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APPLICATIONS OF THE HADAMARD PRODUCT IN GEOMETRIC FUNCTION THEORY

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Summary. Let $\mathscr{A}$ denote the set of functions F holomorphic in the unit disc, normalized clasically: $F(0) = 0$, $F'(0) = 1$, whereas $A \subset \mathscr{A}$ is an arbitrarily fixed subset. In this paper various properties of the classes A_α, $\alpha \in \boldsymbol{C} \setminus \{-1, -\frac{1}{2}, \ldots\}$, of functions of the form $f = F * k_\alpha$ are studied, where

$$F \in A, \quad k_\alpha(z) = k(z, \alpha) = z + \frac{1}{1+\alpha} z^2 + \ldots + \frac{1}{1+(n-1)\alpha} z^n + \ldots,$$

and $F * k_\alpha$ denotes the Hadamard product of the functions F and k_α. Some special cases of the set A were considered by other authors (see, for example, [15], [6], [3]).

Keywords: Hadamard product, class of type A_α, typically real functions.

1. Let $\mathscr{A}$ denote the set of functions F of the form

$$F(z) = z + \sum_{n=2}^{\infty} a_{n,F} z^n, \tag{1}$$

holomorphic in the unit disc $\Delta = \{z \in \boldsymbol{C}: |z| < 1\}$, whereas T is a subset of $\mathscr{A}$ consisting of typically-real functions in Δ (see [12]).

In paper [6], for an arbitrarily fixed $\alpha \in \boldsymbol{R} \setminus \{-1, -\frac{1}{2}, \ldots\}$, the class

$$T_\alpha = \{f \in \mathscr{A}: f = F * k_\alpha,\ F \in T\}$$

was considered, where

$$k_\alpha(z) = k(z, \alpha) = \sum_{n=1}^{\infty} \frac{1}{1 + (n-1)\alpha} z^n, \quad z \in \Delta,$$

and $F * k_\alpha$ denotes the Hadamard product of the functions F and k_α (see, for example, [14], p. 27; [13]).

For nonnegative values of α, the family T_α was introduced earlier by K. Skalska ([15]) in another way.

The aim of this paper is to study various properties of the class

$$A_\alpha = \{f \in \mathscr{A}: f = F * k_\alpha,\ F \in A\} \tag{2}$$

where $A \neq \emptyset$ is an arbitrarily fixed subset of the set $\mathscr{A}$, and $\alpha \in \boldsymbol{C} \setminus \{-1, -\frac{1}{2}, \ldots\}$.

In the subsequent considerations we shall always assume, if not stated otherwise, that α is an arbitrarily fixed complex number different from the numbers $-1, -\frac{1}{2}, \ldots$.

2. It follows directly from the definitions of the family A_α and the Hadamard product that the function f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_{n,f} z^n, \quad z \in \Delta,$$

belongs to the family A_α if and only if there exists $F \in A$ of the form (1) such that

$$a_{n,f} = \frac{a_{n,F}}{1 + (n-1)\alpha}, \quad n = 2, 3, \ldots. \tag{3}$$

So, if the exact estimate $|a_{n,F}| \leqq d_n$ takes place in the class A ($F \in A$), then (3) yields the exact estimate $|a_{n,f}| \leqq d_n / |1 + (n-1)\alpha|$, $f \in A_\alpha$.

Moreover, from formula (3) we obtain that $A_0 = A$.

Also, in a simple way, from (2) we obtain the following properties of the classes A_α.

Theorem 1. *Let $r \in (0, 1)$. If, for each function $F \in A$, the function*

$$F_r(z) = \frac{1}{r} F(rz), \quad z \in \Delta,$$

belongs to the family A, then, for each function $f \in A_\alpha$, the function

$$f_r(z) = \frac{1}{r} f(rz), \quad z \in \Delta,$$

belongs to the family A_α.

Theorem 2. *Let $\theta \in \langle 0, 2\pi)$. If, for each function $F \in A$, the function*

$$F_\theta(z) = \mathrm{e}^{-\mathrm{i}\theta} F(z\mathrm{e}^{\mathrm{i}\theta}), \quad z \in \Delta,$$

belongs to the family A, then, for each function $f \in A_\alpha$, the function

$$f_\theta(z) = \mathrm{e}^{-\mathrm{i}\theta} f(z\mathrm{e}^{\mathrm{i}\theta}), \quad z \in \Delta,$$

belongs to the family A_α.

Theorem 3. *Let $\alpha \in \boldsymbol{R} \setminus \{-1, -\frac{1}{2}, \ldots\}$. If, for each function $F \in A$, the function*

$$G(z) = \overline{F(\bar{z})} = \sum_{n=1}^{\infty} \bar{a}_{n,F} z^n, \quad z \in \Delta,$$

belongs to the family A, then, for each function $f \in A_\alpha$, the function

$$g(z) = \overline{f(\bar{z})} = \sum_{n=1}^{\infty} \bar{a}_{n,f} z^n, \quad z \in \Delta,$$

belongs to the family A_α.

Similarly as in the case $A = T$ (see [15], [6]), the following properties of the families A_α may be proved.

Theorem 4. *A function f belongs to A_α if and only if f is a solution of the differential equation*

$$\alpha z f'(z) + (1-\alpha) f(z) = F(z) \tag{4}$$

where $F \in A$.

Theorem 5. *If $f \in A_\alpha$, then*

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta| = \varrho < 1} k\left(\frac{z}{\zeta}, \alpha\right) F(\zeta) \frac{d\zeta}{\zeta}, \quad |z| < \varrho < 1,$$

where $F \in A$, and vice versa.

Theorem 6. *If $f \in A_\alpha$,* $\operatorname{Re} \alpha > 0$, *then*

$$f(z) = \frac{1}{\alpha} \int_0^1 t^{1/\alpha - 2} F(zt)\, dt, \quad z \in \Delta,$$

where $F \in A$, and vice versa.

Theorem 7. *Let A and B be two fixed subsets of $\mathscr{A}$. If, for any functions $F \in A$, $G \in B$, the function $F * G \in A$, then, for each $f \in A_\alpha$, the function $f * G \in A_\alpha$.*

The above theorems can be used in various problems concerning classes of type A_α. In particular, the properties of solutions of equations of the form (4) were considered in several cases of the classes $A \subset \mathscr{A}$ (for example, in [15], [6], [8], [2]). From Theorems 5 and 6 one often gets structure formulae for the classes A_α (for example, in [15], [6]; see also [10], [2]). On the other hand the properties of the Hadamard product of functions of the form (1) of the classes frequently considered are well-known: CV (the class of convex functions), $ST(1/2)$ (the class of starlike functions of order $1/2$), CC (the class of close-to-convex functions) (see [4], vol. 1, p. 115; vol. 2, p. 2). So, from Theorem 7 and the results of the paper [13] we obtain:

1) for any functions $f \in (CV)_\alpha$, $G \in CV$, the Hadamard product $f * G$ belongs to $(CV)_\alpha$;

2) for any functions $f \in (ST(1/2))_\alpha$, $G \in ST(1/2)$, the Hadamard product $f * G$ belongs to $(ST(1/2))_\alpha$;

3) for any functions $f \in (CC)_\alpha$, $G \in CV$, the Hadamard product $f * G$ belongs to $(CC)_\alpha$.

3. Let H denote the family of all functions holomorphic in the unit disc Δ. The set H with the topology of almost uniform convergence is, of course, a linear topological space.

As is known, certain problems of the geometric theory of analytic functions consist in determining the set Ω of values of a complex continuous functional defined on a given family $A \subset H$. If the set Ω is bounded, closed and connected, then we determine it effectively by characterizing its boundary. To ensure that the set Ω has the above properties, the family A considered should be compact and connected. In other extremal problems, support points and extreme points of the families play an essential part (see, for example, [14], pp. 3, 99; [1]).

Let us recall: a function $F \in A$ is called a support point of a compact subset A of H f and only if there exists a continuous linear functional x^* on H such that, Re x^* is non-constant on A and for each function $G \in A$,

$$\operatorname{Re} x^*(G) \leqq \operatorname{Re} x^*(F) .$$

So, the problem of characterizing the set of the support points of the class $A_\alpha \subset \subset \mathscr{A} \subset H$ seems to be interesting when the characterization of the support points of the family $A \subset \mathscr{A} \subset H$ is known.

In the proof of the theorem solving this problem we shall use the following well-known result of Toeplitz ([16]).

Lemma. *A functional x^* defined on H is linear and continuous if and only if there exists a sequence of complex numbers $\{b_n\}$ such that, for each function $g \in H$,*

$$x^*(g) = \sum_{n=0}^{\infty} a_{n,g} b_n ,$$

$$\limsup_{n\to\infty} |b_n|^{1/n} < 1 .$$

Theorem 8. *A function f_0 is a support point of the set A_α if and only if $f_0 = F_0 * k^\alpha$ where F_0 is a support point of the set A.*

Proof. Let F_0 be a support point of the set A. Then there exists a linear and continuous functional x^* on H such that, for each function $F \in A$,

$$\operatorname{Re} x^*(F) \leqq \operatorname{Re} x^*(F_0) .$$

The above lemma and formula (1) imply that this inequality can be written in the following equivalent form:

$$\text{(5)} \qquad \operatorname{Re}\left(\sum_{n=2}^{\infty} a_{n,F} b_n\right) \leqq \operatorname{Re}\left(\sum_{n=2}^{\infty} a_{n,F_0} b_n\right), \quad F \in A ,$$

where $\{b_n\}$ is a sequence determining the functional x^*.

As $\limsup_{n\to\infty} |b_n[1 + (n-1)\alpha]|^{1/n} < 1$, the sequence $\{b_n[1 + (n-1)\alpha]\}$ also determines a linear and continuous functional on H. Let us denote it by x^*_α. Let f

be an arbitrarily fixed function of the family A_α, whereas $f_0 = F_0 * k_\alpha$. Then there exists exactly one function $F \in A$ such that $f = F * k_\alpha$. Hence, taking formula (3) and inequality (5) into consideration, we obtain

$$\operatorname{Re} x_\alpha^*(f) - \operatorname{Re} x_\alpha^*(f_0) = \operatorname{Re} x_\alpha^*(F * k_\alpha) - \operatorname{Re} x_\alpha^*(F_0 * k_\alpha) =$$

$$= \operatorname{Re}\left(\sum_{n=2}^{\infty} \frac{a_{n,F}}{1 + (n-1)\alpha} b_n[1 + (n-1)\alpha]\right)$$

$$- \operatorname{Re}\left(\sum_{n=2}^{\infty} \frac{a_{n,F_0}}{1 + (n-1)\alpha} b_n[1 + (n-1)\alpha]\right)$$

$$= \operatorname{Re}\left(\sum_{n=2}^{\infty} a_{n,F} b_n\right) - \operatorname{Re}\left(\sum_{n=2}^{\infty} a_{n,F_0} b_n\right) \leqq 0\,,$$

which proves that the function $f_0 = F_0 * k_\alpha$ is a support point of the set A_α. We also note that if $\operatorname{Re} x^*$ is non-constant on A then $\operatorname{Re} x_\alpha^*$ is non-constant on A_α.

The proof of the converse theorem proceeds analogously.

From the linearity and the injectivity of the Hadamard product $F * k_\alpha$ in the space H the following properties of the classes A_α follow.

Theorem 9. *A set A_α is convex in the space H if and only if A is convex in this space.*

Theorem 10. *If a set A is a convex set in space H, then $f \in A_\alpha$ is an extreme point of the set A_α if and only if $f = F * k_\alpha$ where F is extreme point of the set A.*

Next, let us recall that a topological space X is called arcwise connected if, for any two points $x_1, x_2 \in X$, there exists a continuous mapping $\gamma(t)$ of an interval $\langle a, b\rangle$ into the space X such that $\gamma(a) = x_1$, $\gamma(b) = x_2$. Such a mapping will be called a path joining the points x_1 and x_2.

We shall prove the following property of the class A_α.

Theorem 11. *If a set A is arcwise connected, then the set A_α is arcwise connected.*

Proof. Let $f_1, f_2 \in A_\alpha$. Then there exist functions $F_1, F_2 \in A$ such that $f_1 = F_1 * k_\alpha$, $f_2 = F_2 * k_\alpha$, and a path $\Gamma(t) = F(z, t)$, $t \in \langle a, b\rangle$, joining F_1 and F_2. Using the formula given in Theorem 5, we prove in the elementary way that $\gamma(t) = f(z, t) = F(z, t) * k_\alpha(z)$ is a path joining f_1 and f_2, which completes the proof.

Since the arcwise connectedness implies the topological connectedness, Theorem 11 yields that, for the arcwise connected family A, the families A_α are connected.

Similarly, the following property of the families A_α may easily be proved.

Theorem 12. *If A is a compact family, then the families A_α are also compact.*

4. K. Skalska in her paper [15] proved that if $A = T$, then the following inclusions hold:

$$T_\beta \subset T_\alpha \subset T_0 = T\,, \quad 0 < \alpha < \beta\,.$$

In the general case, neither of the inclusions $A_\beta \subset A_\alpha \subset A$, $0 < \alpha < \beta$, need be true. Indeed, let $A = \{z; z + z^2\}$; then $A_\alpha = \{z; z + 1/(1 + \alpha).z^2\}$, so $A_\beta \not\subset A_\alpha \not\subset \not\subset A$ for $0 < \alpha < \beta$. Moreover, if $A = \{z + z^2\}$, then $A_\alpha = \{z + 1/(1 + \alpha).z^2\}$, thus the above inclusions are not true, either, and furthermore, for $\alpha \neq 0$, even $A_\alpha \cap A = \emptyset$.

Next, let $A = S$ where S is the well-known class of univalent functions F of the form (1) in Δ. D. M. Campbell & V. Singh ([2]) proved that then the classes $S_\alpha = A_\alpha$, even for $\alpha = \frac{1}{2}$, include infinite-valent functions. So, $S_\alpha \not\subset S$ for $\alpha = \frac{1}{2}$. Of course, it is also known that $S_1 \not\subset S$ (see [7]). On the other hand Z. Lewandowski, S. Miller, E. Złotkiewicz in their paper [8] proved that if $A = ST$, then $(ST)_\alpha \subset ST$ for all $\alpha \in C$ from the disc $|\alpha - \frac{1}{2}| \leqq \frac{1}{2}$. Another non-trivial example of a family A for which the inclusion $A_\alpha \subset A$ is true for a complex α is the family $B_1(M)$, $M > 1$, (see [4], vol. 2, p. 36) of functions of the form (1) satisfying the inequality

$$|F(z)| < M\,, \quad z \in \Delta\,.$$

Namely, we have the following theorem.

Theorem 13. *If* $M > 1$ *and* $\operatorname{Re} \alpha > 0$, *then*

$$(B_1(M))_\alpha \subset B_1(M)\,.$$

Proof. Let $f \in (B_1(M))_\alpha$ and suppose that, at the same time, $f \notin B_1(M)$. It is easy to verify then that there exists a point $z_0 \in \Delta$ such that

$$\max_{|z| \leqq r} |f(z)| = |f(z_0)| = M\,, \quad r = |z_0|\,.$$

Hence, in view of Jack's lemma ([5]), we obtain that there exists a number $m \geqq 1$ such that

$$z_0 f'(z_0) = m f(z_0)\,.$$

Consequently, in view of Theorem 4 we obtain

$$|\alpha z_0 f'(z_0) + (1 - \alpha) f(z_0)| = |f(z_0)|\,|\alpha(m - 1) + 1| \geqq |f(z_0)| = M$$

in spite of the assumption that $f \in (B_1(M))_\alpha$, which completes the proof.

Now, we shall give a construction of the families A for which both the inclusion relations above will be true. For this purpose, let us consider the operator $D\colon H \to H$ defined by the formula

$$D\,F(z) = z\,F'(z)\,, \quad z \in \Delta\,,$$

and the set $\mathscr{A}' = \{F \in H, F(0) = 1\}$. Let $\mathscr{J}$ denote the class of operators $J\colon \mathscr{A} \to \mathscr{A}'$ satisfying for all $F \in \mathscr{A}$ the condition

(i) $$J(\alpha DF + (1 - \alpha)\,F) = J(F) + \alpha D\,J(F)\,, \quad \alpha \in C\,.$$

Let us observe that, for example, the operators $J_k: \mathscr{A} \to \mathscr{A}'$, $k = 1, 2, 3, 4$, defined by the formulas

$$J_1(F)(z) = F'(0) = 1, \qquad z \in \Delta,$$

$$J_2(F)(z) = F'(z), \qquad z \in \Delta,$$

$$J_3(F)(z) = F(z)/(z), \qquad z \in \Delta,$$

$$J_4(F)(z) = \frac{1}{z}\int_0^z \frac{F(\vartheta)}{\vartheta}\,d\vartheta, \qquad z \in \Delta,$$

belong to the class $\mathscr{J}$.

Let

(6) $$A = \{F \in \mathscr{A},\ \mathrm{Re}\, J(F)(z) > 0,\ z \in \Delta\}$$

where J denotes an arbitrarily fixed operator of the class $\mathscr{J}$.

In the sequel, family (6) will be called a family of type J.

Let us observe that the identity function belongs I to each family A of type J, $(J(I)(z) = 1, z \in \triangle, J \in \mathscr{J})$; moreover, the class A of type J_1 coincides with the whole family $\mathscr{A}$. The well-known families (see [4], vol. 1, p. 101; vol. 2, p. 97)

(7) $$\{F \in \mathscr{A}\colon \mathrm{Re}\, F'(z) > 0,\ z \in \Delta\},$$

(8) $$\left\{F \in \mathscr{A}\colon \mathrm{Re}\,\frac{F(z)}{z} > 0,\ z \in \Delta\right\}$$

are classes of type J_2, J_3, respectively. The family A of type J_4, as far as we know, has not been investigated yet.

The families A_α associated with the classes A of type J have the following properties.

Theorem 14. *If A is a family of type J, then for each $\alpha \in C$, $\mathrm{Re}\,\alpha \geqq 0$, the inclusion $A_\alpha \subset A$ is true.*

Proof. Let $f \in A_\alpha$. Then from (6) and (4) we have

$$\mathrm{Re}\, J(\alpha Df + (1-\alpha)f)(z) > 0, \quad z \in \Delta.$$

This inequality, in view of property (i) of the operator J, is equivalent to

(9) $$\mathrm{Re}\,(p + \alpha Dp)(z) > 0, \quad z \in \Delta,$$

where $p = J(f)$. Using S. Miller's result ([9], Corollary) we get $\mathrm{Re}\, p(z) > 0$, $z \in \Delta$. Therefore, $\mathrm{Re}\, J(f)(z) > 0$, $z \in \Delta$, and, consequently, $f \in A$, which completes the proof.

Theorem 15. *If A is a family of type J and $0 \leqq \alpha \leqq \beta$, then $A_\beta \subset A_\alpha \subset A_0 = A$.*

Proof. Of course, it is sufficient to consider the case $0 < \alpha < \beta$. So, let $0 < \alpha < \beta$, $f \in A_\beta$ and $f \notin A_\alpha$. Then, in view of (4), (6) and property (i), there exists $z_0 \in \Delta$ such that

$$\operatorname{Re} J(f)(z_0) + \beta \operatorname{Re} D\, J(f)(z_0) > 0\,,$$
$$\operatorname{Re} J(f)(z_0) + \alpha \operatorname{Re} D\, J(f)(z_0) \leqq 0\,.$$

Multiplying the first inequality by $\alpha > 0$ and the second inequality by $(-\beta) < 0$ and adding them, we get

$$(\alpha - \beta) \operatorname{Re} J(f)(z_0) > 0\,.$$

Since $\alpha - \beta < 0$, therefore $\operatorname{Re} J(f)(z_0) < 0$ and, consequently, $f \notin A$, which contradicts the relation $A_\beta \subset A$ proved in Theorem 14.

In particular cases, if the family A is of the form (7) or (8), Theorems 14 and 15 give some results from paper [3], (see Sections 4 and 5).

5. Let A be a family of type $J = J_k$, $k = 2, 3, 4$. Then there exists a function $F = F_k$, $k = 2, 3, 4$, of this class, such that

$$J(F)(z) = \frac{1+z}{1-z}\,, \quad z \in \Delta\,. \tag{10}$$

From property (i) of the operator J we get

$$\operatorname{Re} J(\alpha DF + (1-\alpha) F)(z) =$$
$$= \operatorname{Re} (1 + 2\alpha z - z^2)/(1-z)^2 \to -\tfrac{1}{2} \operatorname{Re} \alpha \leqq 0\,,$$

as $z \to -1$, $z \in \Delta$, for each $\operatorname{Re} \alpha \geqq 0$. So, F_k does not belong to the respective class A_α if $\operatorname{Re} \alpha > 0$. Consequently, the classes A_α associated with the families A of type $J = J_k$, $k = 2, 3, 4$, are essential subclasses of the families A.

From the course of the argument carried out we infer that A_α will be an essential subclass of the family A of type J if, for example, we assume in addition that the solution F of equation (10) belongs to A. Then the family A will be called a family of type $\tilde{J}$. So: if A is a family of type $\tilde{J}$, then $A \not\subset A_\alpha$ for $\operatorname{Re} \alpha > 0$.

A family A of type J_1 is not a family of type $\tilde{J}_1$, whereas families A of type J_k, $k = 2, 3, 4$, are families of type $\tilde{J}_k$.

The following property for the families of type $\tilde{J}$ turns out to be true.

Theorem 16. *If A is a family of type $\tilde{J}$, then*

$$A \subset A\,[\Delta_{\Delta_{r(\alpha)}}]_\alpha \quad \text{for} \quad r(\alpha) = \sqrt{(1 + |\alpha|^2)} - |\alpha| \leqq 1$$

where

$$A\,[\Delta_r] = \{f \in \mathscr{A}\colon \operatorname{Re} J(f)(z) > 0, z \in \Delta_r\};\ \Delta_{r(\alpha)} = \{z \in \boldsymbol{C}\colon |z| < r(\alpha)\}\,.$$

Moreover, the disc $\Delta_{r(\alpha)}$ for $\alpha \in \boldsymbol{R}$ cannot be enlarged.

Proof. Let $f \in A$. In view of the definitions of the families A_α and the sets $A[\Delta_r]$, the assertion will be proved if we determine the largest number $r(\alpha) \in (0, 1)$ such that

$$\operatorname{Re} J(\alpha Df + (1 - \alpha) f)(z) > 0, \quad z \in \Delta_{r(\alpha)}.$$

By virtue of property (i) of the operator J, it is sufficient to prove that

$$\operatorname{Re}(p + \alpha Dp)(z) > 0, \quad z \in \Delta_{r(\alpha)}, \tag{11}$$

where $p = J(f)$. Since $f \in A$, therefore p is a Carathéodory function with a positive real part, so ([11], (6.2)) $|z\,p'(z)|/\operatorname{Re} p(z) \leqq 2|z|/(1 - |z|^2)$. Hence

$$\operatorname{Re}(p + \alpha Dp)(z) \geqq \left(1 - \frac{2|\alpha|\, r}{1 - r^2}\right) \operatorname{Re} p(z), \quad |z| = r < 1. \tag{12}$$

But $1 - 2|\alpha|\, r - r^2 > 0$ if and only if $0 < r < r(\alpha) = \sqrt{(1 + |\alpha|^2)} - |\alpha|$, therefore relation (11) follows from (12), which accounts for the inclusion announced in the theorem.

As A is a family of type $\tilde{J}$, the solution F of equation (10) belongs to A. This function turns out to belong to the family $A_\alpha[\Delta_{r(\alpha)}]_\alpha$ and not belong to $A_\alpha[\Delta_r]_\alpha$ for $r > r(\alpha), \alpha \in \mathbf{R}$. Thus the proof is complete.

6. Let A be a family of type $\tilde{J}$ and $\alpha \geqq 0$. In view of Theorems 14 and 15 and the fact that $A \not\subset A_\alpha$ for $\alpha > 0$, the following considerations seem to be interesting.

Let $f \in A$, $\alpha \geqq 0$. Let us put

$$\alpha_f = \{\sup \alpha : f \in A_\alpha\},$$
$$A(\alpha) = \{f \in A : \alpha_f = \alpha\}.$$

Theorem 17. *If A is a family of type $\tilde{J}$, then each class $A(\alpha)$ is nonempty and the following relations hold:*

(13) $f \in A(0)$ *if and only if* $f \notin A_\alpha$ *for each* $\alpha > 0$;

(14) $f \in A(\infty)$ *if and only if* $f \in A_\alpha$ *for each* $\alpha \geqq 0$;

(15) $f \in A(\alpha)$, $\alpha \in (0, \infty)$, *if and only if* $f \in A_\beta$ *for any* $\beta \in \langle 0, \alpha \rangle$ *and* $f \notin A_\beta$ *for each* $\beta > \alpha$.

Proof. As A is of type $\tilde{J}$, then, as we observed earlier, $A(0) \neq \emptyset$. Let $\alpha > 0$ and let $\tilde{F} \in A$ be a solution of equation (10). Let us put $\tilde{f} = \tilde{F} * k_\alpha$. Then, by virtue of (2), $\tilde{f} \in A_\alpha$, so from (4)

$$J(\alpha D\tilde{f} + (1 - \alpha)\tilde{f})(z) = J(\tilde{F})(z) = \frac{1 + z}{1 - z}, \quad z \in \Delta.$$

Hence, in view of (i),

$$\alpha D\, J(\tilde{f})(z) + J(\tilde{f})(z) = \frac{1 + z}{1 - z}, \quad z \in \Delta.$$

Let us consider $\beta > \alpha$. From (i) we get

$$J(\beta D\tilde{f} + (1 - \beta)\tilde{f})(z) = \beta D\, J(\tilde{f})(z) + J(\tilde{f})(z) =$$

$$= \frac{\beta}{\alpha}\frac{1 + z}{1 - z} + \frac{\alpha - \beta}{\alpha} J(\tilde{f})(z) = \frac{\beta}{\alpha}\frac{1 + z}{1 - z} +$$

$$+ \frac{\alpha - \beta}{\alpha} \cdot \frac{1}{\alpha}\int_0^1 t^{1/\alpha - 1}\frac{1 + tz}{1 - tz}\,dt \to \frac{\alpha - \beta}{\alpha} a < 0\,,$$

as $z \to -1$, $z \in \Delta$. Consequently, $\tilde{f} \in A_\alpha$, whence $A(\alpha) \neq \emptyset$. Since the identity function belongs to the family A of type J, it belongs to each class A_α, thus to $A(\infty)$, too. Hence it follows that $A(\infty) \neq \emptyset$.

Now, let us observe that for $\alpha \in (0, \infty)$, conditions (13), (14) and the sufficient condition in (15) follow directly from the definition of the family $A(\alpha)$ and the properties of the family A_α. It only remains to prove the necessary condition in (15).

So, let $f \in A(\alpha)$, $\alpha \in (0, \infty)$. Then the definition of the family $A(\alpha)$ and Theorem 15 imply that $f \notin A_\beta$ for each $\beta > \alpha$, and $f \in A_\beta$ for each $0 \leqq \beta < \alpha$. In view of (4), the last fact is equivalent to

$$\operatorname{Re} J(\beta Df + (1 - \beta)f)(z) > 0\,, \quad z \in \Delta\,,$$

for $\beta \in \langle 0, \alpha)$. Passing to the limit $\beta \to \alpha^-$ in the above inequality, we get

$$\operatorname{Re} J(\alpha Df + (1 - \alpha)f)(z) \geqq 0\,, \quad z \in \Delta\,,$$

which, in view of the extremum principle for harmonic functions, gives

$$\operatorname{Re} J(\alpha Df + (1 - \alpha)f)(z) > 0\,, \quad z \in \Delta\,,$$

and, consequently, $f \in A_\alpha$. Thus the proof is complete.

Theorem 17 evidently yields that

$$A = \bigcup_{\alpha \geqq 0} A(\alpha)\,.$$

Finally, let us observe that the operator $J_g\colon \mathscr{A} \to \mathscr{A}'$ defined by the formula

$$(J_g(F))(z) = \frac{(F * g)(z)}{z}\,, \quad z \in \Delta\,,$$

where g is an arbitrarily fixed function of the family $\mathscr{A}$, belongs to the class $\mathscr{J}$, too. Moreover, putting $g = g_k$, $k = 1, 2, 3, 4$, where

$$g_1(z) = z\,, \quad z \in \Delta\,;$$

$$g_2(z) = \frac{z}{(1 - z)^2}\,, \quad z \in \Delta\,;$$

$$g_3(z) = \frac{z}{1-z}, \quad z \in \Delta ;$$

$$g_4(z) = -\mathrm{Log}(1-z), \quad z \in \Delta ,$$

we get $J_k = J_{g_k}$.

There arises a natural question if J_g is the most general form of the operator $J \in \mathscr{J}$.

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APLIKACE HADAMARDOVA SOUČINU V GEOMETRICKÉ TEORII FUNKCÍ

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Nechť $\mathscr{A}$ je množina funkcí F holomorfních v jednotkovém kruhu a normalizovaných klasickým způsobem: $F(0) = 0$, $F'(0) = 1$, a nechť $A \in \mathscr{A}$ je její libovolná pevně zvolená podmnožina. V článku se studují různé vlastnosti tříd A_α, $\alpha \in C \setminus \{-1, -\frac{1}{2}, \ldots\}$, funkcí tvaru $f = F * k_\alpha$, kde

$$F \in A\,, \quad k_\alpha(z) = k(z, \alpha) = z + \frac{1}{1+\alpha} z^2 + \ldots + \frac{1}{1+(n-1)\alpha} z^n + \ldots,$$

a $F * k_\alpha(z)$ znamená Hadamardův součin funkcí F, k_α. Některé speciální případy množiny A byly vyšetřeny dříve jinými autory (viz např. [15], [6], [3]).

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