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SOME APPLICATIONS OF KURZWEIL-HENSTOCK INTEGRATION

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Summary. Applications of ideas from Kurzweil-Henstock integration to elementary analysis on **R**, mean value theorems for vector valued functions, l'Hospital rule, theorems of Taylor type and path independence of line integrals are discussed.

Keywords: Perron integral, KH-integral, fundamental theorem, l'Hospital, Taylor, differentiation of series.

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1. INTRODUCTION

One of the main features of the Perron integral is that it integrates every derivative without any restriction. It seems that this property was not fully exploited e.g. in teaching of analysis, because of the nonelementary character of Perron's definition. In 1957 Kurzweil [K], in connection with research in differential equations, gave an elementary definition equivalent to the Perron one, moreover the proof of the fundamental theorem became then extremely simple. For Kurzweil's own presentation of the theory see [K1]. Henstock later [H] independently rediscovered Kurzweil's approach and advanced it further [H1–4]. It is the aim of this paper to give several applications of these ideas in elementary analysis. Only the most rudimentary knowledge of K-H integration is needed for understanding of this article.

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2. NOTATION, BASIC FACTS

A partition of a compact interval $[a, b]$ is a set of couples (ξ_k, I_k) such that the points $\xi_k \in [a, b]$, the intervals I_k are non-overlapping and

$$\bigcup_1^n I_k = [a, b]. \tag{1}$$

The set of couples (ξ_k, I_k) satisfying all the above properties except (1) is called a subpartition. A partition is, of course, also a subpartition. A subpartition with the additional property that $\xi_k \in I_k$ is a P-subpartition. We shall be dealing only with P-subpartitions and shall omit the qualifying letter P. Often it will be convenient to have the intervals, $I_k = [u_k, v_k]$, ordered hence for a partition $\Pi \equiv \{\xi_k, [u_k, v_k]\}$ we have

$$\Pi \equiv a = u_1 \leqslant \xi_1 \leqslant v_1 = u_2 \leqslant \xi_2 \leqslant v_2 \leqslant \dots \leqslant \xi_n \leqslant v_n = b.$$

If $\delta \colon [a, b] \mapsto (0, \infty)$ then a partition Π for which

$$\xi_i - \delta(\xi_i) < u_i \leqslant \xi_i \leqslant v_i < \xi_i + \delta(\xi_i)$$

for all i with $1 \leqslant i \leqslant n$ is called a δ-fine partition of $[a, b]$. It is obvious what we mean by a δ-fine subpartition. The set of all δ-fine partitions will be denoted by $\mathscr{P}(\delta)$. Hence, instead of saying that Π is δ-fine, we write $\Pi \in \mathscr{P}(\delta)$. For any positive function δ and any compact interval $[a, b]$ a δ-fine partition of $[a, b]$ always exists. The existence and the use of δ-fine partitions has been traced by Mawhin [Ma] to Cousin in the last century. We shall refer to the statement guaranteeing the existence of a δ-fine partition for any positive function δ as to Cousin's lemma. The contrast between the early discovery of Cousin's lemma and rather late arrival of Kurzweil-Henstock integral is a bit surprising. As Hadamard once said: "In mathematics simple ideas come late."

3. REAL LINE APPLICATIONS

It has been shown, for instance in [V1], that δ-fine partitions can be used advantageously in teaching analysis on the real line.

As a a first example we prove the Bolzano-Cauchy convergence principle. The usual argument can be used to show that a Cauchy sequence must be bounded. Next we show that if $n \mapsto x_n$ is a Cauchy sequence and x is not a limit of $n \mapsto x_n$ then there exists $\delta > 0$ such that the interval $(x - \delta, x + \delta)$ contains only a finitely many x_n. Indeed, there exists a subsequence $k \mapsto x_{n_k}$ and a positive ε_0 such that

$|x - x_{n_k}| \geqslant \varepsilon_0$ for all k. Since $n \mapsto x_n$ is Cauchy there is an integer N such that $|x_n - x_m| < \varepsilon_0/2$ for $n > N$ and $m > N$. Consequently, for some x_{n_k} with $n_k > N$ and all $n > N$ we have $|x - x_n| \geqslant |x - x_{n_k}| - |x_{n_k} - x_n| > \varepsilon_0/2 = \delta$. Assume now, contrary to what we want to prove, that no $x \in [-K, K]$ is a limit of our sequence and that the interval [-K,K] contains all x_n. By Cousin's lemma there exists a δ-fine partition of $[-K, K]$. This partition has finitely many intervals and each of them has in turn only finitely many x_n. This contradicts the fact that $[-K, K]$ contains all x_n.

Cousin's lemma has been shown to be equivalent to the least upper bound axiom and has been used in the proofs of the following theorems: Bolzano-Weierstrass, intermediate value theorem, uniform continuity, uniform approximation of continuous functions by piecewise linear functions, Weierstrass' theorems on boundedness and extreme values of continuous functions and others. We shall not repeat these proofs here but refer to articles by Botsko [Bo], [Bo1], Bullen [B] and the author [V1]. Besides all this there is a host of theorems asserting something about the increment of a function from some information concerning the derivative. The simplest example is: $f' > 0 \Rightarrow f$ increasing, a more sophisticated example is: f absolutely continuous, g increasing and $|f'| \leqslant g$ a.e. implying $|f(b) - f(a)| \leqslant g(b) - g(a)$. We shall prove a most general theorem in this direction but we need some definitions first. We shall say that $F\colon [a, b] \mapsto \mathbf{C}$ varies negligibly on a set $M \subset [a, b]$ if for every $\varepsilon > 0$ there exists a $\delta\colon [a, b] \mapsto (0, \infty)$ such that for every δ-fine subpartition Π_s with all $\xi_i \in M$ we have

$$\sum_{\Pi_s} |F(v_i) - F(u_i)| < \varepsilon. \tag{2}$$

Examples of F varying negligibly on M are:

(1) M countable and F continuous on M;

(2) M of measure zero and F absolutely continuous;

(3) $M = [\alpha, \beta] \subset (a, b)$ and F constant on M and continuous at α and β.

We shall say that F has *the strong Lusin property* if it varies negligibly on every set of measure zero. Instead of saying that F varies negligbly on M or that F has the strong Lusin property we shall say that $F \in VNM$ or $F \in SL$, respectively.

Theorem. Substitute for the Cauchy Mean Value Theorem. *If* $F\colon [a, b] \mapsto \mathbf{R}$ *varies negligibly on* M, g *is increasing and*

$$|F'(x)| \leqslant g'(x) \text{ for } x \notin M \tag{3}$$

then

$$|F(b) - F(a)| \leqslant g(b) - g(a). \tag{4}$$

Corollary. *Inequality* (3) *implies* (4) *if*

a. *F has the strong Lusin property and M is of measure zero;*
b. *F is absolutely continuous and M is of measure zero;*
c. *M is countable and F is continuous on M.*

Proof of the theorem. For every $\varepsilon > 0$ and every $\xi \notin M$ there is a positive δ such that

$$|F(v) - F(u)| < (|F'(\xi)| + \varepsilon)(v - u) \tag{5}$$

and

$$g'(\xi)(v - u) < g(v) - g(u) + \varepsilon(v - u), \tag{6}$$

for

$$\xi - \delta < u \leqslant \xi \leqslant v < \xi + \delta.$$

By (3), (5) and (6) we have

$$|F(v) - F(u)| < g(v) - g(u) + 2\varepsilon(v - u).$$

We define δ on M by using the fact that $F \in VNM$ then we have (2). If now $\Pi \in \mathcal{P}(\delta)$ we obtain

$$\begin{aligned} |F(b) - F(a)| &= \sum_{i=1}^{n} |F(v_i) - F(u_i)| \leqslant \sum_{\xi_i \in M} |F(v_i) - F(u_i)| \\ &\quad + \sum_{\xi_i \notin M} |F(v_i) - F(u_i)| < \sum_{\xi_i \notin M} [g(v_i) - g(u_i) + 2\varepsilon(v_i - u_i)] + \varepsilon \\ &\qquad < g(b) - g(a) + \varepsilon[2(b - a) + 1] \end{aligned}$$

□

4. A Mean Value Theorem

The Mean Value Theorem is, generally speaking, not valid for complex or vector-valued functions. There are two types of valid generalizations. In the first the equation $f(b)-f(a)=f'(c)(b-a)$ is replaced by an inequality, for a complex-valued f this inequality reads $|f(b)-f(a)| \leqslant |f'(c)|(b-a)$. This was already known in 1876 to Darboux and was extended to linear normed spaces by Aziz and Diaz. The assertion in the second generalization states that the quotient $[f(b)-f(a)]/(b-a)$ lies in the convex hull of the range of the derivative. For linear normed spaces this theorem is due to Ważewski but for complex valued function goes back as far as Weierstrass. Both generalizations can be established by the same method used in the proof of the Cauchy mean value theorem below. For sake of simplicity we restrict our attention to a normed space X, denote the norm by $|\cdot|$ and extend the notion of a vector valued F varying negligibly on M by interpreting $|\cdot|$ in (2) as the norm. If $A \subset X$ then $\operatorname{cl} A$ and $\operatorname{co} A$ denotes the topological closure and the convex hull of A, respectively. The $\operatorname{clco} A$ stands for closed convex hull of A which is both the closure of $\operatorname{co} A$ and the smallest closed convex set containing A.

The Cauchy Mean Value Theorem. *If*

(a) $F\colon [a,b] \mapsto X$;
(b) *g is strictly increasing on $[a,b]$;*
(c) *there is a set M such that $F'(x)$ and $g'(x)$ exist with $g'(x) \neq 0$ for $x \notin M$, moreover $[a,b] \setminus M \neq \emptyset$;*
(d) *both F and $g \in VNM$;*
(e) $\mathfrak{D} = \{F'(x)/g'(x);\ x \in [a,b] \setminus M\}$

then

$$\frac{F(b)-F(a)}{g(b)-g(a)} \in \operatorname{clco} \mathfrak{D}.$$

Remark. A statement similar to the Corollary holds, in particular, the theorem remains valid if (d) is replaced by

(d′) *both F and g are continuous on $[a,b]$ and M is countable.*

Proof. Choose $\Xi \in [a,b] \setminus M$. Let $\varepsilon > 0$. By using (d) we define δ on M in such a way that

$$\begin{gathered} \sum_{\Pi_\bullet} |F(v_i) - F(u_i)| < \varepsilon[g(b)-g(a)] \\ \left|\frac{F'(\Xi)}{g'(\Xi)}\right| \sum_{\Pi_\bullet} [g(v_i) - g(u_i)] < \varepsilon[g(b)-g(a)] \end{gathered} \tag{7}$$

for any δ-fine subpartition Π_s for which all $\xi \in M$. It follows from the definition of derivative by simple calculation that for every $\xi \notin M$ there exists a positive δ such that

$$\left|F(v) - F(u) - \frac{F'(\xi)}{g'(\xi)}(g(v) - g(u))\right| < \varepsilon(g(v) - g(u)). \tag{8}$$

for

$$\xi - \delta < u \leqslant \xi \leqslant v < \xi + \delta.$$

Let $\Pi \in \mathscr{P}(\delta)$ and define

$$Z_\Pi = \sum_{\xi_i \notin M} \frac{F'(\xi_i)}{g'(\xi_i)} \frac{g(v_i) - g(u_i)}{g(b) - g(a)} + \frac{F'(\Xi)}{g'(\Xi)} \sum_{\xi_i \in M} \frac{g(v_i) - g(u_i)}{g(b) - g(a)}.$$

Clearly $Z_\Pi \in \operatorname{co} \mathfrak{D}$. We complete the proof by showing that Z_Π comes arbitrarily close to $(F(b) - F(a))/(g(b) - g(a))$. Firstly we have

$$F(b) - F(a) = \sum_{\xi_i \in M} F(v_i) - F(u_i) + \sum_{\xi_i \notin M} F(v_i) - F(u_i)$$

and then by (8)

$$\left|\sum_{\xi_i \notin M} \left(F(v_i) - F(u_i) - \frac{F'(\xi_i)}{g'(\xi_i)}(g(v_i) - g(u_i))\right)\right| < \varepsilon\,(g(b) - g(a))\,.$$

This together with (7) gives

$$\left|\frac{F(b) - F(a)}{g(b) - g(a)} - Z_\Pi\right| < 3\varepsilon.$$

□

C o m m e n t s , c o u n t e r e x a m p l e s . The theorem can be extended to locally convex topological spaces but not much further. There is an example of a function F and a linear topological space X such that $F\colon [a, b] \mapsto X$, the derivative F' equals zero *everywhere* on $[a, b]$ and F is not constant. See [R] or [Ya, p. 14].

Let

$$\mathfrak{Q} = \left\{\frac{F(v) - F(u)}{g(v) - g(u)}\,;\ [u, v] \subset [a, b]\right\}.$$

It is a direct consequence of the above Cauchy Mean Value theorem that $\mathfrak{Q} \subset \operatorname{clco} \mathfrak{D}$ and hence $\operatorname{clco} \mathfrak{Q} \subset \operatorname{clco} \mathfrak{D}$. On the other hand $\mathfrak{D} \subset \operatorname{cl} \mathfrak{Q} \subset \operatorname{clco} \mathfrak{Q}$, consequently $\operatorname{clco} \mathfrak{D} \subset \operatorname{clco} \mathfrak{Q}$. Hence we have:

Corollary to CMVT. $\operatorname{clco}\mathfrak{D} = \operatorname{clco}\mathfrak{Q}$.

If X is finite dimensional then there is a linear variety of smallest possible dimension containing $\operatorname{co}\mathfrak{D}$, call it H. Then $(\operatorname{co}\mathfrak{D})^0$, the interior of $\operatorname{co}\mathfrak{D}$ relative to H, is not empty and we show that

$$q = \frac{F(b) - F(a)}{g(b) - g(a)} \in (\operatorname{co}\mathfrak{D})^0.$$

Assume now, contrary to what we want to prove, that q is not in $(\operatorname{co}\mathfrak{D})^0$. Then there exists a supporting linear variety at q, i.e. there is a linear functional p and a real α such that $p(q) = \alpha$ and $p(x) \leqslant \alpha$ for all $x \in \operatorname{co}\mathfrak{D}$. However, there must be at least one interval $[u, v]$ such that

$$p(\frac{F(v) - F(u)}{g(v) - g(u)}) < \alpha,$$

otherwise H would not be minimal. Now we have[2]

$$\begin{aligned} p(q) = \frac{g(u) - g(a)}{g(b) - g(a)} p\left(\frac{F(u) - F(a)}{g(u) - g(a)}\right) \\ + \frac{g(v) - g(u)}{g(b) - g(a)} p\left(\frac{F(v) - F(u)}{g(v) - g(u)}\right) \\ + \frac{g(b) - g(v)}{g(b) - g(a)} p\left(\frac{F(b) - F(v)}{g(b) - g(v)}\right) < \alpha, \end{aligned}$$

a contradiction.

If X is not finite dimensional then q need not lie in $\operatorname{co}\mathfrak{D}$, the closure is essential. An example showing this is again in [Ya]. The CMVT can be further generalized, any limiting value of $(F(v) - F(u))/(g(v) - g(u))$ can be used in the definition of $\mathfrak{D}$ instead of $F'(\xi)/g'(\xi)$, see e.g. [Mcl]. Unfortunately, our method does not seems to be easily adaptable to such an extension.

5. Fundamental theorem

In Kurzweil-Henstock theory the formula

$$\int_a^b F' = F(b) - F(a)$$

holds for a continuous F if

(i) the derivative F' exists except possibly a countable set M;
(ii) the derivative F' exists almost everywhere on $[a, b]$ and $F \in SL$.

[2] If $u = a$ or $b = v$ the undefined terms should be omitted

There are other conditions under which the equation above holds but these two suffice for our purposes. Of course, if (i) holds then so does (ii), we stated (i) because of its frequent use.

5.1. Differentiation of series. The general theorem on term by term differentiation of series is usually considered to be out of reach of an average undergraduate. For instance, E. Landau—who certainly would not exaggerate difficulty—says in his book [La] that the proof of the theorem on differentiation of series term by term is one of the most difficult in the whole book. We are going to give a simple proof of the following

Theorem. *Let us assume that*

(A) *the condition (ii) is satisfied with F replaced by F_n, for every $n \in \mathbb{N}$;*
(B) *for some $c \in [a, b]$, the sequence $n \mapsto F_n(c)$ converges, say to $F(c)$;*
(C) *the sequence $n \mapsto F'_n$ converges uniformly almost everywhere to g, say.*

Then the sequence $n \mapsto F_n$ converges uniformly on $[a, b]$ and the limit function F is differentiable at every x for which $\lim\limits_{n\to\infty} F'_n(x)$ exists and then $F'(x) = g(x)$.

Remark. The most important case is, of course, when $\lim\limits_{n\to\infty} F'_n$ exists everywhere on (a, b); the function F is also differentiable everywhere on (a, b).

Proof. Firstly

$$F_n(x) = F(c) + \int_a^x F'_n.$$

By uniform convergence

$$F(x) = F(c) + \int_a^x g.$$

Moreover the convergence of F_n to F is clearly uniform. Assume now the existence of

$$\lim_{n\to\infty} F'_n(x).$$

For a positive ε there is a natural n such that $F'_n(x)$ exists and

$$|F'_n(x) - g(x)| < \varepsilon \quad \text{and} \quad \left|\frac{1}{h}\int_x^{x+h} (g(t) - F'_n(t))\,\mathrm{d}t\right| < \varepsilon.$$

Since

$$\frac{F(x+h) - F(x)}{h} - g(x) = \frac{1}{h}\int_x^{x+h} (g(t) - F'_n(t))\,\mathrm{d}t \\ + \frac{F_n(x+h) - F_n(x)}{h} - F'_n(x) + F'_n(x) - g(x),$$

we obtain

$$\left|\frac{F(x+h)-F(x)}{h}-g(x)\right|<2\varepsilon+\left|\frac{F_n(x+h)-F_n(x)}{h}-F_n'(x)\right|.$$

By the choice of x and n there exists a positive δ such that the last term is less than ε for $0<|h|<\delta$. Consequently

$$\left|\frac{F(x+h)-F(x)}{h}-g(x)\right|<3\varepsilon.$$

□

5.2. L'Hospital rule. This rule is not valid for complex valued functions. For a counterexample define

$$f(x)=\sqrt{x},\quad g_1(x)=\sqrt{x}+x\sin x^{-1},\quad g_2(x)=\sqrt{x}+x\cos x^{-1}$$

and have

$$\begin{aligned}\lim_{x\downarrow 0} f(x)&=0,\\ \lim_{x\downarrow 0}(g_1(x)+\mathrm{i}g_2(x))&=0,\\ \lim_{x\downarrow 0}\frac{f(x)}{g_1(x)+ig_2(x)}&=\frac{1}{1+\mathrm{i}},\\ \lim_{x\downarrow 0}\frac{f'(x)}{g_1{}'(x)+\mathrm{i}g_2{}'(x)}&=0.\end{aligned}$$

Using the Fundamental Theorem, we prove below a theorem which states the additional conditions under which l'Hospital's rule remains valid for complex valued functions.

Theorem. *If $f,g\colon(0,b)\mapsto\mathbf{C}$ are continuous,*

$$\lim_{x\downarrow 0} f(x)=\lim_{x\downarrow 0} g(x)=0,$$

the inequality $g(x)\neq 0$ holds for all $x\in(0,b)$, the derivatives f' and g' exist on $[0,b]$ except a countable set M,

(9) $$\limsup_{x\downarrow 0}\frac{\int_0^x|g'(t)|\,\mathrm{d}t}{|g(x)|}=K<\infty,$$

and for every sequence $x_n \to 0$ with $x_n \notin M$ we have

$$\lim_{n\to\infty} \frac{f'(x_n)}{g'(x_n)} = L \in \mathbf{C}, \tag{10}$$

then

$$\lim_{x\downarrow 0} \frac{f(x)}{g(x)} = L.$$

Remark. If M is empty then (10) can be rephrased in the usual way as $\lim_{x\downarrow 0} f'(x)/g'(x) = L$.

Proof. It follows from (10) and (9) that for every positive ε there is a positive δ such that for $0 < x < \delta$

$$|f'(x) - Lg'(x)| < \varepsilon |g'(x)|, \quad x \notin M$$

and

$$\frac{\int_0^x |g'(t)|\,\mathrm{d}t}{|g(x)|} < K + 1.$$

Consequently, for every x with $0 < x < \delta$, we have

$$\left|\frac{f(x)}{g(x)} - L\right| = \frac{\left|\int_0^x (f'(t) - Lg'(t))\,\mathrm{d}t\right|}{|g(x)|} \leqslant \frac{\varepsilon \int_0^x |g'(t)|\,\mathrm{d}t}{|g(x)|} \leqslant \varepsilon(K+1).$$

□

Comments. The limit passage $x \downarrow 0$ can be replaced by any of the following: $x \downarrow a$, $x \uparrow a$, $x \to a$, $x \to \infty$ and $x \to -\infty$ with obvious changes to the theorem. The Fundamental Theorem was used in the proof of l'Hospital rule first probably by Huntington [Hu] and advocated by Boas [Bs]. The assumption (9) was introduced by G. Szabó [S] in case of real and absolutely continuous f and g. A discussion for one-sided derivatives is given in [VN]. As with many other theorems in mathematics, l'Hospital's rule is wrongly named—the theorem was discovered by Johann Bernoulli who communicated it in a letter to marquis l'Hospital.

5.3. The Taylor Theorem. It is sometimes stated even in very good texts that the Lagrange and Cauchy form of remainder are more general than the integral form. This is not as much a statement about the nature of the theorem as it is about the concept of the integral used in these formulae. Let us denote

$$T_n = f(a) + f'(a)(b-a) + \ldots + f^n(a)\frac{(b-a)^n}{n!}$$

and

$$R_{n+1} = \int_a^b f^{(n+1)}(t) \frac{(b-t)^n}{n!}\, dt. \tag{11}$$

The usual Taylor's formula with integral remainder

$$f(b) = T_n + R_{n+1} \tag{12}$$

holds under the mere assumption that $f^{(n)}$ is continuous on $[a,b]$ and $f^{(n+1)}$ exists on (a,b) except possibly a countable set. However, the integral (11) ought to be interpreted as a Kurzweil integral. With our fairly general assumption on $f^{(n+1)}$, the identity (12) is valid neither with Lebesgue nor with Riemann integral. Since the derivative need not exist everywhere on (a,b) the Lagrange (or Cauchy) remainder is not valid either. On the other hand it is possible to prove the mean value theorem of integral calculus for the KH-integral and use it to deduce the Lagrange (or Cauchy) remainder under the additional assumption that $f^{(n+1)}$ exits everywhere on (a,b). It is therefore fair to say that the integral form of the remainder is more general than either Cauchy's or Lagrange's. For proofs and details we refer to [T] or [T1].

Our aim is to estimate the difference

$$S_{n+1} = f(b) - T_{n-1} - f^{(n)}\left(\frac{na+b}{n+1}\right)\frac{(b-a)^n}{n!}.$$

Assuming continuity of $f^{(n+2)}$, Poffald [Po] obtained the formula

$$S_{n+1} = \frac{n}{2(n+1)} f^{(n+2)}(\xi)\frac{(b-a)^{n+2}}{(n+2)!}, \tag{13}$$

for some $\xi \in (a,b)$. We give a different (simpler) proof and remove the continuity assumption. We now have

Modified Taylor's formula. *There exists a positive continuous function K_n, $n \in \mathbf{N}$ with*

$$\int_a^b K_n = \frac{n}{2(n+1)} \frac{(b-a)^{n+2}}{(n+2)!} \tag{14}$$

such that if $f^{(n+1)}$ is continuous on $[a,b]$ and $f^{(n+2)}$ exists on (a,b) except possibly a countable set then

$$S_{n+1} = \int_a^b K_n(t) f^{(n+2)}(t)\, dt. \tag{15}$$

Remark. Let H denote the Heaviside function, i.e. let $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t > 0$. Explicit formula can be given for K_n, namely

$$(16)\quad K_n(t) = \frac{1}{(n+1)!}\left[(b-t)^{n+1} + (b-a)^n H\left(\frac{na+b}{n+1} - t\right)((n+1)t - na - b)\right].$$

Accepting (16) as the definition of K_n, it is easy to verify (14). Equation (16) shows that K_n is continuous on $[a,b]$. By using the formula $[H(\alpha - t)(t-\alpha)]' = H(\alpha - t)$ for $t \neq \alpha$ we see that

$$(17)\qquad K_n'(t) = \frac{1}{n!}\left[-(b-t)^n + H\left(\frac{na+b}{n+1} - t\right)(b-a)^n\right].$$

Clearly, $K_n' > 0$ on $\left(a, \frac{na+b}{n+1}\right)$ and $K_n(a) = 0$, consequently $K_n \geqslant 0$ on $[a,b]$. If $f^{(n+2)}(t)$ exists for every $t \in (a,b)$ then as a derivative it has the intermediate value property. Hence we can use the mean value theorem of integral calculus on the integral in (15) and (14) leads directly to Poffald's result (13). The example of $f(t) = |t|^3/3!$ with $a = -1$, $b = 1$, $n = 1$ shows that if $f^{(n+2)}$ does not exist at one point then formula (13) may fail.

Remark. For the proof we need integration by parts. It follows from the Fundamental Theorem that the the usual formula

$$(18)\qquad \int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG$$

holds if F and G are continuous on $[a,b]$, $F' = f$ and $G' = g$ on $[a,b]$ except a countable set and one of the integrals in (18) exists.

Proof. We employ the definition of T_n and (12) and obtain

$$(19)\qquad S_{n+1} = R_{n+1} - \left[f^{(n)}\left(\frac{na+b}{n+1}\right) - f^{(n)}(a)\right]\frac{(b-a)^n}{n!};$$

and with the use of the fundamental theorem, (11) and (17)

$$(20)\qquad S_{n+1} = R_{n+1} - \frac{(b-a)^n}{n!}\int_a^{\frac{na+b}{n+1}} f^{(n+1)} = -\int_a^b f^{(n+1)}K_n'.$$

Since $K_n(a) = K_n(b) = 0$, integrating the last integral in (20) by parts completes the proof. □

6. A LINE INTEGRAL

In this section the letters G and S will stand for an open set and an open star-shaped set in $\mathbf{R}^n$, $n \geqslant 2$, respectively. We denote by $d(M)$, $A(M)$ and $p(M)$ the diameter, the two dimensional area and the perimeter[3] of M, in that order. A generic point in $\mathbf{R}^n$ will be denoted by $(x^1, x^2, \dots, x^n)$, hence for a mapping $F\colon G \mapsto \mathbf{R}^n$ we have $F = (F^1, F^2, \dots F^n)$ with $F^i\colon G \mapsto \mathbf{R}^1$. Partial derivatives will be denoted by subscripts, consequently $\partial/\partial x_i F^j = F^j_{,i}$. The word ***path*** will be used for a continuous map of bounded variation from an interval in $\mathbf{R}$ into $\mathbf{R}^n$. We shall allow a slight abuse of notation and use the same symbol for a closed path and its geometrical image (correspondingly oriented). By a line integral $\int_\varphi F(x)\,\mathrm{d}x$ or $\int_\varphi \sum_1^n F^i(x)\,\mathrm{d}x^i$ we understand the Kurzweil-Henstock limit of the Riemann sums

$$\sum_k \sum_{i=1}^n F^i(\varphi(\xi_k))(\varphi^i(v_k) - \varphi^i(u_k)).$$

It is a classical result that if F has continuous partial derivatives in S and

$$F^i_{,j}(x) = F^j_{,i}(x) \tag{21}$$

for all $x \in S$ then the line integral is independent of the path and there exists a function U with $U_{,i}(x) = F^i(x)$ for all $x \in S$. Moreover U is obtained by choosing an arbitrary point x_0 in S and integrating F from x_0 to a variable point x along any path in S. Our aim in this section is to reduce the assumption of continuous derivatives to mere differentiabilty of F. Results of this nature can be also obtained by using the work of Jarník, Kurzweil, Schwabik, Mawhin and Pfeffer, however proving the Stokes theorem in sufficient generality is a rather sophisticated matter (perhaps not quite necessary for this purpose), whereas our approach is fairly simple and straightforward. We shall need the following generalization of Cousin's lemma: If T is a triangle and T_k; $k = 1, \dots, r$ are nonoverlapping triangles with $\bigcup_1^r T_k = T$, points y_k belong to T_k and $\delta\colon T \mapsto (0, \infty)$ then we say that the set $\{(y_k, T_k);\ k = 1, 2, \dots, r\}$ is a δ-fine partition of T if $d(T_k) < \delta(y_k)$. *For a triangle $T \in \mathbf{R}^n$ there always exists a δ-fine partition consisting of triangles* similar *to T*. An indirect proof can be given which follows the usual pattern of the one-dimensional bisection argument except that now T would be divided into four similar triangles formed by mid-points of sides of T. We shall say that assumption $\mathscr{D}$ is satisfied in G if F is continuous in G

[3] in the elementary geometrical sense

and there exists a countable set M such that F is differentiable and satisfies (21) in $G \setminus M$.

Lemma. *If $\mathscr{D}$ is satisfied in G and the triangle $T \subset G$ then*

$$\int_T F(x)\,\mathrm{d}x = 0. \tag{22}$$

P r o o f. The elements of M can be enumerated and for $w_m \in M$ and for arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|F(x) - F(w_m)| < \frac{\varepsilon}{2^m} \tag{23}$$

whenever $|x - w_m| < \delta$. For $y \in S \setminus M$ there is a $\delta > 0$ such that for $i = 1, 2, \dots, n$

$$\Big|F^i(x) - F^i(y) - \sum_{j=1}^{n} F^i_{,j}(y)(x^j - y^j)\Big| < \frac{\varepsilon}{n}|y - x| \tag{24}$$

whenever $|x - y| < \delta$. Let $T \subset G$ and $\{y_k, T_k\}$ a δ-fine partition of T with triangles similar to T. If $y_k \in M$ then $y_k = w_m$ for some m and we obtain from (23)

$$\int_{T_k} F(x)\,\mathrm{d}x < \frac{\varepsilon}{2^m} p(T_k) \leqslant \frac{\varepsilon}{2^m} p(T). \tag{25}$$

If $y_k \notin M$ then because of (21)

$$\int_{T_k} \sum_{i=1}^{n} \Big(F^i(y) + \sum_{j=1}^{n} F^i_{,j}(y)(x^j - y^j)\Big)\,\mathrm{d}x^i = 0. \tag{26}$$

This can be seen most easily by realizing that the integrand in (26) has a 'primitive' $\tilde{U}$, where $\tilde{U}(x) = \sum\limits_{i=1}^{n} F^i(y)x^i + \frac{1}{2} \sum\limits_{i,j=1}^{n} F^i_{,j}(y)(x^j - y^j)(x^i - y^i)$. For $y_k \notin M$ we get from (24) and(26) that

$$\Big|\int_{T_k} F(x)\,\mathrm{d}x\Big| < \varepsilon d(T_k)p(T_k). \tag{27}$$

The triangles T_k are similar to T, therefore there exists a constant C depending *only* on T (and independent of k) such that $d(T_k)p(T_k) < CA(T_k)$ for all k. Consequently (27) becomes

$$\Big|\int_{T_k} F(x)\,\mathrm{d}x\Big| < C\varepsilon A(T_k). \tag{28}$$

Obviously

$$\int_T F(x)\,\mathrm{d}x = \sum_{k=1}^{r} \int_{T_k} F(x)\,\mathrm{d}x,$$

which finally by (25) and (28) gives

$$\int_T F(x)\,\mathrm{d}x < \varepsilon(CA(T) + p(T)).$$

□

Theorem. Path independence. *If $\mathscr{D}$ is satisfied in S then there is a function $U\colon S \mapsto \mathbf{R}$ such that*

$$dU(x) = \sum_{i=1}^{n} F^i(x)\,\mathrm{d}x^i \tag{29}$$

for all x in S. If $\varphi\colon [a,b] \mapsto S$ is a path lying in S then

$$\int_\varphi F(x)\,\mathrm{d}x = U(\varphi(b)) - U(\varphi(a)). \tag{30}$$

Proof. It suffices to prove (29), equation (30) then follows by an argument similar to that one used in the proof of the Fundamental Theorem in K-H theory. To see this choose $\varepsilon > 0$. For every $\xi \in [a,b]$ there is a positive δ such that if $|z - \xi| < \delta$ then

$$|U(\varphi(z)) - U(\varphi(\xi)) - \sum_1^n F^i(\varphi(\xi))(\varphi^i(z) - \varphi^i(\xi))| < \varepsilon|\varphi(z) - \varphi(\xi))|.$$

Consequently, if $\xi - \delta < u \leqslant \xi \leqslant v < \xi + \delta$ then

$$\left|U(\varphi(v)) - U(\varphi(u)) - \sum_1^n F^i(\varphi(\xi))(\varphi^i(v) - \varphi^i(u))\right| < \varepsilon \operatorname*{Var}_u^v \varphi. \tag{31}$$

For Π a δ-fine partition of $[a,b]$ we obtain with the help of (31)

$$|U(\varphi(b)) - U(\varphi(a)) - \sum_\Pi \sum_1^n F^i(\varphi(\xi_k))(\varphi^i(v_k) - \varphi^i(u_k))| < \varepsilon \operatorname*{Var}_a^b \varphi.$$

This establishes the implications (29)⇒(30). We denote by $l(x)$ the path whose geometrical image joins the centre of the star-shaped region S with x and define

$$U(x) = \int_{l(x)} F(z)\,\mathrm{d}z.$$

It follows from the Lemma that

$$U(x+h) - U(x) = \int_{\psi} F(z)\,\mathrm{d}z,$$

where $\psi(t) = x + th$, with $0 \leqslant t \leqslant 1$. Routine continuity argument now gives (29). $\square$

G e n e r a l i z a t i o n s . If φ_1 and φ_2 are two paths homotopic with fixed ends in G and assumption $\mathscr{D}$ is satisfied in G then

$$\int_{\varphi_1} F(z)\,\mathrm{d}z = \int_{\varphi_2} F(z)\,\mathrm{d}z.$$

This can be proved in two different ways. Firstly the integral $\int_{\varphi} F(x)\,\mathrm{d}x$ is independent of the path locally, i.e. in some neighborhood of every point in G. Using this one can employ the usual homotopy argument (see e.g. [C]) to obtain the result in the large. Alternatively, one can use Cousin's lemma (with squares rather than triangles) for the homotopy square and proceed similarly as in the Lemma. This would necessitate the use of a differentiable (compare [V2]) homotopy, the general result must then be obtained by an approximation argument.

R e m a r k . The assumption $\mathscr{D}$ can be weakened without essentially changing the proofs. One needs to assume the differentiability on two-dimensional planes only, the modified assumption reads as follows: We say that weak-$\mathscr{D}$ is satisfied in G if F is continuous in G and for every two-dimensional plane P there exists a countable set M_P such that for every $x \in G \cap (P \setminus M_P)$ there exists a symmetric n by n matrix $[a_{ij}]$ with the following property: Given $\varepsilon > 0$ and $x \in P \cap G$ there is a positive δ such that for every y and z in P with $|x-y| < \delta$ and $|x-z| < \delta$ we have

$$\left| \int_{\overline{yz}} \sum_{i=1}^{n} \Big(F^i(u) - F^i(x) - \sum_{j=1}^{n} a_{ij}(u^i - x^i)\Big) du^i \right| < \varepsilon |y-z| \max(|x-y|, |x-z|).$$

The lemma and the theorem on path independence remain valid if the assumption $\mathscr{D}$ is replaced by weak-$\mathscr{D}$. This allows the set where F is not differentiable or equation (21) is not satisfied to be uncountable. It is an interesting problem to determine how big (say in measure theoretic terms) this set can be and still have the theorem on path independence valid.

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