpha * \chi_Q |D^\alpha u|)(x)|^q\, \sigma(x)\, dx\right)^{1/q} \leqq \\
&\leqq \|g_\alpha\|_{mt',\mathbf{R}^n}^{m/p'} \|\varrho^{-p'/p}\|_{t,Q}^{1/p'} \|D^\alpha u\|_{p,\varrho,Q}^{p/s} \|g_\alpha\|_{mt',\mathbf{R}^n}^{m/q} \|\sigma\|_{t,Q}^{1/q} \|D^\alpha u\|_{p,\varrho,Q}^{p/q} = \\
&= \|g_\alpha\|_{mt',\mathbf{R}^n} \|\varrho^{-p'/p}\|_{t,Q}^{1/p'} \|\sigma\|_{t,Q}^{1/q} \|D^\alpha u\|_{p,\varrho,Q}\,.
\end{aligned}$$

When $p = q$, Young's inequality for convolutions gives

(6) $$\begin{aligned}
&\left(\int_{\mathbf{R}^n} \chi_Q(x) |(g_\alpha * \chi_Q |D^\alpha u|)(x)|^p\, \sigma(x)\, dx\right)^{1/p} \leqq \\
&\leqq \|g_\alpha\|_{t',\mathbf{R}^n} \|D^\alpha u\|_{p,\varrho,Q} \|\varrho^{-p'/p}\|_{t,Q}^{1/p'} \|\sigma\|_{t,Q}^{1/q}\,.
\end{aligned}$$

Since

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{q} - \frac{1}{p} = \left(\frac{r}{n} - 1\right) t' + \frac{1}{m}\,,$$

we see that the condition

$$(r - n)\, mt' + n > 0$$

holds; and

$$\frac{1}{p't'} = 1 - \frac{r}{n}\,, \quad \frac{1}{mt'} = 1 - \frac{r}{n} + \frac{1}{qt'}\,.$$

We therefore have, using (4), (5) and (6),

(7) $$\|u - P_{r,Q} u\|_{q,\sigma,Q} \leqq C_1 |Q|^{1/qt'} \left(\frac{q}{m}\right)^{1/mt'} \|\varrho^{-p'/p}\|_{t,Q}^{1/p'} \|\sigma\|_{t,Q}^{1/q} \|\mu\|_{r,p,\varrho,Q}\,,$$

where C_1 depends only on n and r. This inequality, with Q_j in place of Q, shows that for all $u \in C^r(\bar{Q})$ with $\|u\|_{r,p,\varrho,Q} = 1$,

$$\|u - P_n u\|_{q,\sigma,Q} = \{\sum_{j=1}^{2^{nN}} \|\chi_{Q_j}(u - P_{r,Q_j}u)\|^q_{q,\sigma,Q_j}\}^{1/q} \leqq$$
$$\leqq C_1(|Q|\, 2^{-nN})^{1/qt'} \left(\frac{q}{m}\right)^{1/mt'} \{\sum_{j=1}^{2^{nN}} (\|\varrho^{-p'/p}\|^{1/p'}_{t,Q_j} \|\sigma\|^{1/q}_{t,Q_j} \|u\|_{r,p,\varrho,Q_j})^q\}^{1/q}.$$

As ϱ and σ satisfy condition (A), the Lemma now follows.

Lemma 3. *Under the same conditions as Lemma 2, except that the condition* $1 < p \leqq q$ *is replaced by* $1 < q < p$, *we have for all* $u \in C^r(\bar{Q})$ *with* $\|u\|_{r,p,\varrho,Q} = 1$,

$$\|u - P_N u\|_{q,\sigma,Q} \leqq C_1 p^{1/t'} K q^a (|Q|\, 2^{-nN})^{1/q + 1/tp'}.$$

Proof. Using Hölder's inequality and (7) we obtain

$$\|u - P_{r,Q}u\|_{q,\sigma,Q} \leqq \|u - P_{r,Q}u\|_{p,\sigma,Q} (\|\sigma\|_{t,Q} |Q|^{1/t'})^{1/q - 1/p} \leqq$$
$$\leqq C_1 |Q|^{1/t'q}\, p^{1/t'} \|\varrho^{-1/(p-1)}\|^{1/p'}_{t,Q} \|\sigma\|^{1/q}_{t,Q} \|u\|_{r,p,\varrho,Q}.$$

This inequality, with Q_j in place of Q, gives

$$\|u - P_N u\|_{q,\sigma,Q} =$$
$$= \{\sum_{j=1}^{2^{nN}} (C_1 |Q_j|^{1/t'q} p^{1/t'} \|\varrho^{-1/(p-1)}\|^{1/p'}_{t,Q_j} \|\sigma\|^{1/q}_{t,Q_j} \|u\|_{r,p,\varrho,Q_j})^q\}^{1/q} \leqq$$
$$\leqq C_1 p^{1/t'} (|Q|\, 2^{-nN})^{1/t'q} K q^a (|Q|\, 2^{-nN})^{(1/q + 1/p')/t} (\sum_{j=1}^{2^{nN}} \|u\|^q_{r,p,\varrho,Q_j})^{1/q} \leqq$$
$$\leqq C_1 p^{1/t'} K q^a (|Q|\, 2^{-nN})^{1/q + 1/tp'}.$$

Next we introduce several important restrictions on the weight function ϱ, following the approach of Kufner [7].

Definition 1. Let $s: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be continuous ($\boldsymbol{R}^+ = [0, \infty)$). We say that s has *property* (H) if, given any positive constants c_1, c_2 with $c_1 < c_2$, there are positive constants C_1, C_2 such that

$$c_1 \leqq t \leqq \tau \leqq c_2 \quad \text{implies} \quad C_1 \leqq s(t)/s(\tau) \leqq C_2.$$

The function s is said to be of type I if it is non-decreasing on some interval $(0, c)$ and $\lim_{t \to 0+} s(t) = 0$; it is of type II if it is non-increasing on some interval $(0, c)$, $\lim_{t \to 0+} s(t) = \infty$ and $\int_0^c s(t)\, dt < \infty$; and it is of type III if it is non-increasing on some interval $(0, c)$, $\lim_{t \to 0+} s(t) = \infty$ and $\int_0^c s(t)\, dt = \infty$.

Now let ϱ be a weight function of the form

(8) $\qquad \varrho(x) = s(d(x)) \quad (x \in \boldsymbol{R}^n),$

where

$$d(x) = \operatorname{dist}(x, \partial\Omega)$$

and s has property (H). It is known (see Theorems 11.2 and 11.11 of [7]) that if $\partial\Omega$ is of class $C^{0,1}$ then:

(i) $C^\infty(\bar{\Omega})$ is dense in $W_\varrho^{r,p}(\Omega)$ if ϱ is of type I or II;

(ii) $C_0^\infty(\Omega)$ is dense in $W_\varrho^{r,p}(\Omega)$ if ϱ is of type III.

We can now give the main result of this section.

Theorem 1 *Let* $p \in (1, \infty)$, $r \in \boldsymbol{N}$, $t \in (1, \infty]$ *and suppose that* (1) *and* (2) *hold; let* Ω *be a bounded domain in* $\boldsymbol{R}^n$ *with boundary of class* C^r. *Suppose that* ϱ *and* σ *are weight functions which satisfy condition* (A), *and in addition suppose that* $\varrho(x) = s(d(x))$ $(x \in \boldsymbol{R}^n)$, *where* s *has property* (H) *and* ϱ *is of type* I, II *or* III. *Let* $1 < v < n/\{n(1 + a) - r\}$ *and let* ϕ *be the Orlicz function given by*

$$\phi(b) = \exp(b^v) - 1 \quad (b \geqq 0)\,;$$

let

$$I\colon W_\varrho^{r,p}(\Omega) \to L_\sigma^\phi(\Omega)$$

be the natural embedding. Then as $m \to \infty$,

$$a_m(I) = O\left(m^{-1/p't}(\log m)^{-1/v+1/p't'+a}\right).$$

Proof. First consider the case in which s is of type I or II. Let Q be an open cube in $\boldsymbol{R}^n$ such that $\bar{\Omega} \subset Q$ and $|Q| > 1$. By [5], there exists an extension map

$$E\colon W_\varrho^{r,p}(\Omega) \to W_\varrho^{r,p}(Q)$$

such that for all $u \in W_\varrho^{r,p}(\Omega)$,

$$\|u\|_{r,p,\varrho,\Omega} \leqq \|Eu\|_{r,p,\varrho,Q} \leqq K_2\|u\|_{r,p,\varrho,\Omega}\,,$$

where K_2 is a constant independent of u; moreover, if $u \in C^r(\bar{\Omega})$, then $Eu \in C^r(\bar{Q})$. For all $u \in C^r(\bar{\Omega})$, put $\tilde{u} = Eu$ and $\tilde{U} = \tilde{u} - P_N\tilde{u}$. Then by Lemmas 2 and 3,

$$\int_Q \phi(|\tilde{U}(x)|/\lambda)\,\sigma(x)\,dx = \sum_{j=1}^\infty \frac{1}{j!}(\|\tilde{U}\|_{jv,\sigma,Q}\lambda^{-1})^{jv} \leqq$$

$$\leqq 2^{-nN}|Q| \sum_{j=1}^\infty \frac{1}{j!}\{C_2K(jv/p')^a\,(|Q|\,2^{-nN})^{1/tp'}\|\tilde{u}\|_{r,p,\varrho,Q}\,\lambda^{-1}\}^{jv} \times$$

$$\times \left(1 + \frac{jv}{p'}\right)^{(1+jv/p'//t'} \leqq$$

$$\leqq 2^{-nN}|Q|\sum_{j=1}^\infty \frac{1}{j!}\{C_2K(|Q|\,2^{-nN})^{1/tp'}\|\tilde{u}\|_{r,p,\varrho,Q}\lambda^{-1}\}^{jv} \times$$

$$\times\left(1+\frac{jv}{p'}\right)^{(1+at'+jv/p')/t'},$$

where $C_2 = C_1 p^{1/t'}(p')^a$. Put $C_3 = C_2 K t^{(a+1/t'p')/t'}$,

$$\xi = \{C_3(|Q|\,2^{-nN})^{1/tp'}\,\|\tilde{u}\|_{r,p,\varrho,Q}\lambda^{-1}\}^v,$$

$$\mu_1 = 1/t', \quad \mu_2 = (a + 1/t'p')\,v.$$

Observing that $a + 1/t'p' = \{n(1+a) - r\}/n$ and that consequently $\mu_2 < 1$, we see with the aid of Lemma 1 that

$$\int_Q |\tilde{U}(x)/\lambda|\,\sigma(x)\,dx \leqq |Q|\,2^{-nN}(t')^{1/t'}\sum_{j=1}^{\infty}\frac{1}{j!}\xi^j(\mu_1+\mu_2 j)^{\mu_1+\mu_2 j} \leqq$$

$$\leqq |Q|\,2^{-nN}(t')^{1/t'}K_1\exp\{(2e)^{\mu_1/(1-\mu_2)}(e^{\mu_1}\xi)^{1/(1-\mu_2)}\} \leqq 1$$

if

$$(2e)^{\mu_2/(1-\mu_2)}(e^{u_1}\xi)^{1/1-\mu_2} \leqq \log 2^{nN} - \log\{K_1|Q|(t')^{1/t'}\},$$

which is certainly true if

$$\lambda \geqq C_3(2e)^{\mu_2/v}e^{\mu_1/v}(|Q|\,2^{-nN})^{1/tp'}\{\log 2^{nN} -$$

$$-\log(K_1|Q|\,(t')^{1/t'})\}^{-1/v+a+1/t'p'}\|\tilde{u}\|_{r,p,\varrho,Q} =$$

$$= C_4(|Q|\,2^{-nN})^{1/tp'}(\log 2^{nN} - \log C_5)^{-1/v+a+1/t'p'}\|\tilde{u}\|_{r,p,\varrho,Q}, \quad \text{say}.$$

Hence

$$\|\tilde{u} - P_n\tilde{u}\|_{\phi,Q} \leqq$$

$$\leqq C_4(|Q|\,2^{-nN})^{1/tp'}(\log 2^{nN} - \log C_5)^{-1/v+a+1/t'p'}\,\|\tilde{u}\|_{r,p,\varrho,Q}.$$

Now let $u \in C^r(\bar{\Omega})$, so that $\tilde{u}(x) = u(x)$ for all $x \in \Omega$; let

$$P_N u = \sum_{j=1}^{2^{nN}} \chi_{Q_j\cap\Omega}P_{r,Q_j}\tilde{u}.$$

Since

$$\|u - P_N u\|_{\phi,\Omega} \leqq \|\tilde{u} - P_N\tilde{u}\|_{\phi,Q},$$

it follows that

(9) $$\|u - P_N u\|_{\phi,\Omega} \leqq$$

$$\leqq C_4(|Q|\,2^{-nN})^{1/tp'}(\log 2^{nN} - \log C_5)^{-1/v+a+1/t'p'}\,\|\tilde{u}\|_{r,p,\varrho,Q} \leqq$$

$$\leqq C_4K_2(|Q|\,2^{-nN})^{1/tp'}(\log 2^{nN} - \log C_5)^{-1/v+a+1/t'p'}\,\|u\|_{r,p,\varrho,\Omega}.$$

Since $C^r(\bar{\Omega})$ is dense in $W^{r,p}_\varrho(\Omega)$, (9) holds for all $u \in W^{r,p}_\varrho(\Omega)$. As the map $u \mapsto P_N u$ is finite-dimensional, with rank at most $2^{nN}M$, it follows easily that as $m \to \infty$,

$$a_m(I) = O(m^{-1/tp'}(\log m)^{-1/v+a+1/t'p'}).$$

When s is of type III the argument is similar but easier, as E is not needed.

Remarks. 1. Note that

$$\frac{1}{p} = \left(\frac{r}{n} - \frac{1}{t}\right) t' \leqq rn \,,$$

with equality holding only when $t = \infty$; moreover,

$$\frac{1}{p't} = \frac{r}{n} - \frac{1}{p} \quad \text{and} \quad \frac{1}{p't'} = 1 - \frac{r}{n} \,.$$

Thus the conclusion of Theorem 1 may be stated as

$$a_m(I) = O(m^{-(r/n-1/p)}(\log m)^{1-r/n+a-1/\nu})\,.$$

2. A combination of the techniques used above and those used to prove Theorem 1 of [2] may be used to show that, under the same hypotheses as in our Theorem 1, the entropy numbers $e_m(I)$ satisfy the same estimates as for $a_m(I)$.

3. If instead of I we consider the embedding map I_0: $W^{r,p}_{0,\varrho}(\Omega) \to L^{\Phi}_{\sigma}(\Omega)$, then since no use is made of density and extension theorems, it follows that

$$a_m(I_0)\,, \quad e_m(I_0) = O(m^{-(r/n-1/p)}(\log m)^{1-r/n+a-1/\nu})$$

under the same hypotheses as in Theorem 1, save that ϱ and σ may be general weight functions which merely satisfy condition (A) and the requirement that $\varrho \in L^1_{\mathrm{loc}}(\Omega)$; ϱ need not be of the form $\varrho(x) = s(d(x))$; and no restrictions are imposed on $\partial\Omega$.

4. The (A) condition requires that the weights ϱ and σ be defined on the whole of $\boldsymbol{R}^n$. Thus if we are merely given weights defined on Ω, we have to extend them to $\boldsymbol{R}^n$ in a way which will ensure that the extended functions satisfy the (A) condition. To illustrate how this may be done, suppose that ϱ and σ are defined on Ω by

$$\varrho(x) = (d(x))^{\varepsilon_1}\,, \quad \sigma(x) = (d(x))^{\varepsilon_2} \quad (x \in \Omega)\,,$$

where $\varepsilon_1 \geqq 0$ and $-p < \varepsilon_2 \leqq 0$. Let $K > 0$ and define functions s and s_1 by

$$s(t) = \begin{cases} t^{\varepsilon_1} & \text{if} \quad 0 \leqq t \leqq K\,, \\ K^{\varepsilon_2} & \text{if} \quad t > K\,, \end{cases}$$

$$s_1(t) = \begin{cases} t^{\varepsilon_2} & \text{if} \quad 0 < t \leqq K\,, \\ K^{\varepsilon_2} & \text{if} \quad t > K\,. \end{cases}$$

Then if K is large enough, the functions $\tilde{\rho}$, $\tilde{\sigma}$ defined by

$$\tilde{\rho}(x) = s(d(x))\,, \quad \tilde{\sigma}(x) = s_1(d(x)) \quad (x \in \boldsymbol{R}^n)$$

are extensions of ϱ and σ respectively which satisfy the (A) condition with $a = 0$.

In particular, when $\varepsilon_1 = \varepsilon_2 = 0$, so that $\varrho(x) = \sigma(x) = 1$ $(x \in \Omega)$, Theorem 1 shows that the approximation numbers of the embedding I: $W^{r,p}(\Omega) \to L^{\Phi}(\Omega)$ satisfy

$$a_m(I) = O(m^{-(r/n-1/p)}(\log m)^{1-r/n-1/\nu})$$

when $1/p \leqq r/n$. When $1/p = r/n$ this gives Theorem V.6.6 of [4], but under stronger hypotheses on $\partial\Omega$; the same estimate holds for $e_m(I)$, which gives Theorem 3 of [2], again under stronger hypotheses on $\partial\Omega$.

4. THE CASE WHEN Ω IS UNBOUNDED

To deal with this situation the following lemmas are required.

Lemma 4. *Let $u \in C_0^r(\Omega)$. Then there is a constant C_6, depending only on r and n, such that for all $x \in \Omega$,*

$$|u(x)| \leqq C_6 \int_{\Omega \cap B(x,1)} \sum_{i=0}^{r} |D^i u(y)| \, |y - x|^{r-n} \, dy ,$$

where

$$|D^i u(y)|^2 = \sum_{|\alpha| = i} |D^\alpha u(y)|^2$$

and $B(x, 1)$ is the open ball in $\mathbf{R}^n$ with centre x and radius 1.

Lemma 5. *Let $0 < b < n$. Then*

$$\int_{\Omega \cap B(x,1)} |x - y|^{b-n} \, dy \leqq \omega_n b^{-1} (|\Omega \cap B(x, 1)|/\omega_n)^{b/n} ,$$

where ω_n denotes the $(n-1)$-dimensional Lebesgue measure of the unit sphere S^{n-1} in $\mathbf{R}^n$.

These lemmas are proved in Chapter V of [4].

To estimate the approximation and entropy numbers of the embedding

$$I_0 \colon W^{r}_{0,p}(\Omega) \to L^{\phi}_{\sigma}(\Omega)$$

where Ω is unbounded, we first assume that (1) and (2) hold and that the weight functions ϱ and σ satisfy

$$\text{(A}'\text{)} \qquad \sup \{ (|B(x, r)|^{-1} \int_{B(x,r)} (\sigma(x))^t \, dx)^{1/tq} \, . \\ . \, (|B(x, r)|^{-1} \int_{B(x,r)} (\varrho(x))^{-t/(p-1)} \, dx)^{1/tp'} : x \in \mathbf{R}^n, \ r > 0 \} \leqq K q^a$$

for all $q \in [p, \infty)$, where K and a are constants with $0 \leqq a < r/n$. Condition (A′) is plainly equivalent to (A). We also assume that

$$\varrho \in L^1_{\text{loc}}(\Omega) ,$$

and define ϕ by

$$\text{(10)} \qquad \phi(s) = s^\tau \exp(s^\nu) \quad (s \geqq 0) ,$$

where

$$\text{(11)} \qquad 1 < p < \tau < \infty \quad \text{and} \quad 1 < \nu < n/\{n(1 + a) - r\} .$$

Finally, set

$$\eta_\Omega = \sup\{|\Omega \cap B(x, 1)|/|B(x, 1)| : x \in \Omega\}.$$

Lemma 6. *Let Ω be an unbounded open subset of R^n, suppose that* (1), (2) *and* (A') *hold and let ϕ be defined by* (10) *and* (11). *Then for all $u \in W^{r,p}_{0,\varrho}(\Omega)$, we have*

$$\|u\|_{\phi,\sigma,\Omega} \leqq C_{12}\eta_\Omega^{1/tp'}(\log \eta_\Omega^{-1} - \log C_{10})^{1+a-r/n-1/\nu} \|u\|_{r,p,\varrho,\Omega}.$$

Proof. Since $C_0^\infty(\Omega)$ is dense in $W^{r,p}_{0,\varrho}(\Omega)$ it is enough to prove the lemma when $u \in C_0^\infty(\Omega)$. Let $1/s = 1/p - 1/q$, $q > p$; put $a_1 = (r - n)t' + n = n/p$, $a_2 = (1 - 1/q)(r - n)p't' + n = n/q$, $v(y) = \sum_{i=0}^{r} |D^i u(y)|$. From Lemmas 4 and 5 we see that, with all integrals being taken over $\Omega \cap B(x, 1)$,

$$\begin{aligned}
|u(x)| &\leqq C_6 \int v(y)\,|x - y|^{r-n}\,dy \leqq \\
&\leqq C_6\left(\int |v(y)|^p\,|x - y|^{r-n}\,\varrho(y)\,dy\right)^{1/q}\left(\int |v(y)|^p\,\varrho(y)\,dy\right)^{1/s} \times \\
&\times \left(\int |x - y|^{(1-1/q)(r-n)p't'}\,dy\right)^{1/p't'}\left(\int \varrho(y)^{-p't/p}\right)^{1/p't} \leqq \\
&\leqq C_6\eta_\Omega^{1/qp't'}(\omega_n a_2^{-1})^{1/p't'}\,\|v\|^{p/s}_{p,\varrho,\Omega}\,\|\varrho^{-1/(p-1)}\|^{1/p'}_{t,\Omega\cap B(x,1)} \times \\
&\times \left(\int |v(y)|^p\,|x - y|^{r-n}\,\varrho(y)\,dy\right)^{1/q} \leqq \\
&\leqq C_7\eta_\Omega^{1/qp't'}q^{1/p't'}\|\varrho^{-1(p-1)}\|^{1/p'}_{\Omega\cap B(x,1)}\|u\|^{p/s}_{r,p,\varrho,\Omega} \times \\
&\times \left(\int |v(y)|^p\,|x - y|^{r-n}\,\varrho(y)\,dy\right)^{1/q}, \quad \text{say}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|u\|_{q,\sigma,\Omega} &\leqq C_7\eta_\Omega^{1/qp't'}q^{1/p't'}\|u\|^{p/s}_{r,p,\varrho,\Omega} \times \\
&\times \int_\Omega \int_{\Omega\cap B(y,1)} \|\varrho^{-1/(p-1)}\|^{q/p'}_{t,\Omega\cap B(x,1)}\,|v(y)|^p\,|x - y|^{r-n}\,\sigma(x)\,\varrho(y)\,dx\,dy \leqq \\
&\leqq C_7\eta_\Omega^{1/qp't'}q^{1/p't'}\|u\|^{p/s}_{r,p,\varrho,\Omega} \times \\
&\times \left\{\int_\Omega \|\varrho^{-1/(p-1)}\|_{\Omega\cap B(y,2)}\,|v(y)|^p\,\varrho(y)\left(\int_{\Omega\cap B(y,1)} |y - x|^{(r-n)t'}\,dx\right)^{1/t'} \times \right. \\
&\times \left.\left(\int_{\Omega\cap B(y,1)} \sigma^t(x)\,dx\right)^{1/t} dy\right\}^{1/q}.
\end{aligned}$$

By condition (A') and Lemma 5, we have

$$\begin{aligned}
\|u\|_{q,\sigma,\Omega} &\leqq C_7\eta_\Omega^{1/qp't'}q^{1/p't'}\|u\|^{p/s}_{r,p,\varrho,\Omega}\,\eta_\Omega^{1/pt'q}(\omega_n a_1^{-1})^{1/qt'} \times \\
&\times \left\{\int_\Omega (Kq^a|\Omega \cap B(y, 2)|^{(1/p'+1/q)/t})^q\,|v(y)|^p\,\varrho(y)\,dy\right\}^{1/q} \leqq \\
&\leqq C_8\eta_\Omega^{1/q+1/tp'}q^{a+1/p't'}\|u\|_{r,p,\varrho,\Omega},
\end{aligned}$$

where C_8 depends only on n, r and p.

Since $1/p't' = 1 - r/n$, we define positive numbers μ_1 and μ_2 by

$$\mu_1 = n(1 + a) - r, \quad \mu_2 = (1 + a - r/n)\,\nu < 1,$$

and put

$$\xi = \left\{C_8\eta_\Omega^{1/tp'}\left(\frac{n}{n(1 + a) - r}\right)^{1+a-r/n}\|u\|_{r,p,\varrho,\Omega}\,\lambda^{-1}\right\}^\nu =$$

$$= (C_9 \eta_\Omega^{1/tp'} \|u\|_{r,p,\varrho,\Omega} \lambda^{-1})^\nu , \quad \text{say} .$$

By Lemma 1, we have

$$\int_\Omega \phi(|u(x)|/\lambda)\, \sigma(x)\, dx = \sum_{j=0}^{\infty} \frac{1}{j!} \int_\Omega |u(x)\, \lambda^{-1}|^{j\nu+\tau} \sigma(x)\, dx \leqq$$

$$\leqq \eta_\Omega \sum_{j=0}^{\infty} \frac{1}{j!} (C_8 \eta_\Omega^{1/tp'} \|u\|_{r,p,\varrho,\Omega} \lambda^{-1})^{j\nu+\tau} (j\nu + \tau)^{(1+a-r/n)(j\nu+\tau)} \leqq$$

$$\leqq (1 + a - r/n)^{-\mu_1} \eta_\Omega \xi^{a/\nu} \sum_{j=0}^{\infty} (\mu_1 + \mu_2 j)^{\mu_1+\mu_2 j}/j! \leqq$$

$$\leqq K_1 (1 + a - r/n)^{-\mu_1} \eta_\Omega e^{\tau\xi/\nu} \exp\{(2e)^{\mu_2/(1-\mu_2)} (e^{\mu_1}\xi)^{1/(1-\mu_2)}\} .$$

If $\xi \geqq 1$, since $\tau/\nu < \mu_1$ we have

$$e^{\tau\nu/\xi} \exp\{(2e)^{\mu_2/(1-\mu_2)} (e^{\mu_1}\xi)^{1/(1-\mu_2)}\} \leqq \exp\{((2e)^{\mu_2/(1-\mu_2)} + 1) \times (e^{\tau/\nu}\xi)^{1/(1-\mu_2)}\} ;$$

and if $\xi < 1$, then the corresponding estimate has

$$e^{\tau/\nu} \exp[(2e)^{\mu_1/(1-\mu_2)} e^{\mu_1/(1-\mu_2)}] .$$

Thus

$$\int_\Omega \phi(|u(x)|\, \lambda^{-1})\, \sigma(x)\, dx \leqq C_{10} \eta_\Omega \exp(C_{11} \xi^{1/(1-\mu_2)}) ,$$

and from this the Lemma follows easily.

Lemma 7. *Let $R > 0$, $Q_R = (-R/2, R/2)^n$. Then under the assumptions of Lemma 6 we have, for all $u \in W_{0,\varrho}^{r,p}(Q_R)$,*

$$\|u - P_N u\|_{\phi,\sigma,Q_R} \leqq C_{15} (R^n 2^{-nN})^{1/p't} \times$$
$$\times (\log 2^{nN} - \log R^n - \log C_{14})^{1+a-r/n-1/\nu} \|u\|_{r,p,\varrho,Q_R} .$$

Proof. From Lemma 2 we have, for all $q \in [p, \infty)$ and all $u \in C_0^\infty(Q_R)$,

$$\|u - P_N u\|_{q,\sigma,Q_R} \leqq$$
$$\leqq C_1 K^a (R^n 2^{-nN})^{1/q+1/p't} (1 + q'/p)^{(1/q+1/p')/t'} \|u\|_{r,p,\varrho,Q_R} .$$

Thus with $U = u - P_N u$,

$$\int_{Q_R} \phi(|U(x)|\, \lambda^{-1})\, \sigma(x)\, dx \leqq$$

$$\leqq R^n 2^{-nN} \sum_{j=0}^{\infty} \frac{1}{j!} \{C_1 K (j\nu + \tau)^a (R^n 2^{-nN})^{1/p't} \lambda^{-1} \|u\|_{r,p,\varrho,Q_r}\}^{j\nu+\tau} \times$$

$$\times \{1 + (j\nu + \tau)/p'\}^{(1+(j\nu+\tau)/p')/t'} .$$

Put $\mu_1 = (1 + \tau/p' + t'a\tau)/t' > 0$, $\mu_2 = (a + 1/t'p')\,\nu = (1 + a - r/n)\,\nu \in (0, 1)$,

$$\xi = \{C_1 K \lambda^{-1} (R^n 2^{-nN})^{1/p't} (p')^a (t')^{1+a-r/n} \|u\|_{r,p,\varrho,Q_R}\}^\nu .$$

Then using Lemma 1 we obtain

$$\int_{Q_R} \phi(|U(x)| \lambda^{-1}) \sigma(x) \, dx \leqq$$
$$\leqq (t')^{1/t'} R^n 2^{-nN} \xi^{\tau/\nu} \sum_{j=0}^{\infty} \frac{1}{j!} \xi^j (\mu_1 + \mu_2 j)^{\mu_1 + \mu_2 j} \leqq$$
$$\leqq (t')^{1/t'} K_1 R^n 2^{-nN} e^{\tau\xi/\nu} \exp \{(2e)^{\mu_2/(1-\mu_2)} (e^{\mu_1} \xi)^{1/(1-\mu_2)}\} .$$

Using methods similar to those in Lemma 6 we have

$$\int_{Q_R} \phi(|U(x)| \lambda^{-1}) \sigma(x) \, dx \leqq C_{14} R^n 2^{-nN} \exp (C_{13} \xi^{1/(1-\mu_2)}) ,$$

from which it follows easily that

$$\|u - P_N u\|_{\phi,\sigma,Q_R} \leqq C_{15} (R^n 2^{-nN})^{1/tp'} \times$$
$$\times (\log 2^{nN} - \log R^n - \log C_{14})^{1+a-r/n-1/\nu} \|u\|_{r,p,\varrho,Q_R} .$$

Since $C_0^\infty(Q_R)$ is dense in $W_{0,\varrho}^{r,p}(Q_R)$, the proof is complete.

We now come to the main result of this section.

Theorem 2. *Let Ω be an unbounded open set in $\mathbf{R}^n$, let (1) and (2) hold, let the weight functions ϱ and σ satisfy condition (A) with $\varrho \in L^1_{loc}(\Omega)$, and let ϕ be defined by (10) and (11). Let*

$$I_0 : W_{0,\varrho}^{r,p}(\Omega) \to L_\sigma^\phi(\Omega)$$

be the natural embedding and assume that $\eta_R = O(R^{-n})$ as $R \to \infty$, where $\eta_R = \sup \{|\Omega \cap B(x, 1)|/|B(x, 1)| : \max_{1 \leqq i \leqq n} |x_i| > R\}$. Then as $s \to \infty$,

$$a_s(I_0) = O(s^{-(r/n-1/p)/2} (\log s)^{1+a-r/n-1/\nu}) .$$

Proof. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be such that $0 \leqq \psi(x) \leqq 1$ for all $x \in \mathbf{R}^n$, $\psi(x) = 1$ if $\max_{1 \leqq i \leqq n} |x_i| \leqq 4/3$, $\psi(x) = 0$ if $\max_{1 \leqq i \leqq n} |x_i| \geqq 5/3$. For each $k \in \mathbf{N}$ put $\psi_k(x) = \psi(2x/k)$, $\Omega_k = \{x \in \Omega : \max_{1 \leqq i \leqq n} |x_i| < k/2\}$, $\tilde{\Omega}_k = \{x \in \Omega : \max_{1 \leqq i \leqq n} |x_i| > k/2\}$.

For each $u \in C_0^\infty(\Omega)$, let $\tilde{u}(x) = u(x)$ if $u \in \Omega$, $\tilde{u}(x) = 0$ otherwise, $(E_k \tilde{u})(x) = \psi_k(x) \tilde{u}(x)$. Then $E_k \tilde{u} \in C_0^\infty(Q_{2k})$, $\tilde{u} - E_k \tilde{u} \in C_0^\infty(\tilde{\Omega}_k)$. Noticing that there is a constant C_{16} such that for all $x \in \mathbf{R}^n$, all $k \in \mathbf{N}$ and all $\alpha \in \mathbf{N}_0^n$ with $|\alpha| \leqq r$,

$$|D^\alpha \psi_k(x)| \leqq C_{16} ,$$

and using the Leibniz formula, we have

$$\|E_k \tilde{u}\|_{r,p,\varrho,\Omega} \leqq C_{17} \|\tilde{u}\|_{r,p,\varrho,Q_{2k}}$$

and

$$\|\tilde{u} - E_k \tilde{u}\|_{r,p,\varrho,\Omega} \leqq C_{17} \|\tilde{u}\|_{r,p,\varrho,\tilde{\Omega}_k} .$$

For given even $N \in \mathbf{N}$, let $k_0 = 2^{N/2}$ and define a map $\tilde{P}_N$ by

$$\tilde{P}_N u(x) = \begin{cases} P_N(E_{k_0}\tilde{u})(x), & u \in Q_{2k_0} \\ 0 & \text{otherwise}. \end{cases}$$

By Lemmas 6 and 7 we have

$$\begin{aligned} &\|u - \tilde{P}_N u\|_{\phi,\sigma,\Omega} = \|\tilde{u} - \tilde{P}_N u\|_{\phi,\sigma,\Omega} \leqq \\ &\leqq \|\tilde{u} - E_{k_0}\tilde{u}\|_{\phi,\sigma,\Omega_{k_0}} + \|E_{k_0}\tilde{u} - P_N E_{k_0}\tilde{u}\|_{\phi,\sigma,Q_{2k_0}} \leqq \\ &\leqq C_{12}(k_0/2)^{-n/tp'}(\log (k_0/2)^n - \log C_{10})^{1+a-r/n-1/v} \|\tilde{u} - E_{k_0}\tilde{u}\|_{r,p,\varrho,\Omega_{k_0}} + \\ &+ C_{15}((2k_0)^n 2^{-nN})^{1/tp'}(\log 2^{nN} - \log (2k_0)^n - \log C_{14})^{1+a-r/n-1/v} \times \\ &\times \|E_{k_0}\tilde{u}\|_{r,p,\varrho,Q_{2k_0}} \leqq \\ &\leqq C_{12}\, 2^{-n(N/2-1)/tp'}(\log (2^{-n+nN/2}) - \\ &- \log C_{10})^{1+a-r/n-1/v}\, C_{17}\|\tilde{u}\|_{r,p,\varrho,\Omega_{k_0}} + \\ &+ C_{15}\, 2^{-n(N/2-1)/tp'}(\log 2^{nN} - \log 2^n - \log 2^{nN/2} - \\ &- \log C_{14})^{1+a-r/n-1/v} \|\tilde{u}\|_{r,p,\varrho,Q_{2k_0}}\,. \end{aligned}$$

Thus we see that

$$\|u - \tilde{P}_N u\|_{\phi,\sigma,\Omega} \leqq C_{18}\, 2^{-nN(r/n-1/p)/2}(\log 2^{nN})^{1+a-r/n-1/v} \|u\|_{r,p,\varrho,\Omega}$$

when N is large enough. The result follows.

Remarks. 1. When $\varrho(s) = \sigma(x) = 1$ for all $x \in \Omega$, the approximation numbers of the embedding map $I_0 : W^{r,p}_{0,\varrho}(\Omega) \to L^{\phi}_{\sigma}(\Omega)$ satisfy

$$a_s(I_0) = O(s^{-(r/n-1/p)/2}(\log s)^{1-r/n-1/v})\,.$$

In particular, when $1/p = r/n$,

$$a_s(I_0) = O((\log s)^{1-r/n-1/v})\,.$$

2. Under the same hypotheses as Theorem 2, we can also show that $e_s(I_0)$ satisfies the same estimates as $a_s(I_0)$.

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References

[1] *M. S. Birman, M. S. Solomjak:* Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory. Amer. Math. Soc. Transl. (2) *114* (1980), 1—132.

[2] *D. E. Edmunds, R. M. Edmunds:* Entropy and approximation numbers of embeddings in Orlicz spaces. J. Lond. Math. Soc. (2) *32* (1985), 528—538.

[3] *D. E. Edmunds, W. D. Evans:* Orlicz and Sobolev spaces on unbounded domains. Proc. Roy. Soc. *A342* (1975), 373—400.

[4] *D. E. Edmunds, W. D. Evans:* Spectral theory and differential operators. Oxford University Press, Oxford, 1987.
[5] *D. E. Edmunds, A. Kufner, Jiong Sun:* Extensions of functions in weighted Sobolev spaces. Rend. Accad. Nat. Sci. XL Mem. Mat. (5) *14, 17* (1990), 327—339.
[6] *M. Krbec, L. Pick:* On imbeddings between weighted Orlicz spaces. Zeitschrift Anal. Anwend. (1) *10* (1991), 105—116.
[7] *A. Kufner:* Weighted Sobolev spaces. John Wiley and Sons Ltd., 1985.
[8] *A. Kufner:* Equadiff 6, Boundary value problems in weighted spaces. Lecture Notes in Mathematics 1192, Springer-Verlag, Berlin–Heidelberg–New York, 1986, pp. 35—48.
[9] *M. Namasivayam:* Estimates of entropy numbers of embeddings of Sobolev spaces on unbounded domains. Quart. J. Math., Oxford (2) *36* (1985), 231—242.
[10] *E. W. Stredulinsky:* Weighted inequalities and degenerate elliptic partial differential equations. Lecture Notes in Mathematics 1074, Springer-Verlag, Berlin–Heidelberg–New York, 1984.
[11] *J.-O. Strömberg, R. L. Wheeden:* Fractional integrals on weighted H^p and L^p spaces. Trans. Amer. Math. Soc. *287* (1985), 293—321.
[12] *H. Triebel:* Interpolation theory, function spaces, differential operators. North-Holland, Amsterdam, 1978.
[13] *N. S. Trudinger:* On imbeddings into Orlicz spaces and some applications. J. Math. Mech., *17* (1967), 473—484.

Souhrn

APROXIMAČNÍ ČÍSLA A ČÍSLA ENTROPIE VNOŘENÍ DO VÁHOVÝCH ORLICZOVÝCH PROSTORŮ

D. E. EDMUNDS, JIONG SUN

V práci jsou odvozeny horní odhady pro aproximační čísla a čísla entropie vnoření Sobolevových prostorů s vahou do vhodných váhových Orliczových prostorů. Výsledky se týkají případu, kdy definiční oblast je omezená, a některých neomezených oblastí.

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