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A DYNAMICAL SYSTEM IN A HILBERT SPACE WITH A WEAKLY ATTRACTIVE NONSTATIONARY POINT

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Summary. A differential equation in a Hilbert space with all solutions bounded but with no finite nontrivial invariant measure is constructed. In fact, it is shown that all solutions to this equation converge weakly to the origin, nonetheless, there is no stationary point. Moreover, no solution has a non-empty Ω-set.

Keywords: differential equations in Hilbert spaces, invariant measures, Ω-sets

AMS classification: 34G20

In this paper we construct a differential equation with a bounded Lipschitz continuous right-hand side in an infinite-dimensional Hilbert space, all solutions of which converge weakly to the origin, nevertheless, there is no stationary point. Further, it is shown that the Ω-set of every solution is empty and there is no finite nontrivial measure invariant under the flow defined by the equation. The question whether such a system exists was posed to us by J. Zabczyk. Although the problem seems purely deterministic, it is intimately related to the problem of existence of invariant measures for stochastic evolution equations. G. Da Prato, D. Gątarek and J. Zabczyk in [1] (Theorem 4) established a theorem, which in the very particular deterministic case reads as follows: Consider a differential equation

$$\dot{x} = Ax + f(x)$$

in a Hilbert space H, where f is a Lipschitz continuous mapping in H and A generates a *compact* strongly continuous semigroup e^{At} on H. If at least one bounded mild solution to this equation exists, then there exists a (not necessarily unique) invariant measure.

Our results show that the assumption on the compactness of e^{At} in the above theorem cannot be omitted. Moreover, our example indicates that there is no immediate relation between the invariant measures for the Galerkin approximations of an equation and the invariant measures for the equation itself. In Remark following the proof of Theorem 3 we will prove that the finite dimensional approximations to the equation under consideration have invariant measures, these measures have a weak* limit (in the space of finite Borel measures on the Hilbert space, provided this Hilbert space is equipped with its weak topology), nonetheless, the limit measure is not invariant.

Main results

We will prove the following three theorems.

Theorem 1. *Let H be an infinite-dimensional Hilbert space. Then there exists a bounded Lipschitz continuous mapping $f: H \longrightarrow H$ such that all solutions of the equation*

$$\text{(E)} \qquad \dot{x} = f(x)$$

converge weakly to 0 as $t \to \infty$, nevertheless, the equation has no stationary solution, that is, $f(x) \neq 0$ for any $x \in H$.

Let $x(.)$ be a solution of (E). Denote by $\Omega(x(.))$ the *Ω-set* of x, that is, the set of all $y \in H$ such that

$$y = \lim_{n \to \infty} x(t_n)$$

for some $t_n \to \infty$.

Theorem 2. *Let H be a Hilbert space. Let $f: H \longrightarrow H$ be a Lipschitz continuous mapping fulfilling $f(0) \neq 0$. Assume that all solutions of* (E) *converge weakly to 0 as $t \to \infty$. Then $\Omega(z)$ is empty for any solution z of* (E).

Let us note that if $f: H \longrightarrow H$ is Lipschitz continuous, then for any $\xi \in H$ there exists a unique solution $x(.,\xi)$ of the equation (E) fulfilling $x(0,\xi) = \xi$ and defined on the whole real line. Let us denote the flow induced by (E) on H by $(T_t, t \in \mathbf{R})$, that is,

$$T_t : H \longrightarrow H, \quad \xi \longmapsto x(t,\xi), \qquad t \in \mathbf{R}.$$

We say that a nonnegative Borel measure $\nu \neq 0$ on H is invariant with respect to (E) provided $T_t\nu = \nu$ for all $t \in \mathbf{R}$, where we set $T_t\nu(A) = \nu(T_t^{-1}A)$ for any $A \subseteq H$ Borel measurable. As T_t is in fact a homeomorphism, the condition $T_t\nu = \nu$ is equivalent to $\nu(T_t A) = \nu(A)$ for all Borel sets $A \subseteq H$.

Theorem 3. *Let H be a Hilbert space, let $f\colon H \longrightarrow H$ be a Lipschitz continuous mapping. Assume that all solutions of* (E) *have empty Ω-sets. Then there is no (nontrivial nonnegative) finite Borel Radon measure on H invariant with respect to* (E).

Remark. If the space H is separable, then all finite Borel measures on H are Radon by the Ulam theorem (see e.g. [2], Theorem I.3.1), hence if all solutions to (E) in a separable Hilbert space have empty Ω-sets, then there is no finite Borel measure invariant with respect to (E).

Remark. If f is the particular mapping we will construct in the course of the proof of Theorem 1 then the nonexistence of any finite Borel measure invariant with respect to (E) can be proved without the assumption of radonness even in the non-separable case.

The paper is organized as follows. In Section I, the mapping f the existence of which is claimed in Theorem 1 is constructed, in Section II it is established that f is lipschitzian, and in Section III the behaviour of solutions to (E) is investigated. In the last section, the proofs of Theorems 2 and 3 are provided.

I. Construction of the mapping f

First, let us note that it suffices to prove Theorem 1 in the case of a separable space H. Indeed, let H be a non-separable Hilbert space, then we can split H into an orthogonal sum $H = H_1 \oplus H_2$, H_1 being an infinite-dimensional separable Hilbert space. Let $f_1 : H_1 \longrightarrow H_1$ be a Lipschitz mapping such that for the equation

$$\dot{x} = f_1(x) \tag{1.1}$$

in H_1 all the assertions of Theorem 1 hold. Further, define

$$f_2\colon H_2 \longrightarrow H_2, \quad x \longmapsto -\frac{\|x\|x}{1+\|x\|^2}.$$

Then any solution x to the equation

$$\dot{x} = f_2(x) \tag{1.2}$$

in H_2 is of the form

$$x(t) = \frac{\left(\left(t + \frac{1}{\|x(0)\|} - \|x(0)\|\right)^2 + 4\right)^{\frac{1}{2}} - \left(t + \frac{1}{\|x(0)\|} - \|x(0)\|\right)}{2} \frac{x(0)}{\|x(0)\|}$$

$$= \frac{2}{\left(\left(t + \frac{1}{\|x(0)\|} - \|x(0)\|\right)^2 + 4\right)^{\frac{1}{2}} + t + \frac{1}{\|x(0)\|} - \|x(0)\|} \frac{x(0)}{\|x(0)\|},$$

$t \in \mathbf{R}$, provided $x(0) \neq 0$, and $x \equiv 0$ if $x(0) = 0$, hence $\|x(t)\| \longrightarrow 0$ as $t \to +\infty$. Let $Q_i \colon H \longrightarrow H_i$ stand for the orthogonal projection onto H_i, $i = 1, 2$. Define

$$f \colon H \longrightarrow H, \quad x \longmapsto f_1(Q_1 x) \oplus f_2(Q_2 x),$$

then f is Lipschitz continuous and $f(0) = (f_1(0), 0) \neq 0$. If x solves (E), then $Q_1 x$, $Q_2 x$ are solutions to (1.1), (1.2), respectively, hence all solutions to (E) have the required properties.

Therefore, up to the end of the proof of Theorem 1 we will content ourselves to the case of a separable space H. Without loss of generality we will work in the space ℓ_2, where ℓ_2 denotes as usual the Hilbert space of all sequences $\{x_i\}_{i=1}^{\infty}$ of real numbers with finite norm $\|x\| = \left(\sum\limits_{i=1}^{\infty} x_i^2\right)^{1/2}$. We start with introducing some notation.

Notation. Denote by ℓ^+ the set of all elements $x \in \ell_2$ fulfilling $x_1 \geqslant 0$, $x \neq 0$; by $\operatorname{Int} M$ the interior and by $\operatorname{cl} M$ the closure of a set M. Let e be the unit vector $e_1 = 1$, $e_i = 0$ for $i = 2, \ldots$, orthogonal to the set $Z = \{x \in \ell_2 \colon x_1 = 0\}$ and let P be the orthogonal projection on Z; i.e. $Px = x - x_1 e$. Let us fix a real number r, $0 < r < 1$, and define the following domains:

$$\begin{aligned}
A_1 &= \{x \in \ell^+ \colon \|x\| = 1\}, \\
A_2 &= \{x \in \ell_2 \colon x_1 < 0\} \\
A_3 &= \{x \in \operatorname{cl} \ell^+ \colon \|x\| \leqslant r/5\}, \\
A_4 &= \{x \in \ell^+ \colon \|x\| \geqslant r/5, \|Px\| \leqslant r - 2x_1\}, \\
A_5 &= \{x \in \ell^+ \colon \|Px\| \geqslant r - 2x_1, \|x\| \leqslant r\}, \\
A_6 &= \{x \in \ell^+ \colon r \leqslant \|x\| \leqslant 1\}, \\
A_7 &= \{x \in \ell^+ \colon \|x\| > 1\}, \\
Z_1 &= \{x \in \ell_2 \colon x_1 = 0, r/5 \leqslant \|x\| \leqslant 1\}.
\end{aligned}$$

Further, let us introduce a set of mappings:

$$\begin{aligned} P_1 &: \ell^+ \longrightarrow \ell^+, & P_1 x &= \frac{x}{\|x\|}; \\ P_3 &: \ell^+ \longrightarrow \ell^+, & P_3 x &= \frac{rx}{5\|x\|}; \\ P_4 &: \ell^+ \longrightarrow \ell^+, & P_4 x &= \frac{rx}{\|Px\| + 2x_1}; \\ P_5 &: \ell^+ \longrightarrow \ell^+, & P_5 x &= \frac{rx}{\|x\|}. \end{aligned}$$

Note that we have $P_4(\lambda x) = P_4 x$ for all positive λ and $\|PP_4 x\| + 2(P_4 x)_1 = r$, which means that $P_4 x$ is an element of the cone $\{x : \|Px\| + 2x_1 = r\}$.

Assume that the following hypothesis is fulfilled:

(H_1) *Let* q: $\mathrm{cl}\,\ell^+ \longrightarrow \mathbf{R}$ *be a bounded continuous function such that* $q \equiv 0$ *on* $A_1 \cup A_3 \cup A_7 \cup Z_1$.

Now we can define the mapping f. First we define a mapping F: $A_1 \longrightarrow \ell_2$ by

$$(F(x))_i = x_{i-1} - x_i \alpha(x) \qquad \text{for } i = 1, \ldots, \tag{1.3}$$

where we set $x_0 = 0$ and

$$\alpha : A_1 \longrightarrow \mathbf{R}, \quad x \longmapsto \sum_{j=1}^{\infty} x_{j-1} x_j. \tag{1.4}$$

Now we define:

(1.5) On A_1:

$$f(x) = F(x).$$

(1.6) On A_2:

$$f(x) = \min(1, \|Px\|) F(P_1 Px) + \left(\frac{r}{5} - \|Px\|\right)^+ e + \\ + \frac{x_1}{(1 + |x_1|)(1 + \|Px\|)} Px,$$

where $a^+ = \max(a, 0)$ is the positive part of a number a.

(1.7) On A_3:

$$f(x) = \begin{cases} \|Px\| F(P_1 Px) + \left(\frac{r}{5} - \|Px\|\right) e & \text{for } Px \neq 0, \\ \frac{r}{5} e & \text{for } Px = 0. \end{cases}$$

(1.8) On A_4:

$$f(x)=\begin{cases}\|Px\|F(P_1Px)+\left(\frac{r}{5}-\|PP_3x\|\right)\frac{\|x-P_4x\|}{\|P_3x-P_4x\|}e+ & \\ \quad +q(x)x & \text{for } Px\neq 0,\\ \frac{r}{5}\frac{\|x-P_4x\|}{\|P_3x-P_4x\|}e+q(x)x & \text{for } Px=0.\end{cases}$$

(1.9) On A_5:

$$f(x)=\begin{cases}\|PP_4x\|\frac{\|x-P_5x\|}{\|P_4x-P_5x\|}F(P_1Px)+ & \\ \quad +r\frac{\|x-P_4x\|}{\|P_5x-P_4x\|}F(P_1x)+q(x)x & \text{for } x_1>0, Px\neq 0,\\ r\frac{\|x-P_4x\|}{\|P_5x-P_4x\|}F(P_1x)+q(x)x & \text{for } x_1>0, Px=0,\\ rF(P_1x) & \text{for } x_1=0.\end{cases}$$

(Take into account that $\|x\|=r$, $P_1Px=P_1x$ and $\|PP_4x\|=r$ in the last case.)

(1.10) On A_6:

$$f(x)=\|x\|F(P_1x)+q(x)x.$$

(1.11) On A_7:

$$f(x)=F(P_1x).$$

It can be easily checked that f is a well-defined bounded continuous function on ℓ_2 since q fulfils (H_1).

II. Lipschitz continuity of f

Everywhere in this and the next section, the symbols f, F, α will be reserved for the functions defined in Section I. Let us add the following supposition about the function q.

(H_2) *Let $q\colon \mathrm{cl}\,\ell^+\longrightarrow \mathbf{R}$ be a Lipschitz continuous function.*

In the present section we aim at establishing the Lipschitz property of the function f.

Proposition 1. *Assume that the function q fulfils* (H_1) *and* (H_2). *Then $f\colon \ell_2\longrightarrow \ell_2$ is a Lipschitz continuous mapping.*

Proof. To start with, recall that f is continuous on ℓ_2. Let A_i, A_j be neighbouring domains, $x\in \operatorname{Int} A_i$, $y\in \operatorname{Int} A_j$, then one can find $\lambda\in(0,1)$ such that

$\lambda x+(1-\lambda)y \in A_i \cap A_j$. Hence it is sufficient to establish the lipschitzianity of f on A_k, $k=1, \ldots, 7$.

Further, let us note that the inequality

$$\|v\|\,\|P_1 u - P_1 v\| \leqslant 2\|u-v\| \tag{2.1}$$

holds for all $u, v \in \ell^+$. Indeed, we have

$$\begin{aligned}\|v\|\left\|\frac{u}{\|u\|}-\frac{v}{\|v\|}\right\| &= \|v\|\frac{\big\|u\|v\|-v\|u\|\big\|}{\|u\|\,\|v\|}\\ &\leqslant \|v\|\frac{\big\|u(\|v\|-\|u\|)+(u-v)\|u\|\big\|}{\|u\|\,\|v\|}\\ &\leqslant \big|\|v\|-\|u\|\big|+\|u-v\|.\end{aligned}$$

The estimate (2.1) yields

$$\|P_1 u - P_1 v\| \leqslant \frac{2}{\kappa}\|u-v\|, \qquad u, v \in \ell^+,\ \|v\| \geqslant \kappa > 0, \tag{2.2}$$

hence we obtain

(2.3) *The mappings P_i, $i=1, 3, 5$ are Lipschitz continuous on $\ell^+ \setminus \operatorname{Int} A_3$.*

Taking into account that $\|Px\|+2x_1 \geqslant \|x\| \geqslant r/5$ for $x \in \ell^+ \setminus A_3$ we obtain by an analogous argument

(2.4) *The mapping P_4 is Lipschitz continuous on $\ell^+ \setminus \operatorname{Int} A_3$.*

Now, the very definition implies that α is Lipschitz continuous on A_1,

$$|\alpha(x)-\alpha(y)| \leqslant 2\|x-y\|, \qquad |\alpha(x)| \leqslant 1$$

for any $x, y \in A_1$. This yields

$$\|f(x)\| = \|F(x)\| = (1-\alpha(x)^2)^{1/2} \leqslant 1, \qquad x \in A_1, \tag{2.5}$$

and

$$\|f(x)-f(y)\| = \|F(x)-F(y)\| \leqslant 4\|x-y\|, \qquad x, y \in A_1. \tag{2.6}$$

Formula (2.6) means that f is lipschitzian on A_1, moreover (2.6) and (2.1) yield that the following estimates

$$\|x\|\,\|F(P_1 x)-F(P_1 y)\| \leqslant 8\|x-y\|, \tag{2.7}$$

$$\big\|\|x\|F(P_1 x)-\|y\|F(P_1 y)\big\| \leqslant 9\|x-y\|, \tag{2.8}$$

$$\big\|\min(1,\|x\|)F(P_1 x)-\min(1,\|y\|)F(P_1 y)\big\| \leqslant 9\|x-y\| \tag{2.9}$$

hold for all $x, y \in \ell^+$. These estimates and the Lipschitz continuity of q immediately imply

(2.10) *The mapping f is Lipschitz continuous on each of the sets A_1, A_3, A_6 and A_7.*

Further, we want to prove

(2.11) *The function f is lipschitzian on A_2.*

Due to the obvious inequality $|a^+ - b^+| \leqslant |a-b|$ valid for any two real numbers a, b and to (2.9) it suffices to investigate the third term on the right-hand side in the definition of f on A_2. Thus, take $x, y \in A_2$ and set for brevity $\Gamma = (1+|x_1|)(1+\|Px\|)$, $G = (1+|y_1|)(1+\|Py\|)$, then

$$\begin{aligned}
\left\| \frac{x_1 Px}{\Gamma} - \frac{y_1 Py}{G} \right\| &= \frac{1}{\Gamma G} \|(Gx_1)Px - (\Gamma y_1)Py\| \\
&\leqslant \frac{1}{\Gamma G}\|Gx_1(Px - Py)\| + \frac{\|Py\|}{\Gamma G}|Gx_1 - \Gamma y_1| \\
&\leqslant \|Px - Py\| + \frac{1}{\Gamma(1+|y_1|)}\Big\{|Gx_1 - (1+\|Px\|)(1+|y_1|)x_1| + \\
&\qquad + \big|(1+\|Px\|)(1+|y_1|)x_1 - \Gamma x_1\big| + \Gamma|x_1 - y_1|\Big\} \\
&\leqslant \|Px - Py\| + \frac{|x_1|}{\Gamma}\big|\|Px\| - \|Py\|\big| + \frac{1}{1+|y_1|}\big||y_1| - |x_1|\big| + \frac{1}{1+|y_1|}|x_1 - y_1| \\
&\leqslant 2\|Px - Py\| + 2|x_1 - y_1|,
\end{aligned}$$

and (2.11) follows.

To proceed further, let us realize that

$$\begin{aligned}
\|P_3x - P_4x\| &= \left\| \frac{rx}{5\|x\|} - \frac{rx}{\|Px\| + 2x_1} \right\| \\
&= \frac{r}{5(\|Px\| + 2x_1)} \big|5\|x\| - \|Px\| - 2x_1\big| \\
&\geqslant \frac{r}{15\|x\|}\big|5\|x\| - 3\|x\|\big| = \frac{2r}{15},
\end{aligned}$$

as, obviously, $\|Px\| + 2x_1 \leqslant 3\|x\|$, $x \in \ell^+$. By (2.3) and (2.4), the mappings P_3, P_4 are lipschitzian on A_4, so the above estimate yields easily that the function $x \mapsto 1/\|P_3 - P_4\|$ is Lipschitz continuous on A_4, hence also

$$x \longmapsto \left(\frac{r}{5} - \|PP_3x\|\right) \frac{\|x - P_4x\|}{\|P_3x - P_4x\|}$$

is Lipschitz continuous on A_4 as a product of bounded Lipschitz functions. Using this together with (2.8) and (H_2) we get

(2.12) ***The mapping f is Lipschitz continuous on A_4.***

It remains to investigate the behaviour of f on A_5. We start with establishing the following two estimates:

$$\|P_4x - P_5x\| \geqslant \frac{1}{3}x_1, \qquad x \in A_5, \tag{2.13}$$

$$\big|r - \|PP_4x\|\big| \leqslant 2x_1, \qquad x \in A_5. \tag{2.14}$$

The proof of (2.13) is straightforward:

$$\begin{aligned}\|P_4x - P_5x\| &= \left\| \frac{rx}{\|Px\| + 2x_1} - \frac{rx}{\|x\|} \right\| \\ &= \frac{r}{\|Px\| + 2x_1}\big|\|Px\| + 2x_1 - \|x\|\big| \\ &\geqslant \frac{rx_1}{\|Px\| + 2x_1} \geqslant \frac{rx_1}{3\|x\|} \geqslant \frac{1}{3}x_1;\end{aligned}$$

we have used the obvious fact that $\|x\| \leqslant \|Px\| + x_1 \leqslant \|Px\| + 2x_1 \leqslant 3\|x\| \leqslant 3r$ for every $x \in A_5$. Further,

$$\big|r - \|PP_4x\|\big| = r\left|1 - \frac{\|Px\|}{\|Px\| + 2x_1}\right| = \frac{2rx_1}{\|Px\| + 2x_1} \leqslant 2x_1,$$

since $\|Px\| + 2x_1 \geqslant r$ on A_5.

As the next step, let us realize that the points x, P_4x, P_5x lie on a line, hence $\|P_4x - P_5x\| - \|x - P_4x\| = \|x - P_5x\|$, $x \in A_5$, so we can write

$$f(x) = \begin{cases} f^*(x) + \|PP_4x\|F(P_1Px) + q(x)x & \text{for } Px \neq 0, \\ f^*(x) + q(x)x & \text{for } Px = 0 \end{cases} \tag{2.15}$$

provided $x \in \operatorname{Int} A_5$, where we set

$$f^*(x) = \begin{cases} \big(rF(P_1x) - \|PP_4x\|F(P_1Px)\big)\dfrac{\|x - P_4x\|}{\|P_5x - P_4x\|} & \text{for } Px \neq 0, \\[2ex] rF(P_1x)\dfrac{\|x - P_4x\|}{\|P_5x - P_4x\|} & \text{for } Px = 0. \end{cases}$$

The term $q(x)x$ on the right-hand side of (2.15) is Lipschitz continuous by (H_2). From (2.4) and (2.8) it is easy to see that the term $\|PP_4x\|F(P_1Px)$ (defined as 0 if $Px = 0$) is Lipschitz continuous as well if we take into account that $\|PP_4x\| \leqslant \|Px\|$ on A_5 (since $\|Px\| + 2x_1 \geqslant r$ for $x \in A_5$). Let us investigate the function f^*. Define

$$h(x) = \begin{cases} rF(P_1x) - \|PP_4x\|F(P_1Px) & \text{for } Px \neq 0, \\ rF(P_1x) & \text{for } Px = 0. \end{cases}$$

Again h is a (bounded) Lipschitz continuous function on A_5. We can split

$$\begin{aligned}
f^*(x) - f^*(y) &= \frac{\|x - P_4x\|}{\|P_5x - P_4x\|} h(x) - \frac{\|y - P_4y\|}{\|P_5y - P_4y\|} h(y) \\
&= \frac{\|y - P_4y\|}{\|P_5y - P_4y\|}\Big(h(x) - h(y)\Big) + \left(\frac{\|x - P_4x\|}{\|P_5x - P_4x\|} - \frac{\|y - P_4y\|}{\|P_5y - P_4y\|}\right) h(x) \\
&\equiv U(x,y) + V(x,y)
\end{aligned}$$

for $x, y \in \operatorname{Int} A_5$. The points P_4y, y, P_5y lie on a line, so $\|y - P_4y\| \leqslant \|P_5y - P_4y\|$ for $y \in A_5$, therefore

$$\|U(x,y)\| \leqslant \|h(x) - h(y)\|. \tag{2.16}$$

By (2.3), (2.4) there is a constant K such that for all $x, y \in \operatorname{Int} A_5$ the estimate

$$\left| \frac{\|x - P_4x\|}{\|P_5x - P_4x\|} - \frac{\|y - P_4y\|}{\|P_5y - P_4y\|} \right| \leqslant \frac{K\|x - y\|}{\|P_5x - P_4x\|} \tag{2.17}$$

holds. Indeed, one has

$$\begin{aligned}
&\left| \frac{\|x - P_4x\|\,\|P_5y - P_4y\| - \|y - P_4y\|\,\|P_5x - P_4x\|}{\|P_5x - P_4x\|\,\|P_5y - P_4y\|} \right| \\
&\qquad \leqslant \frac{\|x - y - P_4(x-y)\|}{\|P_5x - P_4x\|} \\
&\qquad\quad + \frac{\|y - P_4y\|}{\|P_5x - P_4x\|\,\|P_5y - P_4y\|} \Big|\|P_5y - P_4y\| - \|P_5x - P_4x\|\Big| \\
&\qquad \leqslant \frac{\|x - y - P_4(x-y)\|}{\|P_5x - P_4x\|} + \frac{\|P_5(y-x) - P_4(y-x)\|}{\|P_5x - P_4x\|},
\end{aligned}$$

since, as we have already noted, $\|y - P_4y\| \leqslant \|P_5y - P_4y\|$. Further, choose $x \in \operatorname{Int} A_5$; one can assume $Px \neq 0$ (the opposite case being simpler), then

$$\begin{aligned}
\|h(x)\| &\leqslant \big|r - \|PP_4x\|\big|\,\|F(P_1x)\| + \|PP_4x\|\,\|F(P_1x) - F(P_1Px)\| \\
&\leqslant \big|r - \|PP_4x\|\big| + \|Px\|\,\|F(P_1x) - F(P_1Px)\| \\
&\leqslant \big|r - \|PP_4x\|\big| + 8\|x - Px\| \leqslant 10x_1
\end{aligned}$$

by (2.5), (2.14) and (2.7). (Recall that $\|PP_4x\| \leqslant \|Px\|$.) The above estimate together with (2.17) yields

$$\begin{aligned}
\|V(x,y)\| &= \|h(x)\| \left| \frac{\|x - P_4x\|}{\|P_5x - P_4x\|} - \frac{\|y - P_4y\|}{\|P_5y - P_4y\|} \right| \\
&\leqslant 10x_1 K \frac{\|x - y\|}{\|P_5x - P_4x\|}
\end{aligned}$$

for all $x, y \in \operatorname{Int} A_5$. Applying (2.13) we obtain

$$V(x,y) \leqslant 30K\|x-y\|, \qquad x, y \in \operatorname{Int} A_5.$$

Invoking the estimate (2.16) we conclude that

(2.18) *The function f is Lipschitz continuous on A_5.*

This completes the proof of Proposition 1. $\square$

III. The behaviour of solutions of (E)

In this section we will use also the following assumption:

(H_3) *Let $q\colon \operatorname{cl} \ell^+ \longrightarrow \mathbf{R}$ be such that $q(x) \geqslant 8x_1/r$ for every $x \in A_5$ and that $\inf\{q(x)\colon x_1 \geqslant \delta > 0, r/5+\delta \leqslant \|x\| \leqslant 1-\delta\} > 0$ for all $\delta > 0$.*

It is worth noticing that functions satisfying the hypotheses (H_1), (H_2) and (H_3) do exist. For example, we can set

$$q(x) = \min\left\{\lambda(1-\|x\|)^+, \beta\left(\|x\|-\frac{r}{5}\right)^+, \frac{8x_1}{r}\right\}, \qquad x \in \operatorname{cl}\ell^+,$$

where $\lambda = 8/(1-r)$ and $\beta > 0$ will be specified later. Obviously, $1-\|x\| \geqslant 1-r$ and $8x_1/r \leqslant 8$ on A_5, hence

$$\lambda(1-\|x\|) \geqslant \frac{8x_1}{r}, \qquad x \in A_5.$$

Since $\inf\{\|x\|\colon x \in A_5\} \geqslant r/\sqrt{5}$, we obtain

$$1-\|x\| \leqslant 1-\frac{r}{\sqrt{5}}, \qquad \|x\|-\frac{r}{5} \geqslant \frac{r}{5}(\sqrt{5}-1)$$

for $x \in A_5$, so β can be chosen such that

$$\lambda(1-\|x\|) \leqslant \beta\left(\|x\|-\frac{r}{5}\right), \qquad x \in A_5,$$

which yields $q(x) = 8x_1/r$ for $x \in A_5$.

We aim at establishing a proposition on the weak convergence of the solutions of (E) to the origin.

Proposition 2. *Let the assumptions (H_1), (H_2), (H_3) be fulfilled. Then all solutions of* (E) *converge weakly to* 0 *as $t \to \infty$.*

The proof will be done in several steps. First, we prove that the set A_1 is invariant, which makes it possible to use the particular form of the mapping F and to derive a formula for the solution $x(.,\xi)$ with $\xi \in A_1$. (Recall that $x(.,\xi)$ denotes the solution to (E) fulfilling $x(0,\xi)=\xi$.) This formula implies easily that Proposition 2 holds for solutions starting in A_1. According to the definition of f this result extends immediately to solutions $x(.,\xi)$ with $\xi \in A_7 \cup A_6 \cup Z_1$. In fact, the sets A_7, A_6 and Z_1 are positively invariant, and projecting the solutions starting at their points onto A_1 one obtains again a solution to (E). Further, we will prove successively: if $\xi_2 \in A_2$, then $x(.,\xi_2)$ enters A_3 in a finite time, if $\xi_3 \in A_3 \setminus Z_1$, then $x(.,\xi_3)$ enters $A_4 \cap \operatorname{Int} \ell^+$ in a finite time, if $\xi_4 \in A_4 \cap \operatorname{Int} \ell^+$, then $x(.,\xi_4)$ enters $A_5 \cap \operatorname{Int} \ell^+$ in a finite time, and finally, if $\xi_5 \in A_5 \cap \operatorname{Int} \ell^+$, then $x(.,\xi_5)$ enters $A_6 \cap \operatorname{Int} \ell^+$ in a finite time, which will complete the proof.

First, let us note that we can define the mapping F by (1.3) for any $x \in \ell_2$, $\|x\|=1$, and that the mapping P_1, $P_1 x = \|x\|^{-1}x$ is well-defined on $\ell_2 \setminus \{0\}$. We prove several useful identities for these extended functions. Namely, we claim that

$$(x, F(P_1 x)) = 0, \tag{3.1}$$

$$(Px, F(P_1 x)) = x_1^2 \frac{\alpha(P_1 x)}{\|x\|} \tag{3.2}$$

hold for all $x \in \ell_2$, $x \neq 0$, and

$$(e, F(P_1 P x)) = 0, \tag{3.3}$$

$$(x, F(P_1 P x)) = 0 \tag{3.4}$$

hold for all $x \in \ell_2$, $Px \neq 0$. Indeed, take an arbitrary $x \in \ell_2$, $x \neq 0$. Set $z = P_1 x$, hence $\|z\| = 1$ and by the definition of F one obtains

$$\begin{aligned}(x, F(P_1 x)) &= \|x\|(P_1 x, F(P_1 x)) = \|x\|(z, F(z)) = \|x\| \sum_{i=1}^{\infty} z_i F(z)_i \\ &= \|x\| \sum_{i=1}^{\infty} z_i \left(z_{i-1} - z_i \sum_{j=1}^{\infty} z_{j-1} z_j \right) = \|x\|(\alpha(z) - \|z\|^2 \alpha(z)) = 0.\end{aligned}$$

Further,

$$\begin{aligned}(Px, F(P_1 x)) &= (x - x_1 e, F(P_1 x)) = -x_1 (e, F(P_1 x)) \\ &= -x_1 (F(P_1 x))_1 = x_1 (P_1 x)_1 \alpha(P_1 x) = x_1^2 \frac{\alpha(P_1 x)}{\|x\|},\end{aligned}$$

thus (3.2) is valid. As $(Px)_1 = 0$ we have $(F(P_1Px))_1 = 0$ and (3.3) follows. Finally, we get

$$(x, F(P_1Px)) = (Px + x_1e, F(P_1Px)) = (Px, F(P_1Px)) + x_1(e, F(P_1Px)) = 0$$

by (3.1) and (3.3). Subtracting (3.2) from (3.1) and taking into account that $e = (x - Px)/x_1$ for arbitrary $x \neq 0$ with $x_1 \neq 0$ we obtain

$$(e, F(P_1x)) = -x_1\frac{\alpha(P_1x)}{\|x\|}; \tag{3.5}$$

obviously, (3.5) holds for any $x \in \ell_2$, $x \neq 0$.

The next step is to prove

(3.6) *The sets Z_1 and $A_7 \cap Z$ are invariant, that is, $T_t(Z_1) \subseteq Z_1$, $T_t(A_7 \cap Z) \subseteq A_7 \cap Z$ for all $t \in \mathbf{R}$.*

Towards this end, let us define an auxiliary mapping

$$g(x) = \min(1, \|x\|)F(P_1x), \qquad x \in \ell_2,\ \|x\| > \frac{r}{10}.$$

Certainly, g is a bounded Lipschitz mapping (cf. the proof of (2.9) which can be easily modified to the present situation); furthermore, $g = f$ on $A_1 \cup A_7 \cup Z_1$. Let $\tilde{x}$ be a solution to (E) fulfilling $\tilde{x}(t_0) \in Z_1 \cup (A_7 \cap Z)$ for some $t_0 \in \mathbf{R}$. Let y be a solution of the problem

$$\dot{y} = g(y), \quad y(t_0) = \tilde{x}(t_0). \tag{3.7}$$

There exists $\delta > 0$ such that y is defined on $I \equiv (t_0 - \delta, t_0 + \delta)$. (In particular, $\|y(t)\| > r/10$ for $t \in I$.) We have

$$\begin{aligned}\frac{\mathrm{d}\|y(t)\|^2}{\mathrm{d}t} &= 2(y(t), \dot{y}(t)) = 2(y(t), g(y(t)))\\ &= 2\min(1, \|y(t)\|)\,(y(t), F(P_1y(t)))\\ &= 0\end{aligned}$$

by (3.1), for all $t \in I$. This means

$$\|y(t)\| = \|y(t_0)\| = \|\tilde{x}(t_0)\|, \qquad t \in I, \tag{3.8}$$

moreover,

$$\begin{aligned}\dot{y}_1(t) &= (e, g(y(t))) = \min(1, \|y\|)\,(e, F(P_1y(t)))\\ &= -y_1(t)\min\left(1, \frac{1}{\|y(t)\|}\right)\alpha(P_1y(t))\end{aligned} \tag{3.9}$$

by (3.5). As $y_1(t_0) = 0$, the uniqueness of solutions to (3.9) yields that $y_1(t) = 0$ for any $t \in I$. This together with (3.8) implies that $y(t) \in Z_1 \cup (A_7 \cap Z)$. Due to the fact that $f = g$ on $Z_1 \cup (A_7 \cap Z)$ and to the uniqueness of solutions to (E) we have $y(t) = \tilde{x}(t)$, $t \in I$. Therefore, we have proved that if $\tilde{x}(t_0) \in Z_1 \cup (A_7 \cap Z)$ for some $t_0 \in \mathbf{R}$, then there exists $\delta = \delta(\tilde{x}(t_0)) > 0$ such that

$$\tilde{x}_1(t) = 0, \quad \|\tilde{x}(t)\| = \|\tilde{x}(t_0)\| \tag{3.10}$$

for $t \in (t_0 - \delta, t_0 + \delta)$. Finally, set

$$t_i = \inf\{s \in \mathbf{R} \colon \tilde{x} \text{ fulfils (3.10) on } [s, t_0]\},$$
$$t_s = \sup\{s \in \mathbf{R} \colon \tilde{x} \text{ fulfils (3.10) on } [t_0, s]\}.$$

Suppose that, say, $t_i > -\infty$, then $x(t_i) \in Z_1 \cup (A_7 \cap Z)$ by continuity, so $\tilde{x}$ fulfils (3.10) on $(t_i - \delta(\tilde{x}(t_i)), t_0]$, which contradicts the definition of t_i. Hence (3.10) holds for any $t \in \mathbf{R}$, moreover, the second equality in (3.10) implies that not only $Z_1 \cup (A_7 \cap Z)$ but both the sets Z_1, $A_7 \cap Z$ are invariant.

Using an analogous argument we will prove

(3.11) *The sets A_1 and A_7 are invariant.*

Consider a solution $\tilde{x}$ to (E) such that $\tilde{x}(t_0) \in A_1 \cup A_7$ for some $t_0 \in \mathbf{R}$. We can assume that $\tilde{x}_1(t_0) > 0$, the other case being covered by (3.6). Again, let y be a solution to (3.7), find $\Delta > 0$ so that y may be defined on $J \equiv (t_0 - \Delta, t_0 + \Delta)$ and $\|y(t)\| \geqslant r/5$ for $t \in J$. Then

$$\frac{\mathrm{d}\|y(t)\|^2}{\mathrm{d}t} = 2\big(y(t), \dot{y}(t)\big) = 2\big(y(t), F(P_1 y(t))\big) = 0$$

by (3.1), so $\|y(t)\| = \|y(t_0)\|$ for $t \in J$. Furthermore, $y_1(t) > 0$ for $t \in J$, since otherwise y will reach the set $A_1 \cap Z$ or $A_7 \cap Z$ which is impossible according to (3.6). Hence we can complete the proof of (3.11) proceeding as in the proof of (3.6).

Now we will investigate the solutions to (E) with initial data from A_1. Let $\xi \in A_1$, then the solution $x = x(.,\xi)$ starting at ξ fulfils $x(t) \in A_1$ for all $t \in \mathbf{R}$ by (3.11), hence

$$x(t,\xi) = \xi + \int_0^t F(x(s,\xi))\,\mathrm{d}s, \qquad t \geqslant 0. \tag{3.12}$$

As F is given by (1.3) we can check easily that (3.12) holds if and only if

$$x_i(t,\xi) = \sum_{j=1}^{i} \xi_j \frac{t^{i-j}}{(i-j)!} \exp\Big\{-\int_0^t \alpha(x(s,\xi))\,\mathrm{d}s\Big\}, \qquad i = 1, 2, \ldots,\ t \in \mathbf{R}.$$

(Recall that α denotes the function defined by (1.4) and we set $x_0 \equiv 0$.) Let k be the index such that $\xi_1 = \ldots = \xi_{k-1} = 0$, $\xi_k \neq 0$. Then $x_i(.\,,\xi) \equiv 0$ for $1 \leqslant i \leqslant k-1$ and for every $l \geqslant 0$ we have

$$\lim_{t\to\infty} \frac{x_{k+l}(t)}{x_{k+l+1}(t)} = \lim_{t\to\infty} \frac{\sum_{j=k}^{k+l} \frac{\xi_j t^{k+l-j}}{(k+l-j)!}}{\sum_{j=k}^{k+l+1} \frac{\xi_j t^{k+l+1-j}}{(k+l+1-j)!}} = 0. \tag{3.13}$$

As $\|x(.\,,\xi)\| \equiv 1$, (3.13) yields that $\lim_{t\to\infty} x_i(t) = 0$, $i \in \mathbb{N}$. We have proved

(3.14) *If* $\xi \in A_1$ *then* $\operatorname{w-lim}_{t\to\infty} x(t,\xi) = 0$.

Here and in the sequel, we denote by w-lim the limit with respect to the weak topology of the space ℓ_2.

The result just proved has immediate consequences. First, take $\chi \in A_7$. By (3.11) (or by (3.6), if $\chi_1 = 0$) we know that

$$\|x(s,\chi)\| = \|\chi\|, \qquad x_1(s,\chi) \geqslant 0, \qquad \text{for } s \geqslant 0, \tag{3.15}$$

hence setting $y(t) = P_1 x(t\|\chi\|, \chi)$, $t \geqslant 0$, we obtain a function fulfilling $y(t) \in A_1$, $t \geqslant 0$. We prove that y solves (E). Indeed,

$$\begin{aligned}
\frac{\mathrm{d}y(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x(t\|\chi\|)}{\|x(t\|\chi\|)\|}\right) \\
&= \frac{1}{\|x(t\|\chi\|)\|^2}\Big\{\|x(t\|\chi\|)\|\,\|\chi\| F(P_1 x(t\|\chi\|)) \\
&\quad - \frac{\|\chi\|}{\|x(t\|\chi\|)\|}\Big(x(t\|\chi\|), F(P_1 x(t\|\chi\|))\Big) x(t\|\chi\|)\Big\} \\
&= F(P_1 x(t\|\chi\|)) = F(y(t))
\end{aligned}$$

by (3.15) and (3.1). Hence (3.14) yields that

$$y(t) = \frac{x(t\|\chi\|)}{\|x(t\|\chi\|)\|} = \frac{x(t\|\chi\|)}{\|\chi\|}$$

converges weakly to 0 as $t \to \infty$. We conclude

(3.16) *Let* $\chi \in A_7$, *then* $\operatorname{w-lim}_{t\to\infty} x(t,\chi) = 0$.

The same procedure yields also

(3.17) *If* $\xi \in Z_1$ *then* $\operatorname{w-lim}_{t\to\infty} x(t,\xi) = 0$.

Further, we want to apply an analogous trick to the solutions starting at A_6. Towards this end we establish

(3.18) *The set A_6 is positively invariant, that is, $T_t(A_6) \subseteq A_6$ for every $t \geqslant 0$.*

Taking $\zeta \in A_6 \cap \operatorname{Int} \ell^+$, we may assume $\|\zeta\| < 1$ since otherwise $\zeta \in A_1$. First, let us suppose that $\|\zeta\| > r$, i.e. $\zeta \in \operatorname{Int} A_6$, and set

$$\hat{t} = \sup\{s \geqslant 0 : x(r, \zeta) \in A_6 \cap \operatorname{Int} \ell^+ \text{ for all } r \in [0, s)\}.$$

As $\operatorname{Int} A_6$ is open, necessarily $\hat{t} > 0$. By (H_3) we have (setting $x = x(.\,, \zeta)$ for brevity)

$$\begin{aligned}
\frac{d\|x(t)\|^2}{dt} &= 2(x(t), \dot{x}(t)) \\
&= 2\|x(t)\|(x(t), F(P_1x(t))) + 2(x(t), q(x(t))x(t)) \\
&= 2q(x(t))\|x(t)\|^2 \geqslant 0
\end{aligned}$$

for $t \in [0, \hat{t})$. It follows that $\|x(.)\|$ is nondecreasing on $[0, \hat{t})$. Assume that $\hat{t} < \infty$, then $\|x(\hat{t})\| \geqslant \|x(0)\| \geqslant r$, but one can have neither $x_1(\hat{t}) \leqslant 0$ (as the set Z_1 is invariant) nor $\|x(\hat{t})\| \geqslant 1$ (as A_1 is invariant). Hence $x(\hat{t}) \in \operatorname{Int} A_6$, which easily yields a contradiction that proves (3.18) in the case $\|\zeta\| > r$. Finally, note that if $\|\zeta\| = r$ then

$$\frac{d\|x(.\,, \zeta)\|^2}{dt}(0) > 0$$

by (H_3), so $x(s_0, \zeta) \in \operatorname{Int} A_6$ for some $s_0 > 0$ and we can apply the above procedure.

Now, set $y(t) = P_1x(t, \zeta)$, $t \geqslant 0$. Again $y(t) \in A_1$ for all $t \geqslant 0$ and

$$\begin{aligned}
\frac{dy(t)}{dt} &= \frac{1}{\|x(t)\|^2}\Big\{\|x(t)\|(\|x(t)\|F(P_1x(t)) + q(x(t))x(t)) \\
&\qquad - \frac{1}{\|x(t)\|}\Big(x(t), F(P_1x(t)) + q(x(t))x(t)\Big)\, x(t)\Big\} \\
&= F(P_1x(t)) = F(y(t)),
\end{aligned}$$

thus y solves (E) and by (3.14) we have $\text{w-lim}\, y(t) = 0$. This implies

(3.19) *Let $\zeta \in A_6 \cap \operatorname{Int} \ell^+$. Then $\operatorname*{w-lim}_{t \to \infty} x(t, \zeta) = 0$.*

In the rest of the proof we aim at establishing that all solutions with initial conditions from A_i, $i = 2, \ldots, 5$, enter the set A_6 in a finite time.

(3.20) *For any $\xi \in A_2$ one can find $t_0 \in \mathbf{R}_+$ such that $x_1(t_0, \xi) = 0$, $\|Px(t_0, \xi)\| < \frac{r}{5}$; that is, $x(t_0, \xi) \in A_3$.*

Let us set

$$t_0 = \sup\{s \geqslant 0 : x(r, \xi) \in A_2 \text{ for all } r \in [0, s)\}.$$

The set A_2 is open, so obviously $t_0 > 0$. We want to prove that $t_0 < \infty$ and has the desired properties. To this end, note that the following estimates hold:

$$\begin{aligned}\frac{\mathrm{d}\|Px(t)\|^2}{\mathrm{d}t} &= 2(Px(t), P\dot{x}(t)) = 2(Px(t), \dot{x}(t)) \\ &= \frac{2x_1(t)}{(1+|x_1(t)|)(1+\|Px(t)\|)}\|Px(t)\|^2 \leqslant 0\end{aligned} \tag{3.21}$$

for $t \in I \equiv [0, t_0)$ by (1.6), (3.1) and (3.3), and

$$\dot{x}_1(t) = (e, \dot{x}(t)) = \Big(\frac{r}{5} - \|Px(t)\|\Big)^+ \geqslant 0, \qquad t \in I. \tag{3.22}$$

We have to discuss two cases. First, let $\|P\xi\| < r/5$. As $\|Px(.)\|$ is nonincreasing by (3.21), we obtain

$$\dot{x}_1(t) \geqslant \frac{r}{5} - \|P\xi\| > 0, \qquad t \in I$$

in accordance with (3.22), hence there is $t_1 < \infty$ such that $x_1(t_1) = 0$; obviously $t_1 = t_0$. Second, let $\|P\xi\| \geqslant r/5$. Set

$$t_2 = \inf\Big\{s \geqslant 0 \colon \|Px(s,\xi)\| = \frac{r}{5}\Big\}.$$

Then $x_1(t,\xi) = \xi_1 < 0$ for $t \leqslant t_2$, hence the inequality in (3.21) is strict, in fact,

$$\frac{\mathrm{d}\|Px(t)\|}{\mathrm{d}t} \leqslant \frac{\xi_1}{1+|\xi_1|}\,\frac{r}{5+r} < 0 \tag{3.23}$$

for $t \leqslant t_2$. The estimate (3.23) yields that for some $u < t_0$ one has $\|Px(u,\xi)\| < r/5$ and the first part of the proof applies.

Further, we want to prove

(3.24) *If $\varrho \in A_3$ is such that $\|P\varrho\| < \frac{r}{5}$, then there exists $t_0 \in \mathbf{R}_+$ such that $x(t_0, \varrho) \in A_4 \cap \operatorname{Int} \ell^+$.*

The proof is analogous to that of (3.20), so we only sketch it. Let $I \subseteq \mathbf{R}_+$ be an arbitrary interval such that $x(t,\varrho) \in A_3$ for $t \in I$. Using (1.7), (3.1) and the orthogonality of e to the range of P we obtain

$$\frac{\mathrm{d}\|Px(t)\|^2}{\mathrm{d}t} = 2(Px(t), \dot{x}(t)) = 0 \tag{3.25}$$

for any $t \in I$. Moreover,

$$\dot{x}_1(t) = (e, \dot{x}(t)) = \frac{r}{5} - \|Px(t)\|$$

by (3.3). As $\|P\varrho\| < r/5$, the formula (3.25) implies that $\inf\{\dot{x}_1(t): t \in I\} > 0$ and (3.24) follows. (Note that the restriction $\|P\varrho\| < r/5$ is inessential, since if $\varrho \in A_3$, $\|P\varrho\| = r/5$, then $\varrho \in Z_1$ and we can invoke (3.17).)

(3.26) *For every $\psi \in A_4 \cap \ell^+$ there exists $t_0 > 0$ such that $x(t_0, \psi) \in A_5 \cap \operatorname{Int} \ell^+$.*

We can proceed as in the preceding proofs. Again, let $I \subseteq \mathbf{R}_+$ be an arbitrary interval such that $x(t, \psi) \in A_4 \cap \operatorname{Int} \ell^+$ for $t \in I$, then

$$\dot{x}_1(t) = (e, \dot{x}(t)) = \left(\frac{r}{5} - \|PP_3x(t)\|\right) \frac{\|x(t) - P_4x(t)\|}{\|P_3x(t) - P_4x(t)\|} + q(x(t))x_1(t) \tag{3.27}$$

for $t \in I$ by (3.3),

$$\frac{\mathrm{d}\|x(t)\|^2}{\mathrm{d}t} = 2\left(\frac{r}{5} - \|PP_3x(t)\|\right) \frac{\|x(t) - P_4x(t)\|}{\|P_3x(t) - P_4x(t)\|} x_1(t) + 2q(x(t))\|x(t)\|^2$$

for $t \in I$ by (3.4), and

$$\frac{\mathrm{d}\|Px(t)\|^2}{\mathrm{d}t} = 2q(x(t))\|Px(t)\|^2, \qquad t \in I, \tag{3.28}$$

by (3.1). Taking into account the assumption (H_3) and the fact that $\|PP_3y\| < r/5$ for $y \in A_4$, $y_1 > 0$, we see that the functions $\|x(.)\|$, $\|Px(.)\|$ and x_1 are strictly increasing on I, therefore the proof of (3.26) can be easily completed.

The last step is a bit more complicated.

(3.29) *Take $\xi \in A_5 \cap \ell^+$ arbitrary, then there exists $t_0 > 0$ such that $x(t_0, \xi) \in A_6 \cap \operatorname{Int} \ell^+$.*

First, note that if $x(u, \xi) \in A_4 \cap A_5$ for a $u \geqslant 0$, then

$$\frac{\mathrm{d}(\|Px(.,\xi)\| + 2x_1(.,\xi))}{\mathrm{d}t}(u) \geqslant 0$$

by (3.27) and (3.28), so the solution $x(.,\xi)$ can never return to A_4. By (3.6), the set Z_1 is inaccesible for $x(.,\xi)$ as well. Let us assume that

$$x(t, \xi) \in A_5 \qquad \text{for all } t \geqslant 0; \tag{3.30}$$

we will show that this assumption leads to a contradiction. Using (1.9) one obtains

$$\begin{aligned}\frac{\mathrm{d}\|Px(t)\|^2}{\mathrm{d}t} &= 2(Px(t), \dot{x}(t)) \\ &= 2r\frac{\|x(t) - P_4x(t)\|}{\|P_5x(t) - P_4x(t)\|}(Px(t), F(P_1x(t))) + 2q(x(t))(Px(t), x(t)) \\ &= 2r\alpha(P_1x(t))\frac{x_1^2(t)}{\|x(t)\|}\frac{\|x(t) - P_4x(t)\|}{\|P_5x(t) - P_4x(t)\|} + 2q(x(t))\|Px(t)\|^2\end{aligned}$$

for any $t \geqslant 0$ by (3.1) and (3.2). (We have assumed that $Px(.) \neq 0$, the opposite case can be treated similarly.) This yields

$$\begin{aligned}\frac{\mathrm{d}\|Px(t)\|}{\mathrm{d}t} &= r\alpha(P_1x(t))\frac{\|x(t)-P_4x(t)\|}{\|P_5x(t)-P_4x(t)\|}\,\frac{x_1^2(t)}{\|x(t)\|\,\|Px(t)\|} + q(x(t))\|Px(t)\| \\ &\geqslant -\frac{x_1^2(t)}{\|x(t)\|\,\|Px(t)\|} + q(x(t))\|Px(t)\|.\end{aligned}$$

Moreover,

$$\begin{aligned}\dot{x}_1(t) &= -r\alpha(P_1x(t))\,\frac{\|x(t)-P_4x(t)\|}{\|x(t)\|\,\|P_5x(t)-P_4x(t)\|}\,x_1(t) + q(x(t))x_1(t) \\ &\geqslant -\frac{rx_1(t)}{\|x(t)\|} + q(x(t))x_1(t)\end{aligned}$$

by (3.3) and (3.5), so it follows that

$$\begin{aligned}\frac{\mathrm{d}\big(\|Px(t)\|+2x_1(t)\big)}{\mathrm{d}t} &\geqslant -\frac{x_1^2(t)}{\|x(t)\|\,\|Px(t)\|} - \frac{2rx_1(t)}{\|x(t)\|} \\ &\quad + q(x(t))\big(\|Px(t)\|+2x_1(t)\big) \\ &\geqslant -\frac{x_1^2(t)}{\|x(t)\|\,\|Px(t)\|} - \frac{2rx_1(t)}{\|x(t)\|} + q(x(t))r.\end{aligned} \tag{3.31}$$

Choose $\gamma \in (0, r^2]$ such that for $y \in A_5$ fulfilling $0 < y_1 < \gamma$ one has $\|y\| > r/2$ and $\|Py\| > r/2$. Since $q(y) \geqslant 8y_1/r$ on A_5 we obtain

$$\frac{\mathrm{d}\big(\|Px(t)\|+2x_1(t)\big)}{\mathrm{d}t} > 0 \qquad \text{if } \ 0 < x_1(t) < \gamma \tag{3.32}$$

by (3.31). Further we have

$$\frac{\mathrm{d}\|x(t)\|^2}{\mathrm{d}t} = 2q(x(t))\|x(t)\|^2$$

by (3.1) and (3.4), thus

$$\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t} = q(x(t))\|x(t)\|. \tag{3.33}$$

As $q \geqslant 0$ by (H_3), (3.33) yields that $\|x(.\,,\xi)\|$ is nondecreasing on $\mathbf{R}_+$. Furthermore, if we suppose that $\liminf\limits_{t\to\infty} x_1(t) > 0$ then (H_3) implies that $\inf\{\, q(x(t)) : t \geqslant u\} > 0$ for some $u \geqslant 0$, hence (3.33) yields $\lim_{t\to\infty} \|x(t)\| = \infty$, but this contradicts the assumption (3.30). Thus we have

$$\liminf_{t\to\infty} x_1(t) = 0. \tag{3.34}$$

If $\limsup_{t\to\infty} x_1(t) < \gamma$, then there exists $u \in \mathbf{R}_+$ such that $x_1(t) < \gamma$ for all $t \geqslant u$. As obviously $\|Px(u)\| + 2x_1(u) \geqslant r$, we obtain by (3.32) that one can find $\lambda > 0$ satisfying

$$\|Px(t)\| + 2x_1(t) \geqslant \|Px(u+1)\| + 2x_1(u+1) \geqslant r + \lambda, \qquad t \geqslant u+1.$$

Since $\|Px(t)\| \leqslant r$ for every $t \geqslant 0$ we obtain $x_1(t) \geqslant \lambda/2 > 0$ for all $t \geqslant u+1$, but this contradicts (3.34). Hence one has $\limsup_{t\to\infty} x_1(t) \geqslant \gamma$ and it is possible to find $t_n, s_n \in \mathbf{R}_+$, $t_n \to \infty$, $s_n \to \infty$, such that $x_1(t_n) = x_1(s_n) = \gamma/2$, $x_1(t) < \gamma/2$ for $t \in (t_n, s_n)$, $x_1(t) \geqslant \gamma/2$ for $t \in (s_n, t_{n+1})$, $n \in \mathbf{N}$. The sequence $\{\|x(t_n)\| : n \in \mathbf{N}\}$ is nondecreasing by virtue of (3.33), the sequence $\{x_1(t_n) : n \in \mathbf{N}\}$ is constant, hence also $\{\|Px(t_n)\| : n \in \mathbf{N}\}$ is nondecreasing. By (3.32) there exists $\mu > 0$ such that $\|Px(t_2)\| + 2x_1(t_2) \geqslant r + \mu$, since necessarily $x_1(s) < \gamma$ for all s in some neighbourhood of t_2. Hence we have

$$\|Px(t)\| + 2x_1(t) > \|Px(t_n)\| + 2x_1(t_n) \geqslant r + \mu, \qquad t \in (t_n, s_n),\ n \in \mathbf{N}.$$

Using again the estimate $\|Px(t)\| < r$ we obtain

$$x_1(t) \geqslant \frac{\mu}{2}, \qquad t \in (t_n, s_n),\ n \in \mathbf{N}.$$

Therefore

$$x_1(t) \geqslant \min\left(\frac{\mu}{2}, \frac{\gamma}{2}\right) \qquad \text{for all } t \geqslant t_2.$$

This contradicts (3.34), so we see that the assumption (3.30) can never be satisfied, the solution $x(\cdot, \xi)$ leaves A_5 in a finite time and (3.29) follows.

Having established (3.29) we have completed the proof of Proposition 2. Moreover, Proposition 2 implies that $f(x) \neq 0$ for all $x \in \ell_2$, $x \neq 0$. As $f(0) = re/5$ by (1.7), the proof of Theorem 1 is completed as well. □

IV. Proofs of Theorems 2 and 3

Proof of Theorem 2. Let $x^0(t)$ be the solution of (E) fulfilling $x^0(0) = 0$. Since $f(0) \neq 0$ there exists $t^0 > 0$ such that $x^0(t^0) \neq 0$. Suppose now that there exists a solution $x(.)$ with a nonempty Ω-set, i.e. we can find $y \in \ell_2$, $t_n \to \infty$ such that $y = \lim x(t_n)$. Since $x(t_n)$ converges weakly to 0 we can conclude that $y = 0$. Let v_n be the solutions to (E) with $v_n(0) = x(t_n)$; obviously $v_n(t) = x(t + t_n)$, $t \geqslant 0$. Continuous dependence on initial data yields $v_n(t^0) = x(t_n + t^0) \longrightarrow x^0(t^0) \neq 0$, $n \to \infty$, which is a contradiction with the weak convergence of $\{x(t_n + t^0)\}_{n=1}^{\infty}$ to 0. □

Proof of Theorem 3. Assume that $\mu \neq 0$ is a nonnegative finite Borel Radon measure invariant with respect to (E). The Radon property of μ implies that there exists a compact set K in H such that $\mu(K) > 0$. Denote by $d(y, K)$ the distance of a point y to the set K. As before, denote by $x(.\,, y)$ the solution of (E) fulfilling $x(0, y) = y$ and by $(T_t,\ t \in \mathbb{R})$ the flow induced by (E). Further, set

$$\varepsilon(y) = \liminf_{t\to\infty} d(x(t,y),K),$$

$$\tau(y) = \inf\left\{t > 0 \colon d(x(s,y),K) \geqslant \min\left(\frac{\varepsilon(y)}{2}, 1\right) \text{ for all } s \geqslant t\right\},$$

$$K_n = \{y \in K : \tau(y) \leqslant n\},$$

for $y \in H$ and $n \in \mathbb{N}$. Obviously, the functions ε, τ are Borel measurable, $\tau(y) < \infty$ for all $y \in K$.

Assume that there is $y_0 \in K$ such that $\varepsilon(y_0) = 0$. Then we can find a sequence of real numbers $t_n > n$ and elements $u_n \in K$ such that $\|u_n - x(t_n, y_0)\| \leqslant 1/n$. Since K is compact we can find a subsequence $\{u_s\}$ of $\{u_n\}$ such that $u_s \longrightarrow u \in K$, $s \to \infty$. Certainly $\lim x(t_s, y_0) = u$ and we have $u \in \Omega(x(.\,, y_0))$. This contradicts the assumptions of Theorem 3, hence we have $\varepsilon(y) > 0$ for every $y \in K$.

Obviously, the sets K_n are Borel as the function τ is Borel measurable. Thus there exists n such that $\mu(K_n) > 0$. Set $B_1 = K_n$, $B_{i+1} = \{x(n,y)\colon y \in B_i\} = T_n(B_i)$, $i = 1, 2, \ldots$; we claim that $B_i \cap B_j = \emptyset$ for $i \neq j$. By uniqueness of solutions, T_n is a one-to-one mapping, thus $B_i \cap B_j = \emptyset$, $i < j$, is equivalent to $B_1 \cap B_{j-i+1} = \emptyset$. Now, for $j > 1$ we have $B_j = \{x(n(j-1), y)\colon y \in K_n\}$. By the definition of K_n this yields $d(z, K) \geqslant \min(\varepsilon(T^{-1}_{n(j-1)}z)/2, 1) > 0$ for any $z \in B_j$, so $B_j \subseteq H \setminus K$, i.e. $K_n = B_1$ is disjoint with B_j. Hence, by the σ-additivity and invariantness of μ we have

$$\mu(H) \geqslant \sum_{j=1}^{\infty} \mu(B_j) = \sum_{j=1}^{\infty} \mu(T_{n(j-1)}(B_1)) = \sum_{j=1}^{\infty} \mu(B_1) = \infty,$$

which is a contradiction.

Remark. Now we will show that one cannot use Galerkin approximations to decide whether there is an invariant measure for the equation (E).

Consider the equation (E) with the right-hand side defined in Section I. Let $\{e_i\}_{i=1}^{\infty}$, $e_i = \{\delta_{ij}\}_{j=1}^{\infty}$, δ_{ij} being the Kronecker symbol, be the standard orthonormal basis of ℓ_2. Let $\mathcal{V}_n$ be the linear space spanned by $\{e_1, \dots, e_n\}$, let Π_n be the orthogonal projection onto $\mathcal{V}_n$. Consider the Galerkin approximations

$$\dot{y}_n = (\Pi_n f)(y_n) \tag{4.1}$$

to (E), $n \geqslant 1$. As $(\Pi_n f)(e_n) = 0$, the point e_n is singular, therefore the system (4.1) admits at least one invariant measure, namely, the Dirac measure concentrated at the point e_n. Denote this measure by μ_n, let μ_0 be the Dirac measure concentrated at the origin. Let us denote by ℓ_2^w the space ℓ_2 endowed with its weak topology, let $\mathcal{C}_w$ be the space of all bounded continuous real functions on ℓ_2^w with the supremal norm, let $\mathcal{C}_w^*$ stand for its dual space. (Note that Borel probabilities on ℓ_2 belong to the space $\mathcal{C}_w^*$.) As $e_n \longrightarrow 0$ in the weak topology of ℓ_2, we have (see [2], Corollary 2 to Theorem I.3.5) $\mu_n \longrightarrow \mu_0$ in the weak* topology of the space $\mathcal{C}_w^*$. Obviously, the measure μ_0 cannot be invariant for the equation (E). □

Finally, we will sketch the proof of the statement contained in the second remark following Theorem 3. Let H be a non-separable Hilbert space, let f, f_i, H_i, Q_i, $i = 1, 2$, have the same meaning as in Section I. In particular, $f(x) = f_1(Q_1 x) \oplus f_2(Q_2 x)$, $x \in H$, $f_2(y) = -\|y\|y/(1 + \|y\|^2)$, $y \in H_2$. Let $\mu \neq 0$ be a finite Borel measure on H invariant with respect to (E). We will prove that

$$\mu\{x \in H : \|Q_2 x\| \leqslant R\} = \mu(H) \qquad \text{for every } R \geqslant 0. \tag{4.2}$$

Assume that $\mu\{x \in H : \|Q_2 x\| \leqslant R_1\} < \mu(H)$ for some $R_1 > 0$. Then there exists $R_2 > R_1$ such that denoting $L_1 = \{x \in H : R_1 < \|Q_2 x\| \leqslant R_2\}$ we obtain $\mu(L_1) > 0$. Given R_1, R_2 we can define by induction the following quantities:

$$\begin{aligned}
R_{2n} &= R_{2n-3}, \\
s_n &= \frac{1}{R_{2n}} - R_{2n} - \left(\frac{1}{R_{2n-2}} - R_{2n-2}\right), \\
R_{2n-1} &= \frac{1}{2}\left\{\left(\left(s_n + \frac{1}{R_{2n-3}} - R_{2n-3}\right)^2 + 4\right)^{1/2} - \left(s_n + \frac{1}{R_{2n-3}} - R_{2n-3}\right)\right\}, \\
t_n &= \sum_{k=2}^{n} s_k
\end{aligned}$$

for $n \in \mathbf{N}$, $n \geqslant 2$. Set $L_n = T_{t_n}(L_1)$, $n \in \mathbf{N}$, $n \geqslant 2$. Let us realize that if x is a solution to (E) then $\|Q_2 x(\cdot)\|$ solves the equation

$$\dot{\gamma} = -\frac{\gamma^2}{1+\gamma^2}. \tag{4.3}$$

If γ_-, γ_+ are two solutions of (4.3), $\gamma_-(0) < \|Q_2 x(0)\| \leqslant \gamma_+(0)$, then, by uniqueness, $\gamma_-(t) < \|Q_2 x(t)\| \leqslant \gamma_+(t)$ for all $t \geqslant 0$. This easily yields $L_n \subseteq \{x \in H : R_{2n-1} < \|Q_2 x\| \leqslant R_{2n}\}$, $n \geqslant 2$, so $L_n \cap L_m = \emptyset$, $m \neq n$. Since μ is invariant we have $\mu(L_n) = \mu(L_1)$ and as before we obtain $\mu(H) = \infty$, which is impossible. Therefore (4.2) holds and $\mu(H) = \mu(H_1)$. But H_1 is separable, thus μ is a Radon measure and Theorem 3 applies.

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References

[1] *G. Da Prato, D. Gątarek, J. Zabczyk*: Invariant measures for semilinear stochastic equations, Stochastic Anal. Appl. *10* (1992), 387–408.

[2] *N. N. Vakhaniya, V. I. Tarieladze, S. A. Chobanyan*: Probability distributions in Banach spaces, Nauka, Moscow, 1985. (In Russian.)

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