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CONVEX ISOMORPHISMS OF DIRECTED MULTILATTICES

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Summary. By applying the notion of the internal direct product decomposition we investigate the relations between convex isomorphisms and direct product decompositions of directed multilattices.

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To each direct product decomposition of a partially ordered set L and each element $s^0 \in L$ there corresponds an internal direct product decomposition of L with the central element s^0 (for the definition of this notion cf. Section 1 below; it is analogous to the corresponding notion for groups (cf., e.g., Kurosh [7], p. 104)).

The following result will be proved. Let L be a directed set and $\varphi_1 : L \longrightarrow A \times B$, $\varphi_2 : L \longrightarrow A \times C$ internal direct product decompositions with the same central element. Then for each $x \in L$ the component of x in A with respect to φ_1 is the same as the component of x in A with respect to φ_2.

Let us remark that an analogous result does not hold for internal direct decompositions of groups.

By applying internal direct product decompositions we shall investigate convex isomorphisms of directed multilattices. This notion was introduced for lattices by Marmazeev [8]. In [9] he studied convex automorphisms of a lattice L under the assumption that L satisfies the following conditions:

(i) Each bounded chain in L is finite.

(ii) L is a direct product of a finite number of directly indecomposable lattices.

Kolibiar and Lihová [6] investigated convex automorphisms of a lattice L under the assumption that the condition (ii) holds.

In the present paper we generalize the main result from [6] (Theorem 10) in two directions. It will be proved that this result is true in the case when L is a direct product of directly indecomposable lattices; the number of these lattices may be arbitrary. Next it will be shown that the result remains valid for the case of directed multilattices.

1. Internal direct product decompositions of a directed set

A direct product of partially ordered sets L_i ($i \in I$) will be denoted by $\prod_{i\in I} L_i$. If $I = \{1, 2, \dots, n\}$, then we apply also the notation $L_1 \times L_2 \times \dots \times L_n$.

If φ is an isomorphism of a partially ordered set L onto a direct product $\prod_{i\in I} L_i$, then we say that the morphism

$$\varphi\colon L \longrightarrow \prod_{i\in I} L_i \tag{1}$$

is a direct product decomposition of L.

The existence of isomorphic refinements of any two direct product decompositions of a connected partially ordered set was proved by Hashimoto [3].

A partially ordered set K is called directly indecomposable if, whenever K is isomorphic to some direct product $\prod_{i\in I} K_i$, then there is $i(1) \in I$ such that card $L_i = 1$ for each $i \in I \setminus \{i(1)\}$. In such a case K is isomorphic to $L_{i(1)}$.

Let us remark that if (1) is valid and if there is $I(1) \subset I$ such that card $L_i = 1$ for each $i \in I(1)$, then there is a direct product decomposition $\varphi_1 : L \longrightarrow \prod_{i\in I\setminus I(1)} L_i$.

If we consider the direct product decomposition (1) and if $x \in L$, $i \in I$, then the component of x in the direct factor L_i will be denoted by $x(L_i, \varphi)$.

By simple examples we can verify that if

$$\psi\colon L \longrightarrow \prod_{i\in I(1)} L'(i) \tag{1'}$$

is another direct product decomposition of L and if there are $i \in I$, $i(1) \in I(1)$ such that $L_i = L_{i(1)}$, then there can exist $x \in L$ with

$$x(L_i, \varphi) \neq x(L_{i(1)}, \psi). \tag{2}$$

Let (1) be valid, card $I > 1$ and let $i \in I$. Put $L'_i = \prod_{j\in I\setminus\{i\}} L_j$. Then there is a direct product decomposition

$$\psi_i\colon L \longrightarrow L_i \times L'_i \tag{1''}$$

where, for each $x \in L$, $x(L_i, \psi_i) = x(L_i, \varphi)$ and $x(L_i', \psi_i) = (\dots, y_j, \dots)$ $(j \in I \setminus \{i\}), y_j = x(L_j, \varphi)$ for each $j \in I \setminus \{i\}$.

Hence the rather "unpleasant" relation (1′) can occur also for direct product decompositions with two factors.

Let s^0 be a fixed element of L and let us consider the direct product decomposition (1). For each $x \in L$ and $i \in I$ we denote

$$[x](L_i, \varphi) = \{y \in L : y(L_j, \varphi) = x(L_j, \varphi) \text{ for each } j \in I \setminus i\},$$

$$L_i^0 = [s^0](L_i, \varphi).$$

For each $x \in L$ and $i \in I$ there is a unique element y_i in L_i^0 such that

$$x(L_i, \varphi) = y_i(L_i, \varphi).$$

Then the mapping

$$\varphi^0 : L \longrightarrow \prod_{i \in I} L_i^0 \tag{3}$$

defined by $\varphi^0(x) = (\dots, y_i, \dots)_{i \in I}$ is also a direct product decomposition of L. It will be called the internal direct product decomposition of L (corresponding to φ). The element s^0 is said to be the central element of the internal direct product decomposition φ^0.

It is evident that for each $i \in I$, L_i is isomorphic to L_i^0. Hence if we are interested only in considerations "up to isomorphisms", then we need not distinguish between (1) and (3).

We shall prove the following result:

(A). *Let L be a directed set. Suppose that two internal direct product decompositions are given,*

$$\psi_1 : L \longrightarrow \prod_{i \in I} A_i, \quad \psi_2 : L \longrightarrow \prod_{j \in J} B_j,$$

such that there exist $i(1) \in I$ and $j(1) \in J$ with $A_{i(1)} = B_{j(1)}$. Then for each $x \in L$ the relation

$$x(A_{i(1)}, \psi_1) = x(B_{j(1)}, \psi_2)$$

is valid.

We have already remarked above that this does not hold in general for direct product decompositions which are not internal.

Let us consider a direct product decomposition

$$\chi: L \longrightarrow \prod_{k \in K} C_k \tag{4}$$

and suppose that (under the notation as in (1))

(i) for each $i \in I$ there is a subset $K(i)$ of K and a direct product decomposition $\chi_i : L_i \longrightarrow \prod_{k \in K(i)} C_k$;

(ii) under the notation as in (i), for each $x \in L$ and $k \in K(i)$ the relation

$$x(C_k, \chi) = (x(L_i, \varphi))(C_k, \chi_i) \tag{5}$$

is valid.

Then χ is said to be a refinement of φ.

While in [3] only isomorphic direct product decompositions are constructed, by applying the proof from [3] and (A) we obtain

(B). *Any two internal direct product decompositions of a directed set have a common refinement.*

Similarly as in the case of (A), the assertion analogous to (B) does not hold in general for internal direct product decompositions of groups.

2. Proofs of (A) and (B)

From the considerations in Section 1 (cf. (1) and (1″)) we obtain that to prove (A) it suffices to take into account two-factor internal direct product decompositions.

Again, let L be a directed set. Let us have a direct product decomposition

$$\varphi: L \longrightarrow X \times Y \tag{1}$$

and let

$$\varphi^0: L \longrightarrow X^0 \times Y^0 \tag{1'}$$

be the corresponding internal direct product decomposition with the central element s^0.

2.1. Lemma. *For each $t \in L$, $t(X^0, \varphi^0)$ is the unique element of L lying in the set $X^0 \cap [t](Y, \varphi)$.*

Proof. This is an immediate consequence of the definition of φ^0. □

The following lemma is easy to verify.

2.2. Lemma. *Let $x_i \in X$, $y_i \in Y$ $(i = 1, 2)$, $x_1 \leqslant x_2$, $y_1 \leqslant y_2$. Then (x_1, y_2) is the unique relative complement of (x_2, y_1) with respect to the interval $[(x_1, y_1), (x_2, y_2)]$ of $X \times Y$.*

2.3. Lemma. *Let $v \in L$, $s^0 \leqslant v$, $\varphi^0(v) = (a, b)$. Then a is the greatest element of the set $X^0 \cap [s^0, v]$.*

P r o o f. We have

$$\varphi^0(s^0) = (s^0, s^0), \quad \varphi^0(a) = (a, s^0), \quad \varphi^0(b) = (s^0, b).$$

Thus $s_0 \leqslant a \leqslant v$. Clearly $a \in X^0$. Let $z \in X^0 \cap [s^0, v]$. Hence $z = z(X^0, \varphi^0) \leqslant v(X^0, \varphi^0) = a$. □

For z_1, z_2 and z_3 in L the notation $z_1 \wedge_0 z_2 = z_3$ means that z_3 is the greatest lower bound of the set $\{z_1, z_2\}$ in L; the notation $z_1 \vee_0 z_2 = z_3$ has the dual meaning. (The symbols $\wedge$ and $\vee$ are reserved for other purposes; cf. Section 3.)

2.4. Lemma. *The set Y^0 is uniquely determined by X^0 and s^0.*

P r o o f. Let us denote by Z the set of all $z \in L$ such that there exist $z_1, z_2 \in L$ with

$$z_1 \leqslant z \leqslant z_2, \quad z_1 \leqslant s^0 \leqslant z_2,$$

$$z_1 \vee_0 x_1 = s^0 \quad \text{for each } x_1 \in X^0 \quad \text{with } x_1 \leqslant s^0,$$

$$z_2 \wedge_0 x_2 = s^0 \quad \text{for each } x_2 \in X^0 \quad \text{with } x_2 \geqslant s^0.$$

Then Z is uniquely determined by X^0 and s^0.

Let $z \in Z$ and let z_1, z_2 be as above. By 2.1 we have $z_2(X^0, \varphi^0) = s^0$, hence z_2 belongs to Y^0. Applying the duality we obtain that z_1 belongs to Y^0. It is obvious that Y^0 is a convex subset of L and hence $z \in Y^0$. Therefore $Z \subseteq Y^0$.

Now let $y_0 \in Y^0$. Since L is directed, Y^0 is directed as well. Thus there are y_1 and y_2 in Y^0 such that

$$y_1 \leqslant y_0 \leqslant y_2, \quad y_1 \leqslant s^0 \leqslant y_2.$$

Then both y_1 and y_2 belong to Z. It is clear that Z is a convex subset of L. Hence $y_0 \in Z$ and thus $Y^0 \subseteq Z$, completing the proof. □

Since $X^0 = [s^0](X, \varphi)$ and $Y^0 = [s^0](Y, \varphi)$ and since s^0 is an arbitrary element of L with no specific properties, we have

2.4.1. Lemma. *Let $t \in L$. Then the set $[t](Y, \varphi)$ is uniquely determined by the set $[t](X, \varphi)$.*

2.5. Lemma. *Let $t \in L$, $t \geqslant s^0$. Then the set $[t](X, \varphi)$ is uniquely determined by X^0 and t.*

Proof. In view of 2.3 (with X and Y interchanged) there exists $b \in Y^0$ such that $b = \max(Y^0 \cap [s^0, t])$. Moreover, according to 2.4, b is uniquely determined by X^0 and s^0; also $t(Y^0, \varphi^0) = b$. Clearly $b \in [t](X, \varphi)$.

(a) Put $A = \{x \in [t](X, \varphi) : t \geqslant b\}$. For $v \in L$ we have

$$v \in A \Longleftrightarrow v \geqslant b \quad \text{and } b = \max(Y^0 \cap [s_0, v]).$$

Hence A is uniquely determined by X^0 and s^0.

(b) Put $B = \{b' \in [t](X, \varphi) : b' \leqslant b\}$.

Let $b' \in B$. Put $b'(X^0, \varphi^0) = x$. Then $\varphi^0(b') = (x, b)$. Since $b' \leqslant b$ and $\varphi^0(b) = (s^0, b)$ we obtain that $x \leqslant s^0$. Thus in view of 2.2, b' is the relative complement of s^0 in the interval $[x, b]$, where $x \in X^0, x \leqslant s^0$.

Let Z be the set of all $z \in L$ such that z is the relative complement of s^0 in an interval $[x', b]$, where $x' \in X^0$ and $x' \leqslant s^0$. From 2.2 we infer that $z \in B$.

We have verified that $B = Z$. Therefore B is uniquely determined by X^0 and s^0.

(c) Let Z' be the convex subset of L generated by $A \cup B$. Thus in view of (a) and (b), Z' is uniquely determined by X^0 and s^0.

Since $A \cup B \subseteq [t](X, \varphi)$ and $[t](X, \varphi)$ is a convex subset of L we obtain that $Z' \subseteq [t](X, \varphi)$. Let $y \in [t](X, \varphi)$. Since L is directed, $[t](X, \varphi)$ is directed as well. Thus from $y, b \in [t](X, \varphi)$ we get that there are $y_1, y_2 \in [t](X, \varphi)$ such that

$$y_1 \leqslant y \leqslant y_2, \quad y_1 \leqslant b \leqslant y_2.$$

The second relation implies that $y_1 \in B$ and $y_2 \in A$. Thus $y \in Z'$ and $Z' = [t](X, \varphi)$, completing the proof. □

Similarly as in 2.4.1 we now have

2.5.1. Lemma. *Let $t_1, t_2 \in L$, $t_1 \leqslant t_2$. Then $[t_2](X, \varphi)$ is uniquely determined by t_2 and $[t_1](X, \varphi)$.*

The assertion dual to 2.5.1 is also valid.

2.6. Lemma. *Let $t \in L$. Then the set $[t](X,\varphi)$ is uniquely determined by t and X^0.*

Proof. Since L is directed there is $t' \in L$ with

$$t' \leqslant s^0,\ t' \leqslant t.$$

In view of the assertion dual to 2.5.1 the set $[t'](X,\varphi)$ is uniquely determined by t' and X^0. Next, by 2.5.1 the set $[t](X,\varphi)$ is uniquely determined by t and $[t'](X,\varphi)$. Hence $[t](X,\varphi)$ is uniquely determined by X^0 and t. □

Lemmas 2.4.1 and 2.6 yield

2.7. Lemma. *Let $t \in L$. Then the set $[t](Y,\varphi)$ is uniquely determined by t and X^0.*

Proof of (A). We have already noticed above that for verifying the validity of (A) it suffices to consider internal direct product decompositions with two factors. Hence let us again deal with the internal direct product decomposition (1′). We have to verify that for each $t \in L$ the component of t in X^0 is uniquely determined by t and X^0.

The element $t(X^0,\varphi^0)$ is the unique element of L lying in the intersection

$$X^0 \cap [t](X,\varphi),$$

hence in view of 2.7, $t(X^0,\varphi^0)$ is uniquely determined by t and X^0. □

If the relation (1) from Section 1 is valid and $A \subseteq I$, $i \in L$ then we denote

$$A(L_i,\varphi) = \{a(L_i,\varphi) : a \in A\}.$$

The set $A(L_i,\varphi)$ is partially ordered by the partial order inherited from L_i.

Proof of (B). Let us have two internal direct product decompositions

(2) $$\varphi^0 : L \longrightarrow \prod_{i\in I} X_i^0,$$

(3) $$\varphi^{01} : L \longrightarrow \prod_{j\in J} Y_j^0$$

with the same central element s^0. In view of Hashimoto's construction (Theorem 1, [3]) we obtain direct product decompositions

(4) $$\psi : L \longrightarrow \prod_{i\in I, j\in J} X_i^0(Y_j^0,\varphi^{01}),$$

(5) $$\psi_1 : L \longrightarrow \prod_{i\in I, j\in J} Y_j^0(X_i^0, \varphi^0).$$

Here, (4) is a refinement of (2) and (5) is a refinement of (3); both (4) and (5) are internal direct product decompositions of L with the same central element s^0.

In view of this fact we have

(6) $$X_i^0(Y_j^0, \varphi^{01}) = Y_j^0(X_i^0, \varphi^0);$$

namely, in the notation applied in the proof of Theorem 1, [3] it was proved there that $S_i^j = S_j^i$ (under different denotation of indices), and since the direct product decompositions are internal with the same central element s^0, the relations $S_i^j = X_i^0(Y_j^0, \varphi^{01})$ and $S_j^i = Y_j^0(X_i^0, \varphi^0)$ are valid. Next, by applying (A) we infer that the mappings ψ and ψ_1 coincide. This completes the proof. □

Let us remark that if we consider an internal direct product decomposition (2) and if x is an element of L, then in view of (A) the component of x in X_i^0 can be denoted simply by $x(X_i^0)$; we suppose that the central element s^0 is fixed.

Next, when considering refinements of direct product decompositions (cf., e.g., (1) and (4) in Section 1) we shall write

$$x(C_k) = x(L_i)(C_k)$$

instead of (5) in Section 1 under the assumption that φ and χ are internal direct product decompositions.

From (B) we obtain as a corollary:

(C). *Let L be a directed set and let (2) be an internal direct product decomposition of L such that all X_i^0 are directly indecomposable. Let*

(7) $$\psi^0 : L \longrightarrow X^0 \times Y^0$$

be an internal direct decomposition of L. Suppose that φ^0 and ψ^0 have the same central element s^0. Then φ^0 is a refinement of ψ^0. Thus there are nonempty subsets $I(1)$ and $I(2)$ of I with $I(1) \cap I(2) = \emptyset$, $I(1) \cup I(2) = I$ such that there exist internal direct product decompositions

$$\varphi_1 : X^0 \longrightarrow \prod_{i\in I(1)} X_i^0, \quad \varphi_2 : Y^0 \longrightarrow \prod_{i\in I(2)} X_i^0$$

with the same central element s^0.

3. Auxiliary results on directed multilattices

The notion of a multilattice was introduced by Benado [1]. It is defined as follows.

Let P be a partially ordered set. For $x, y \in P$ we denote by $L(x, y)$ and $U(x, y)$ the system of all lower bounds or all upper bounds of the set $\{x, y\}$ in P, respectively. P is said to be a multilattice if, whenever $x, y \in L$ and $z \in L(x, y)$, then there is z_1 in $L(x, y)$ such that z_1 is a maximal element of $L(x, y)$ and $z \leqslant z_1$, and if the corresponding dual condition concerning $U(x, y)$ also holds.

In what follows we assume that P is a directed multilattice. For $x, y \in L$ let $x \wedge y$ be the system of all maximal elements of $L(x, y)$; similarly, we denote by $x \vee y$ the system of all minimal elements of $U(x, y)$. Both $x \wedge y$ and $x \vee y$ are nonempty.

A nonempty subset P' of P will be called an m-subset of P if, whenever x and y belong to P', then both $x \wedge y$ and $x \vee y$ are subsets of P'. Let $C(P)$ be the system of all convex m-subsets of P.

Each lattice can be viewed as a directed multilattice. If P is a lattice and X is a subset of P then X is a convex m-subset of P if and only if X is a convex sublattice of P.

In this section we shall deal with directed multilattices P and P_1 which are defined on the same underlying set and satisfy the condition

$$C(P) = C(P_1). \tag{1}$$

The partial order in P or in P_1 will be denoted by $\leqslant$ and $\leqslant_1$, respectively. If $x, y \in P$ and $x \leqslant y$, then $[x, y]$ is the corresponding interval in P; if $x \leqslant_1 y$, then $[x, y]_1$ has the analogous meaning with respect to P_1. Next, for a and b in P the symbols $a \wedge_1 b$, $a \vee_1 b$, $L_1(a, b)$ and $U_1(a, b)$ have the obvious meanings.

3.1. Lemma. *Let $x, y, z \in P$, $x \leqslant z \leqslant y$, $x \leqslant_1 y$. Then $x \leqslant_1 z$ and $z \leqslant_1 y$.*

Proof. We consider the system $C(P)$ to be partially ordered by inclusion. The least element of $C(P)$ containing both x and y is $[x, y]$. Thus in view of (1) the relation $[x, y] = [x, y]_1$ is valid. Hence $z \in [x, y]_1$. □

3.2. Corollary. *Let x, y be as in 3.1. If $z_1, z_2 \in [x, y]$ and $z_1 \leqslant z_2$, then $z_1 \leqslant_1 z_2$.*

By a similar argument we obtain

3.3. Lemma. *Let $x, y \in P$, $x \leqslant y$, $y \leqslant_1 x$. If $z_1, z_2 \in [x, y]$ and $z_1 \leqslant z_2$, then $z_2 \leqslant_1 z_1$.*

3.4. Lemma. *Let $x, y \in P$, $x \leqslant y$, $u \in x \wedge_1 y$, $v \in x \vee_1 y$. Then $[x, y] = [u, v]_1$.*

Proof. $[x,y]$ is the least element of $C(P)$ containing both x and y. Similarly, $[u,v]_1$ is the least element of $C(P_1)$ which contains u and v. Since $x, y \in [u,v]_1$, in view of (1) we obtain that $[x,y] \subseteq [u,v]_1$. If $X \in C(P_1)$ and $x, y \in X$, then u and v belong to X; thus $u, v \in [x,y]$ and then $[u,v]_1 \subseteq [x,y]$, completing the proof. □

3.5. Corollary. *Let $x, y \in P$, $x \leqslant y$. Then* $\operatorname{card}(x \wedge_1 y) = \operatorname{card}(x \vee_1 y) = 1$.

3.6. Lemma. *Let x, y, u and v be as in 3.4. Next let*

$$u^*, u_1^* \in [x,y], \quad x \geqslant_1 u^*, \quad x \leqslant_1 u_1^*.$$

Then $u^* \leqslant u$ and $u_1^* \leqslant v$.

Proof. In view of 3.4 we have $u^* \in [u,v]_1$, hence $u \leqslant_1 u^*$. From the relations

$$x \geqslant_1 u^* \geqslant_1 u, \quad x \leqslant u$$

and from 3.3 (with P and P_1 interchanged) we infer that $u^* \leqslant u$. The relation $u_1^* \leqslant v$ can be verified analogously. □

3.7. Lemma. *Let x, y, u and v be as in 3.4. Next let*

$$v^*, v_1^* \in [x,y], \quad y \leqslant_1 v^*, \quad y \geqslant_1 v_1^*.$$

Then $v^* \geqslant v$ and $v_1^* \geqslant u$.

The proof is analogous to that of 3.6. □

3.8. Lemma. *Let $a, b, t \in P$, $t \leqslant a, t \leqslant b$, $t \leqslant_1 a$, $t \leqslant_1 b$, $t_2 \in a \vee b$. Then $a \leqslant_1 t_2$ and $b \leqslant_1 t_2$.*

Proof. If a and b are comparable in P, then the assertion is implied by 3.1. Thus we can suppose that a and b are incomparable in P.

From 3.1 we infer that

$$a <_1 t_2 \iff b <_1 t_2.$$

By way of contradiction, assume that neither $a <_1 t_2$ nor $b <_1 t_2$ is valid. Then in view of 3.5 there are uniquely determined elements a_1 and b_1 in P such that $a_1 \in a \vee_1 t_2$ and $b_1 \in b \vee_1 t_2$. Hence according to 3.4

$$a \leqslant a_1 < t_2, \quad b \leqslant b_1 < t_2,$$

$$a \leqslant_1 a_1 >_1 t_2, \quad b \leqslant_1 b_1 >_1 t_2.$$

In view of 3.5 there are uniquely determined elements u and v in P with $u \in t \wedge_1 t_2$ and $v \in t \vee_1 t_2$. According to 3.2 the relations $a_1 \leqslant v$ and $b_1 \leqslant v$ are valid. Next, 3.4 yields that $v \leqslant a_1$ and $v \leqslant b_1$. Therefore $a_1 = v = b_1$. Then $a_1 \in a \vee b < t_2$, which is a contradiction. □

There are three obvious modifications of 3.8 (the first is obtained by duality, and then we obtain the other two cases by interchanging P and P_1). When applying any of these modifications we shall refer to 3.8. Similarly we proceed by quotations of obvious modifications of the subsequent lemmas.

3.9. Lemma. *Let a, b, t and t_2 be as in 3.8. Let $t_1 \in a \wedge b$. Then $t_1 \leqslant_1 a$ and $t_1 \leqslant_1 b$.*

Proof. This is a consequence of 3.8. □

4. The relations R_1 and R_2

We apply the same assumptions as in Section 3. For $a, b \in P$ we write aR_1b if there is $t \in P$ such that the assumptions from 3.8 are satisfied. Next we write aR_2b if there is $t \in P$ such that

$$t \leqslant a, \quad t \leqslant b, \quad t \geqslant_1 a, \quad t \geqslant_1 b.$$

From 3.8 we infer that the relations R_1 and R_2 can be defined also by applying the corresponding dual conditions.

4.1. Lemma. *Let $a, b, c \in P$, aR_1b and bR_1c. Then aR_1c.*

Proof. There exist elements t_1 and t_2 in P such that

$$t_1 \leqslant a, \quad t_1 \leqslant b, \quad t_1 \leqslant_1 a, \quad t_1 \leqslant_1 b,$$

$$t_2 \leqslant b, \quad t_2 \leqslant c, \quad t_2 \leqslant_1 b, \quad t_2 \leqslant_1 c.$$

Let $t_3 \in t_1 \wedge t_2$. According to 3.9 (by applying the elements t_1, t_2 and b) we obtain that $t_3 \leqslant_1 t_1$ and $t_3 \leqslant_1 t_2$. Thus aR_1 holds. □

Similarly we can verify

4.2. Lemma. *Let $a, b, c \in P$, aR_2b and bR_2c. Then aR_2c.*

Since the relations R_1 and R_2 are obviously reflexive and symmetric, in view of 4.1 and 4.2 they are equivalence relations on P. Let R_m be the greatest equivalence relation on P.

4.3. Lemma. $R_1 \vee R_2 = R_m$.

Proof. Let $x, y \in P$, $x \leqslant y$. In view of 3.4 there is $c \in [x, y]$ such that xR_1c and cR_2y, hence $x(R_1 \vee R_2)y$. Now it suffices to apply the fact that P is directed. □

4.4. Lemma. *Let $i \in \{1,2\}$, $a, b \in P$, aR_ib. Then there exist elements t and t' in P such that $t \leqslant t'$, $a, b \in [t, t']$ and tR_it'.*

Proof. This is a consequence of the definition of R_i and of 3.8. □

4.5. Lemma. *Let x_0, x_1, $x_2 \in P$, $x_0R_1x_1$ and $x_1R_2x_2$. Assume that both the pairs x_0, x_1 and x_1, x_2 are comparable in P and in P_1. Then there is $y \in P$ such that*

(i) *x_0R_2y and yR_1x_2;*
(ii) *both the pairs x_0, y and y, x_2 are comparable in P and in P_1.*

Proof. (a) If $x_0 \leqslant x_1 \leqslant x_2$, then in view of 3.5 there is a uniquely determined element $y \in P$ with $y \in x_0 \wedge_1 x_2$. Thus according to 3.4, y satisfies (i) and (ii). The other cases under the assumption that the set $\{x_0, x_1, x_2\}$ is linearly ordered in P are analogous.

(b) Now assume that the set $\{x_0, x_1, x_2\}$ is not linearly ordered in P. E.g., suppose that $x_0 \geqslant x_1$ and $x_1 \leqslant x_2$, $x_0 \neq x_2$. Then the set $\{x_0, x_1, x_2\}$ is linearly ordered in P_1 and we can apply the same method as in (a) by using 3.4 with P and P_1 interchanged. The remaining cases when $\{x_0, x_1, x_2\}$ are not linearly ordered in P are analogous. □

4.6. Lemma. $R_1R_2 = R_m$.

Proof. Let $x, y \in P$. Choose $u \in x \wedge y$. Lemma 3.4 yields that there exist elements p and q in P such that

$$p \in [u, x],\ xR_1p,\ pR_2u,$$

$$q \in [u, y],\ yR_2q,\ qR_1u.$$

Hence by 4.5 there is $v \in P$ such that pR_1v and vR_2q. Thus by 4.1 and 4.2, xR_1v and vR_2y. Therefore xR_1R_2y. □

4.6.1. Corollary. $R_1R_2 = R_2R_1$.

Proof. Analogously to 4.6 we have $R_2R_1 = R_m$, hence $R_1R_2 = R_2R_1$. □

4.7. Lemma. *Let $a, b \in P$. Then there are $t_1, t_2 \in P$ such that*

$$aR_1t_1,\quad t_1R_2b,$$
$$aR_2t_2,\quad t_2R_2b.$$

Proof. This is a consequence of 4.6 and 4.6.1. □

4.8. Lemma. *Let $a, b \in P$, aR_1b and aR_2b. Then $a = b$.*

Proof. In view of aR_1b there is $t_1 \in P$ with $t_1 \in a \wedge b$ such that $t_1 \leqslant_1 a$ and $t_1 \leqslant_1 b$. Similarly, from aR_2b we obtain that there is $t_2 \in a \wedge b$ such that $t_2 \geqslant_1 a$ and $t_2 \geqslant_1 b$. According to 3.9 we have, at the same time, $t_2 \leqslant_1 a$ and $t_2 \leqslant_1 b$. Hence $a = t_2 = b$. □

For $x \in P$ we denote

$$x(R_1) = \{x_1 \in P \colon xR_1x_1\}, \quad x(R_2) = \{x_1 \in P \colon xR_2x_1\}.$$

Now we shall deal with the sets $x(R_1)$, where x runs over P. Analogous results hold for the sets $x(R_2)$.

For x and y in P we write $x(R_1) \leqslant y(R_1)$ if there are $x_1 \in x(R_1)$ and $y_1 \in y(R_1)$ such that $x_1 \leqslant y_1$.

4.9. Lemma. *Let $x, y \in P$, $x(R_1) \leqslant y(R_1)$. Then there is $x_3 \in P$ such that $x(R_1) = x_3(R_1)$, $x_3 \leqslant y$ and $x_3 \geqslant_1 y$.*

Proof. There are $x_1 \in x(R_1)$ and $y_1 \in y(R_1)$ such that $x_1 \leqslant y_1$. In view of 3.4 there is $x_2 \in P$ such that $x_1 \leqslant x_2 \leqslant y_1$, $x_1 \leqslant_1 x_2$, $x_2 \geqslant_1 y_1$. Hence $x_2(R_1) = x(R_1)$. Choose $u \in y_1 \wedge y$. According to the definition of R_1 and in view of 3.9 we have $y_1(R_1) = u(R_1)$. Consider the elements x_2, y_1 and u. Then 3.5 yields that $x_2 \wedge u$ is a one-element set; we denote $x_2 \wedge u = \{u_1\}$. Next, $x_2R_1u_1$ and u_1R_2u.

Now let us consider the elements u_1, u and y. By applying 3.5 we get that there is $x_3 \in (u_1 \vee_1 y) \cap [u_1, y]$ such that $u_1 \leqslant x_3 \leqslant y$, $u_1R_1x_3$ and x_3R_2y. Clearly x_3R_1x. □

By similar considerations we obtain

4.10. Lemma. *Let $x, y \in P$, $x(R_1) \leqslant y(R_1)$. Then there is $y_3 \in P$ such that $y(R_1) = y_3(R_1)$, $x \leqslant y_3$ and $x \geqslant_1 y_3$.*

4.11. Lemma. *Let $x, y, z \in P$, $x(R_1) \leqslant y(R_1)$ and $y(R_1) \leqslant z(R_1)$. Then $x(R_1) \leqslant z(R_1)$.*

Proof. In view of 4.9 and 4.10 there are elements $x_3 \in x(R_1)$ and $z_1 \in z(R_1)$ such that $x_3 \leqslant y$ and $y \leqslant z_1$. Hence $x(R_1) \leqslant z(R_1)$. □

4.12. Lemma. *Let $x, y \in P$, $x(R_1) \leqslant y(R_1)$ and $y(R_1) \leqslant x(R_1)$. Then $x(R_1) = y(R_1)$.*

Proof. By way of contradiction, assume that $x(R_1) \neq y(R_1)$. Since $x(R_1) \leqslant y(R_1)$, according to 4.10 there is $y_1 \in P$ such that $y_1 \in y(R_1)$ and $x < y_1$, $x >_1 y_1$. Next, since $y_1(R_1) \leqslant x(R_1)$ and $y_1(R_1) \neq x(R_1)$, according to 4.9 there is $x_1 \in x(R_1)$ such that $y_1 < x_1$ and $x_1 >_1 y_1$. Therefore $x < x_1$ and $x >_1 x_1$. Hence xR_2x_1. According to 4.8, $x = x_1$. Thus $x = y_1$ and so $x(R_1) = y_1(R_1) = y(R_1)$. □

Put $A = \{x(R_1) : x \in P\}$. With respect to the relation $\leqslant$ on A defined above (which is obviously reflexive), A is a partially ordered set (cf. 4.11 and 4. 12).

Now we denote $B = \{x(R_2) : x \in P\}$ and define the relation $\leqslant$ on B analogously as we did for A. Then B is a partially ordered set as well.

By a method analogous to that used for proving 4.9 and 4.10 we get

4.13. Lemma. *Let $x, y \in P$, $x(R_2) \leqslant y(R_2)$. Then there are elements $y_1 \in y(R_2)$ and $x_1 \in x(R_2)$ such that*

$$x \leqslant y_1, \quad x \leqslant_1 y_1,$$

$$x_1 \leqslant y, \quad x_1 \leqslant_1 y.$$

4.14. Lemma. *Let $x, y \in P$, $x(R_1) \leqslant y(R_1)$ and $x(R_2) \leqslant y(R_2)$. Then $x \leqslant y$.*

Proof. By 4.9 and 4.13 there are elements x_1 and x_2 in P such that

$$x \leqslant x_1, \quad x \geqslant_1 x_1 \quad \text{and} \quad x_1R_1y,$$

$$x \leqslant x_2, \quad x \leqslant_1 x_2 \quad \text{and} \quad x_2R_2y.$$

Then in view of 3.4 and 3.5 (with P and P_1 interchanged) there is a unique element z in $x_1 \vee x_2$; moreover, x_1R_1z and x_2R_2z. Hence zR_1y and zR_2y. Hence according to 4.8, $y = z$. Therefore $x \leqslant y$. □

Consider the mapping $\varphi : P \longrightarrow A \times B$ such that $\varphi(a) = (a(R_1), a(R_2))$ for each $a \in P$. It is evident that if $a, b \in P$ and $a \leqslant b$, then $(a(R_1), a(R_2)) \leqslant (b(R_1), b(R_2))$.

4.15. Lemma. *The mapping $\varphi : P \longrightarrow A \times B$ is a direct product decomposition of the multilattice P.*

Proof. This is a consequence of 4.3, 4.8, 4.6 and 4.14. □

By defining the relations R_1 and R_2 we have viewed the multilattice P as beeing basic. Let us now define relations R'_1 and R'_2 by starting with the multilattice P_1 instead of P; i.e., when defining R'_1 and R'_2 we proceed by the same method as when defining R_1 and R_2 with the distinction that the relations $\leqslant$ and $\leqslant_1$ are interchanged.

Since the assumptions of 3.8 are symmetric with respect to $\leqslant$ and $\leqslant_1$ we obtain immediately that the relations R_1 and R'_1 coincide.

Next, for a and b in P_1 we put aR'_2b if there is t_1 in P_1 such that $t_1 \leqslant_1 a$, $t_1 \leqslant_1 b$, $t_1 \geqslant a$, $t_1 \geqslant b$. However, from the modification of 3.8 (by applying duality and by interchanging P and P_1) we obtain that aR'_2b implies aR_2b; similarly we can prove that aR_2b implies aR'_2b. Thus R_2 and R'_2 coincide as well.

Let A and B be as above. For $i \in \{1, 2\}$ we define a binary relation $\leqslant_1$ on A as follows: for a and b in A we put $a(R'_i) \leqslant_1 b(R'_i)$ if there are $a_1 \in a(R'_i)$ and $b_1 \in b(R'_i)$ such that $a_1 \leqslant_1 b_1$. The set A with this relation will be denoted by A_1. Analogously we define the partially ordered set B_1. Similarly as in 4.15 we can prove

4.15'. Lemma. *A_1, B_1 are partially ordered sets and the mapping*

$$\varphi \colon P_1 \longrightarrow A_1 \times B_1$$

defined by $\varphi(a) = (a(R'_1), b(R'_2))$ is a direct product decomposition of P_1.

Next, from 4.9 and 4.13 we immediately obtain

4.16. Lemma. *Let $a, b \in P$. Then*

$$a(R_1) \leqslant b(R_1) \Longleftrightarrow a(R'_1) \geqslant_1 b(R'_1),$$

$$a(R_2) \leqslant b(R_2) \Longleftrightarrow a(R'_2) \leqslant_1 b(R'_2).$$

For each partially ordered set L we denote by L^d the partially ordered set which is dual to P. Then 4.16 yields

4.17. Corollary. $A_1 = A^d$ and $B_1 = B$.

By summarizing, from 4.15, 4.15′, 4.17 and by constructing the corresponding internal direct product decompositions we infer

4.18. Theorem. *Let P and P_1 be directed multilattices defined on the same underlying sets such that $C(P) = C(P_1)$. Let $s^0 \in P$. Then there exist internal direct product decompositions*

$$\varphi^0 \colon P \longrightarrow A^0 \times B^0, \quad \varphi^0 \colon P_1 \longrightarrow (A^0)^d \times B^0$$

with the central element s^0.

For results related to 4.18 cf. [2] (for the case of finite lattices), [4] (for the case of distributive lattices) and [5] (for the case of lattices).

4.19. Lemma. (i) *If L is a multilattice, then $C(L) = C(L^d)$.* (ii) *Let X, Y be multilattices and $Z \subseteq X \times Y$. Let Z_1 and Z_2 be the projections of Z into X or Y, respectively. Then $Z \in C(X \times Y)$ iff $Z_1 \in C(X)$ and $Z_2 \in C(Y)$.*

The proof is easy, it is omitted.

Let us remark that in (ii) above the two-factor direct product decomposition can be replaced by a direct product decomposition with an arbitrary number of direct factors.

4.20. Corollary. *Let P and P_1 be multilattices defined on the same underlying set. Let $s^0 \in P$. Assume that there exist internal direct decompositions*

$$\varphi^0 \colon P \longrightarrow A^0 \times B^0, \quad \varphi^0 \colon P_1 \longrightarrow (A^0)^d \times B^0$$

with the central element s^0. Then $C(P) = C(P_1)$.

5. Convex isomorphisms

We assume that P and P' are directed multilattices.

5.1. Definition. *A mapping f of P onto P' is called a convex isomorphism if*

(i) *f is a bijection;*
(ii) *for each $X \subseteq P$, $X \in C(P) \Longleftrightarrow f(X) \in C(P')$.*

For the case of lattices, this definition is due to Marmazeev [8]. If $P = P'$, then under the assumptions as in 5.1, f is called a convex automorphism.

For each positive integer n put $\overline{n} = \{1, 2, \ldots, n\}$. The following is the main result of [6].

5.2. Theorem([6], Theorem 10.). *Let L be a lattice which can be decomposed into a direct product $L = L_1 \times L_2 \times \ldots \times L_n$, where all L_i are directly indecomposable. Convex automorphisms of L are just the mappings obtained as follows: we take a permutation π of the set $\overline{n}$ such that there exist bijections $f_i \colon L_i \longrightarrow L_{\pi(i)}$, each of them being either an isomorphism or a dual isomorphism, and set $f(x)_{\pi(i)} = f_i(x_i)$ for any $x \in L$.*

(In 5.2, for each $x \in L$ and each $i \in \overline{n}$, x_i denotes the component of x in L_i.)

5.3. Theorem. *Let P and P' be directed multilattices which can be decomposed into direct products*

$$\varphi\colon P \longrightarrow \prod_{i\in I} P_i, \quad \varphi'\colon P' \longrightarrow \prod_{j\in J} P'_j.$$

Assume that $\pi\colon I \longrightarrow J$ is a bijection and that for each $i \in I, f_i\colon P_i \longrightarrow P_{\pi(i)}$, is a bijection which is either an isomorphism or a dual isomorphism. Then f is a convex isomorphism.

Proof. Without loss of generality we can assume that φ and φ' are internal direct product decompositions with the same central element s^0. Let $I(1)$ be the set of all $i \in I$ such that $f_i\colon P_i \longrightarrow P_{\pi(1)}$ is an isomorphism. Put

$$A = \{x \in P\colon x(P_i) = s^0 \quad \text{for each} \quad i \in I \setminus I(1)\},$$
$$B = \{x \in P\colon x(P_i) = s^0 \quad \text{for each} \quad i \in I(1)\}.$$

Then in view of 4.19 and 4.20 (cf. also the remark after 4.19) we obtain that f is a convex isomorphism. □

5.4. Definition. *Let P and P' be directed multilattices. A bijection $f\colon P \longrightarrow P'$ is said to be a similarity mapping from P to P' if, whenever P can be decomposed into a direct product*

$$\varphi\colon P \longrightarrow \prod_{i\in I} P_i \tag{1}$$

where all P_i are directly indecomposable, then

(i) *there exists a direct product decomposition $\varphi'\colon P' \longrightarrow \prod_{i\in I} P'_i$ such that all P'_i are directly indecomposable;*
(ii) *for each $i \in I$ there exists a bijection $f_i\colon P_i \longrightarrow P'_i$ such that f_i is either an isomorphism or a dual isomorphism;*
(iii) *for each $x \in P$ and each $i \in I$, $f(x)(P'_i) = f_i(x(P_i))$.*

5.5. Theorem. *Let P and P' be a directed multilattices and let $f\colon P \longrightarrow P'$ be a convex isomorphism. Then f is a similarity mapping from P to P'.*

Proof. Let (1) be valid where all P_i are directly indecomposable. Let $a_0 \in P$. Without loss of generality we can assume that φ is an internal direct product decomposition with the central element s^0.

For $x, y \in P$ we put $x \leqslant_1 y$ if and only if $f(x) \leqslant f(y)$. Then $\leqslant_1$ is a partial order on P; the set P with this partial order $\leqslant_1$ will be denoted by P_1. The mapping f^{-1}: $P' \longrightarrow P_1$ is an isomorphism. P_1 is a directed multilattice satisfying

$$C(P_1) = C(P).$$

Hence for P and P_1 we can apply the results from Section 4. According to 4.18 there are internal direct product decompositions

$$\varphi^0 \colon P \longrightarrow A^0 \times B^0, \quad \varphi^0 \colon P_1 \longrightarrow (A^0)^d \times B^0$$

with the same central element s^0.

In view of Theorem (C) in Section 2, φ is a refinement of φ^0 and there are internal product decompositions

$$\varphi_0^1 \colon A^0 \longrightarrow \prod_{i \in I(1)} P_i, \quad \varphi_0^2 \colon B^0 \longrightarrow \prod_{i \in I(2)} P_i$$

with the same central element s^0 such that $I(1) \cap I(2) = \emptyset$ and $I(1) \cup I(2) = I$.

For each $i \in I$ we put $Q_i = (P_i)^d$ if $i \in I(1)$, and $Q_i = P_i$ if $i \in I(2)$. We obtain internal product decompositions

$$\varphi_0^1 \colon (A^0)^d \longrightarrow \prod_{i \in I(1)} Q_i, \quad \varphi_0^2 \colon B^0 \longrightarrow \prod_{i \in I(2)} Q_i$$

with the same central element s^0. By applying this and using the mappings φ^0 and φ we get an internal direct product decomposition

$$\varphi \colon P_1 \longrightarrow \prod_{i \in I} Q_i \tag{2}$$

with the central element s^0. Here all Q_i are directly indecomposable.

Put $(s^0)' = f(s^0)$ and $P_i' = f(Q_i)$ for each $i \in I$. Since f is an isomorphism of P_1 onto P' there is an internal direct decomposition with the central element $(s^0)'$

$$\varphi' \colon P' \longrightarrow \prod_{i \in I} P_i'$$

and all P_i' are directly indecomposable.

For each $i \in I$ and each $y \in P_i$ we put $f_i(y) = f(y)$. We obtain a bijection $f_i \colon P_i \longrightarrow P_i'$. If $i \in I(1)$, then f_i is a dual isomorphism; for $i \in I(2)$, f_i is an isomorphism. Next, by applying the fact that f is an isomorphism of P_1 onto P' and by using (1), (2) we get that for each $x \in P$ and each $i \in I$ the relation

$$f(x)(P_i') = f(x(Q_i)) = f_i(x(Q_i)) = f_i(x(P_i))$$

is valid. □

5.6. Theorem. *Let P be a multilattice which has a direct product decomposition $\varphi\colon P \longrightarrow \prod_{i\in I} P_i$ such that all P_i are directly indecomposable. Let f be a convex automorphism of P. Then there exist*

(i) *a bijection $\pi\colon I \longrightarrow I$,*

(ii) *bijections $f_i\colon P_i \longrightarrow P_{\pi(i)}$ where for each $i \in I$, f_i is either an isomorphism or a dual isomorphism, such that $f(x)_{\pi(i)} = f_i(x_i)$ for each $x \in P$.*

Proof. The assertion is trivial in the case $\operatorname{card} P = 1$. Thus we can assume that $\operatorname{card} P > 1$. Then without loss of generality we can suppose that φ is an internal direct product decomposition with a central element s^0 and that $\operatorname{card} P_i > 1$ for each $i \in I$.

We apply 5.3 where we put $P' = P$. In view of the constructions in the proof of 5.4 we have an internal direct product decomposition (with the central element s^0)

$$\varphi'\colon P \longrightarrow \prod_{i\in I} P_i'$$

such that for each $i \in I$ there is a bijection $f_i\colon P_i \longrightarrow P_i'$, this bijection being either an isomorphism or a dual isomorphism. Hence all P_i' are directly indecomposable.

According to (B) there exists an internal direct product decomposition ψ of P with the central element s^0 such that ψ is a refinement of both φ and φ'. Again without loss of generality we can assume that each direct factor standing in ψ fails to be a one-element set. Then, since all P_i are directly indecomposable, by applying (A) we obtain that $\varphi = \psi$; similarly, $\varphi' = \psi$. Thus $\varphi = \varphi'$.

Hence for each $i \in I$ there exists $\pi(i) \in I$ such that $P_i' = P_{\pi(i)}$. Then $\pi\colon I \longrightarrow I$ is a bijection and the condition (ii) is satisfied. Next, in view of 5.5 and the condition (iii) in 5.4 we have (under the obvious notation)

$$f(x)_{\pi(i)} = f_i(x_i) \quad \text{for each} \quad x \in P.$$

□

Theorem 5.2 is a consequence of 5.3 and 5.6.

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