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THE HOPF BIFURCATION THEOREM FOR PARABOLIC EQUATIONS WITH INFINITE DELAY

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Summary. The existence of the Hopf bifurcation for parabolic functional equations with delays of maximum order in spatial derivatives is proved. An application to an integrodifferential equation with a singular kernel is given.

Keywords: Hopf bifurcation, parabolic functional equation, infinite delay.

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1. INTRODUCTION

This paper is mainly concerned with periodic solutions of the functional differential equation in a Banach space X

$$\dot{u}(t) = A\,u(t) + Lu_t + f(\mu, u(t), u_t)\,.$$

Here u_t denotes the shift of u given by $u_t(s) = u(t + s)$ for $s \in \mathbf{R}^-$, A is a generator of an analytic semigroup e^{At} on X, L is a continuous linear operator from an appropriate function space Y into X, $f \in C^2([-1, 1] \times \mathscr{D}(-A^\alpha) \times Y, X)$. The existence of periodic solutions is proved by using classical methods of the Hopf bifurcation. There exists a lot of literature dealing with the Hopf bifurcation problems for ordinary differential equations with finite or infinite delays (see e.g. Hale [5], Cushing [3], Simpson [7], Stech [10], Staffans [9]). Yoshida and Kishimoto [14], [15] have studied partial differential equations with finite time delays. There are only few papers dealing with the infinite dimensional problem with infinite delay. Tesei [11] has considered the special problem

$$u_t = \Delta u + \lambda u - au \int_{-\infty}^{t} \Theta^2(t - s)\, \mathrm{e}^{-\Theta(t-s)}\, u(s)\, \mathrm{d}s$$

and, after rewriting the equation to the system without delay, he applied the standard theorem [1]. Yamada and Niikura [13] have dealt with the system

$$\dot{u}(x, t) = D(\alpha)\, \Delta u(x, t) + C(\alpha)\, u(x, t) +$$
$$+ \int_{-\infty}^{t} K(t - \tau, \alpha)\, u(x, \tau)\, \mathrm{d}\tau + f(u_t, \alpha)(x)\,,$$

$u = (u^1, \ldots, u^N)$, $x \in \Omega \subset \mathbf{R}^n$ with Dirichlet or Neumann boundary conditions. Making use of the special type of the system they proceeded similarly as in [1]. Da Prato and Lunardi [4] were concerned with the equation

$$\dot{u}(t) = f(\lambda, u(t)) + \int_{-\infty}^{t} g(\lambda, t - s, u(s))\, \mathrm{d}s\,,$$

where

$$f\colon (-1, 1) \times D \to X, \quad g\colon (-1, 1) \times (0, \infty) \times D \to X$$

are C^∞ functions, D is continuously imbedded in X, but there is a mistake in a transformation of the equation there.

The present paper generalizes the results [4] [11] [13], needing less restrictions on the delay term Lu_t and on the smoothness of the data. The operator L and the nonlinearity f can act on the spaces of functions with values in $\mathscr{D}(A)$, and certain singular kernels are also admissible provided L has the integral form $Lu_t = \int_{-\infty}^{t} k(t - s)\, A^\alpha u(s)\, \mathrm{d}s$. In the proof we follow the idea of Crandall and Rabinowitz [1] with help of the Fourier series method which leads directly to the decomposition of the space of periodic functions before an application of the Implicit Function Theorem.

2. NOTATION AND ASSUMPTIONS

Let X be a real Banach space and let

(H1) $\quad A$ be a generator of an analytic semigroup $T(t)$ in X.

We shall use the usual complexification of the space and the operators without changing the notation. We shall explain the precise meaning at places, where misunderstanding can occur. Using a shift if necessary, we can define all fractional powers $(-A)^\alpha$. We denote by X^α the domain of $(-A)^\alpha$ endowed with the graph norm $\|\cdot\|_\alpha$ ($X^0 = X$, $\|\cdot\|_0 = \|\cdot\|$). The operators $(\lambda - A)^{-1}$ then satisfy the estimates (see e.g. [6])

$$\|(\lambda - A)^{-1} x\|_\alpha \leqq \frac{C(\alpha)}{|\lambda + a|^{1-\alpha}} \|x\|\,, \quad 0 \leqq \alpha \leqq 1 \tag{2.1}$$

whenever λ does not belong to the sector $\{\lambda \in \mathbf{C}, |\arg(\lambda + a)| \geqq \omega\}$, which contains the spectrum of A. Throughout the paper we shall denote by C any constant.

Let $\alpha \in [0, 1]$ be fixed and let Y^α be the space of continuous functions defined on $\mathbf{R}^- = (-\infty, 0)$ with values in X^α and such that

$$\|u\|_{Y^\alpha} = \sup_{t \in \mathbf{R}^-} \| |t - 1|^{-1} u(t)\|_\alpha < \infty\,.$$

Let

(H2) $\quad L$ be a continuous linear operator from Y^α into X.

For $\lambda \in \mathbf{C}$, $\operatorname{Re} \lambda \geqq 0$ we denote by $B(\lambda)$ the operator defined by

$$\mathscr{D}(B(\lambda)) = X^\alpha , \quad B(\lambda)\, x = L(\tau \to e^{\tau\lambda} x) .$$

We shall assume that there are $\beta \geqq 0$, $\delta > 0$ such that $\alpha - \beta < \delta < 1$ and

(H3) (i) $\|B(ik)\, x\| \leqq Ck^{-\beta} \|x\|_\alpha$ for $x \in X^\alpha$,

(ii) for any $u \in C_{2\pi}^{\delta+\beta-\alpha}(X^\alpha)$ the function $\varphi(t) = Lu_t$ belongs to $C_{2\pi}^{\delta}(X)$ ($C_{2\pi}^{k+\delta}(X^\alpha)$ denotes the space of X^α-valued 2π-periodic functions with δ-Hölder continuous derivatives of the order k with the usual norm). (H1) and (H3) imply the existence of a constant $N \geqq 0$ such that the operator $(ik - A - B(ik))$ admits a continuous inverse $D(ik)$ for $k \in \mathbf{Z}$, $|k| > N$,

$$(2.2) \qquad D(ik) = (ik - A - B(ik))^{-1} = (ik - A)^{-1} \sum_{n=0}^{\infty} [B(ik)(ik - A)^{-1}]^n .$$

There is $K > 0$ such that

$$(2.3) \qquad \|D(ik)\|_{L(X,X^1)} \leqq K \quad \text{whenever} \quad D(ik) \quad \text{exists in} \quad L(X, X^1) \quad \text{and} \quad k \in \mathbf{Z}, \quad |k| \leqq N .$$

(2.3) together with (2.1), (2.2) yield that there is $C > 0$ such that

$$(2.4) \quad \|D(ik)\, x\|_\nu \leqq C/|k|^{1-\nu} \|x\| , \quad 0 \leqq \nu \leqq 1 , \quad k \in \mathbf{Z} , \quad k \neq 0$$

whenever $D(ik)$ exists. It will be shown later that the operator L, given by $L\psi = \int_0^\infty t^{-\gamma} e^{-pt} A\psi(-t)\, dt$, satisfies the assumption (H3) if $\gamma < \frac{1}{3}$, $p > 0$.

Further hypotheses concern the spectrum of the operators $A + B(ik)$.

(H4) (i) $(ik - A - B(ik))^{-1}$ exist in $L(X, X^1)$ for $k \in \mathbf{Z}$, $|k| \neq 1$,

(ii) i is an algebraically simple isolated eigenvalue of $A + B(i)$ with the eigenvector x_0.

It is easily seen that $-i$ is a simple isolated eigenvalue of $A + B(-i)$ with the eigenvector $\bar{x}_0$. There exists a closed subspace of $X^c = X \oplus iX$, denoted by $\tilde{X}$, such that $X^c = \tilde{X} \oplus \mathbf{C}x_0$ and the operators $(i - A - B(i))$, $(i + A + B(-i))$ are invertible on $\tilde{X}$. The last assumption on the operator L will be the following:

(H5) If $L(\tau \to \tau e^{i\tau} x_0) = ax_0 + x_1$ where $x_1 \in \tilde{X}$, then $\operatorname{Re} a \neq 1$.

A function $f = f(\mu, x, p)$ will be supposed to satisfy the following conditions:

(H6) (i) $f \in C^2(\Omega, X)$, where Ω is a neighbourhood of zero in $\mathbf{R} \times X^\alpha \times Y^\alpha$,

(ii) $f(\mu, 0, 0) = 0$, $f_x(0, 0, 0) = 0$, $f_p(0, 0, 0) = 0$ if $(\mu, 0, 0) \in \Omega$.

The assumptions (H4) (i), (H6) (ii) imply (see [2]) that there are continuously differentiable functions $x(\mu)$, $\lambda(\mu)$ defined for small μ such that

$$(2.5) \qquad (A + B(i) + f_x(\mu, 0, 0))\, x(\mu) + f_p(\mu, 0, 0)\, p(\mu) = \lambda(\mu)\, x(\mu) , \quad x(0) = x_0 , \quad \lambda(0) = i , \quad p(\mu)(t) = e^{it} x(\mu) .$$

Following Hopf we shall assume

(H7) $\quad \operatorname{Re} \lambda'(0) \neq 0$.

Remark. The assumption (H3) is satisfied when $\alpha = 0$ (equations in [11], [13]). For $\alpha < 1$ only the smoothing property (H3) (ii) is needed ($\beta = 0$). On the other hand, (H3) (ii) is automatically satisfied when $\beta \geqq \alpha$. (H4) (ii) together with (H6) (ii) means that i is a characteristic eigenvalue for the linearized equation. (H5) and the transversality condition (H7) assure the invertibility of the operator $\mathscr{G}$ (see (3.5)), and consequently allow to apply the Implicit Function Theorem.

At the end of this section we collect some properties of Fourier coefficients of 2π-periodic Hölder continuous functions. A proof of the following lemma can be found e.g. in [12].

Lemma 1. *Let* $v_k = (1/2\pi) \int_0^{2\pi} e^{-iks} v(s)\, ds$ *be the Fourier coefficients of the function* v. *Then for each* $\delta \in (0, 1)$ *there exists* $C(\delta) > 0$ *such that*

(2.6)
$$|k|^{\delta} \|v_k\| \leqq C(\delta) \|v\|_{C^{\delta}(X)} \quad \text{if} \quad v \in C^{\delta}_{2\pi}(X),$$
$$|k|^{1+\delta} \|v_k\| \leqq C(\delta) \|v\|_{C^{1+\delta}(X)} \quad \text{if} \quad v \in C^{1+\delta}_{2\pi}(X),$$

(2.7)
$$\text{if} \quad v(t) = \sum_{k=-\infty}^{\infty} v_k e^{ikt} \quad \text{with} \quad \|v_k\| \leqq C|k|^{-(1+\delta)}, \quad \text{then} \quad v \in C^{\delta}_{2\pi}(X).$$

3. THE BIFURCATION THEOREM

The assumptions stated above imply that the equation

(3.1)
$$\dot{u}(t) = A\, u(t) + L u_t + f(\mu, u(t), u_t)$$

linearized about $u = 0$ for $\mu = 0$ has a nontrivial 2π-periodic solution $u(t) = e^{it} x_0$. We shall now seek nontrivial $2\pi\, \varrho$-periodic solutions of (3.1) with ϱ near 1 and (μ, u) near $(0, 0)$. With the substitution $\tau = \varrho^{-1} t$ the equation (3.1) can be rewritten in the form

(3.2)
$$\dot{u}(t) = \varrho A\, u(t) + \varrho\, L(u_\varrho)_t + \varrho\, f(\mu, u(t), (u_\varrho)_t)$$
$$\text{with} \quad u_\varrho(s) = u\left(\frac{s}{\varrho}\right).$$

Now we shall look for 2π-periodic solutions of the equation

(3.3)
$$F(\varrho, \mu, u) = 0$$

where $F(\varrho, \mu, u)(t) = \dot{u}(t) - \varrho A\, u(t) - \varrho\, L(u_\varrho)_t - \varrho\, f(\mu, u(t), (u_\varrho)_t)$ is regarded as a mapping of $U \times Z$ into $C^{1+\delta}_{2\pi}(X)$ where U is a neighbourhood of $(1, 0)$ in $\mathbf{R}^2$ and $Z = C^{2+\delta}_{2\pi}(X) \cap C^{1+\delta}_{2\pi}(X^1)$. Some properties of F are given in

Lemma 2. *Let the assumptions* (H1), (H2), (H6) *be fulfilled. Then F has continuous derivatives* F_{uu}, $F_{\varrho u}$, $F_{\mu u}$ *and*

(i) $F(\varrho, \mu, 0) = 0$ *for* $(\varrho, \mu) \in U$,

(ii) $F_u(\varrho, \mu, 0)\, v(t) = \dot{v}(t) - \varrho A\, v(t) - \varrho\, L(v_\varrho)_t - \varrho f_x(\mu, 0, 0)\, v(t) - \varrho f_p(\mu, 0, 0)\, (v_\varrho)_t$,

(iii) $F_{\varrho u}(1, 0, 0)\, v(t) = -A\, v(t) - Lv_t + L(\mathrm{id}\, \dot{v}_t)$
(id *denotes the identity map* $\mathrm{id}(t) = t$),

(iv) $F_{\mu u}(1, 0, 0)\, v(t) = -f_{\mu x}(0, 0, 0)\, v(t) - f_{\mu p}(0, 0, 0)\, v_t$.

Proof. The computation of the derivatives is routine, provided we realize that

$$\frac{\partial}{\partial \varrho} u_\varrho(t) = -\frac{t}{\varrho^2}\, \dot{u}\left(\frac{t}{\varrho}\right) = -\frac{1}{\varrho}\left[(\mathrm{id}\, \dot{u})_\varrho\, (t)\right].$$

Next we examine the operator $F_u(1, 0, 0)\colon Z \to C_{2\pi}^{1+\delta}(X)$, $F_u(1, 0, 0)\, v(t) = \dot{v}(t) - A\, v(t) - Lv_t$. Denote $v_0(t) = \mathrm{Re}\,(\mathrm{e}^{\mathrm{i}t} x_0)$, $v_1(t) = \mathrm{Im}\,(\mathrm{e}^{\mathrm{i}t} x_0)$ where x_0 is given by $Ax_0 + B(\mathrm{i})\, x_0 = \mathrm{i} x_0$, $V = \mathrm{lin}\,\{v_0, v_1\}$. Note that $\dot{v}_0(t) = -v_1(t)$.

Lemma 3. *Let* (H3), (H4) *be satisfied*, $\alpha \leqq 1$, $\alpha - \beta < \delta < 1$. *Then*

(i) $\mathcal{N}(F_u(1, 0, 0)) = V$,

(ii) $\mathcal{R}(F_u(1, 0, 0)) \oplus V = C_{2\pi}^{1+\delta}(X)$.

Proof. Let $v \in Z$. Then $v(t) = \sum_{k=-\infty}^{\infty} v_k \mathrm{e}^{\mathrm{i}kt}$, $v_{-k} = \bar{v}_k$.

$$F_u(1, 0, 0)\, v(t) = \sum_{k=-\infty}^{\infty} (\mathrm{i}k - A - B(\mathrm{i}k))\, v_k \mathrm{e}^{\mathrm{i}kt}.$$

The assumptions (H4) (i), (ii) imply that the complex null space $\mathcal{N}^c(F_u(1, 0, 0)) = \mathrm{lin}\,\{x_0 \mathrm{e}^{\mathrm{i}t}, \bar{x}_0 \mathrm{e}^{-\mathrm{i}t}\}$ and hence the real null space $\mathcal{N}(F_u(1, 0, 0)) = V$. To prove (ii), let $y \in C_{2\pi}^{1+\delta}(X)$, $y(t) = \sum_{k=-\infty}^{\infty} y_k \mathrm{e}^{\mathrm{i}kt}$. We can decompose $y_1 = \tilde{y}_1 + a x_0$ where $\tilde{y}_1 \in \tilde{X}$, $a \in \mathbf{C}$. Then $y(t) = \tilde{y}(t) + a x_0 \mathrm{e}^{\mathrm{i}t} + \bar{a} \bar{x}_0 \mathrm{e}^{-\mathrm{i}t} = \tilde{y}(t) + a_1\, v_0(t) - a_2\, v_1(t)$ with $a_1 = 2 \mathrm{Re} a$, $a_2 = 2\, \mathrm{Im}\, a$. We prove that $\tilde{y} \in \mathcal{R}(F_u(1, 0, 0))$. Set $v_k = D(\mathrm{i}k)\, y_k = (\mathrm{i}k - A - B(\mathrm{i}k))^{-1}\, y_k$ for $k \neq \pm 1$, $v_1 = D(\mathrm{i})\, \tilde{y}_1$, $v_{-1} = \bar{v}_1$. By virtue of (2.4) and (2.6) we get $\|v_k\|_\nu \leqq C|k|^{\nu-1}\, \|y_k\| \leqq C|k|^{-(2+\delta-\nu)}$. If we take $\nu = 1$ and $\nu = 0$, we obtain the estimates $\|v_k\|_1 \leqq C|k|^{-(1+\delta)}$, $\|\mathrm{i}k v_k\| \leqq C|k|^{-(1+\delta)}$. Now (2.7) implies that $v = \sum_{k=-\infty}^{\infty} v_k \mathrm{e}^{\mathrm{i}kt} \in C_{2\pi}^{\delta}(X^1) \cap C_{2\pi}^{1+\delta}(X)$ and $\dot{v}(t) - A\, v(t) - Lv_t = \tilde{y}(t)$. To prove the regularity of v, set $u(t) = \dot{v}(t)$, $z(t) = \sum_{k=-\infty}^{\infty} ik\, B(ik)\, v_k e^{ikt}$. Then (H3) (i) and (2.4) with $v = \alpha$ imply that $\|ikB(ik) v_k\| \leqq C|k|^{-(\beta+\delta+1-\alpha)}$ and, consequently, $z \in C_{2\pi}^{\beta+\delta-\alpha}(X)$. Owing to the equality $B(ik)\, D(ik) = -I + (ik - A)\, D(ik)$, the function u, $u(t) = \sum_{k=-\infty}^{\infty} ikv_k e^{ikt}$ satisfies the equation

$$u(t) = e^{At} u(0) + \int_0^t e^{A(t-s)}(z(s) + \dot{y}(s))\, ds\,. \tag{3.4}$$

With help of regularity theorems which state that a solution of (3.4) belongs to the space $C^{1+\delta}((\varepsilon, T), X) \cap C^{\delta}((\varepsilon, T), X^1)$ whenever $z + \dot{y} \in C^{\delta}((0, T), X)$ (see e.g. [8]) we get $u \in C^{\delta+\beta-\alpha}([2\pi, 4\pi], X^1)$ and, owing to the periodicity, $u \in C_{2\pi}^{\delta+\beta-\alpha}(X^1)$. It is easily seen that $z(t) = Lu_t$, and (H3) (ii) yields $z \in C_{2\pi}^{\delta}(X)$. Again, the periodicity and the regularity of solutions of (3.4) imply $u \in C_{2\pi}^{1+\delta}(X) \cap C_{2\pi}^{\delta}(X^1)$ and, consequently, $v \in Z$, $F_u(1, 0, 0)\, v = \tilde{y}$.

Now we define the map G:

$$G(s, \varrho, \mu, v) = \begin{cases} s^{-1}F(\varrho, \mu, s(v_0 + v)) & \text{for} \quad s \neq 0\,, \\ F_u(\varrho, \mu, 0)\,(v_0 + v) & \text{for} \quad s = 0\,. \end{cases}$$

By Lemma 2, G is a mapping of class C^1 from a neighbourhood of $(0, 1, 0, 0)$ in $\mathbf{R}^3 \times Z_1$, where $Z_1 = \mathscr{R}(F_u(1, 0, 0))$, to $C_{2\pi}^{1+\delta}(X)$. Obviously $G(0, 1, 0, 0) = 0$ and the Frèchet derivative of the map $(\varrho, \mu, v) \to G(s, \varrho, \mu, v)$ at $(0, 1, 0, 0)$ is the linear map

$$\begin{aligned} \mathscr{G}(\hat{\varrho}, \hat{\mu}, \hat{v})\,(t) = \hat{v}'(t) - A\,\hat{v}(t) - L\hat{v}_t - \hat{\varrho}(A\, v_0(t) + \\ + Lv_{0t} - L(\mathrm{id}\, \dot{v}_{0t})) - \hat{\mu}(f_{\mu x}(0, 0, 0)\, v_0(t) + f_{\mu\varphi}(0, 0, 0)\, v_{0t})\,. \end{aligned} \tag{3.5}$$

We claim that $\mathscr{G}$ is an isomorphism. Once this is shown, the fact that $G(0, 1, 0, 0) = 0$ and the Implicit Function Theorem imply that the solutions (s, ϱ, μ, v) of $G = 0$ near $(0, 1, 0, 0)$ are given by continuously differentiable functions $\varrho(s)$, $\mu(s)$, $v(s)$. Then setting $u(s)\,(t) = s((v_0 + v(s))\,(t))$ we see that $(\varrho(s), \mu(s), u(s))$ is the desired curve of solutions of $F = 0$.

We shall discuss each part of $\mathscr{G}$ separately.

$$\begin{aligned} &\dot{\hat{v}}(t) - A\,\hat{v}(t) - L\hat{v}_t = F_u(1, 0, 0)\,\hat{v}(t)\,, \\ &\hat{\varrho}(A\, v_0(t) + Lv_{0t} - L(\mathrm{id}\, \dot{v}_{0t})) = \hat{\varrho}(\dot{v}_0(t) - L(\mathrm{id}\, \dot{v}_{0t})) = \\ &= \hat{\varrho}(-v_1(t) + L(\mathrm{id}\, v_{1t})) = \hat{\varrho}(-v_1(t) + \mathrm{Im}(e^{it} L(\tau \to \tau e^{i\tau} x_0))) = \\ &= \hat{\varrho}(-v_1(t) + \mathrm{Im}(e^{it}(ax_0 + x_1)) = \hat{\varrho}(-v_1(t) + a_1\, v_1(t) + a_2\, v_0(t) + \\ &+ \mathrm{Im}\, e^{it} x_1) = \hat{\varrho}((a_1 - 1)\, v_1(t) + a_2\, v_0(t) + F_u(1, 0, 0)\, \varphi(t)) \end{aligned}$$

where

$$\varphi(t) = F_u(1, 0, 0)^{-1}\, (\mathrm{Im}\, e^{it} x_1)$$

We have used here the notation from (H5) with $a_1 = \mathrm{Re}\, a$, $a_2 = \mathrm{Im}\, a$.

By differentiating (2.5) at $\mu = 0$, multiplying by e^{it} and computing the real part we obtain

$$\hat{\mu}(f_{\mu x}(0,0,0)\, v_0(t) + f_{\mu p}(0,0,0)\, v_{0t}) =$$
$$= -\hat{\mu}(\operatorname{Re} e^{it}(A + B(i) - i)\, x'(0) - \operatorname{Re} \lambda'(0)\, v_0(t)) =$$
$$= \hat{\mu}\, F_u(1,0,0)\, \psi(t) + \hat{\mu} \operatorname{Re} \lambda'(0)\, v_0(t)$$

where $\psi(t) = -\operatorname{Re}(e^{it} x'(0))$. Now we can write the operator $\mathscr{G}$ as follows:

$$\mathscr{G}(\hat{\varrho}, \hat{\mu}, \hat{v})(t) = F_u(1,0,0)(\hat{v} - \hat{\varrho}\, \varphi - \hat{\mu}\psi)(t) +$$
$$+ v_1(t)(1 - a_1)\, \hat{\varrho} + v_0(t)(-\hat{\mu} \operatorname{Re} \lambda'(0) - \hat{\varrho} a_2)\,.$$

From this expression, Lemma 3, (H5) and (H7) it is easily seen that $\mathscr{G}$ is a one to one mapping from $\mathbf{R}^2 \times \mathscr{R}(F_u(1,0,0))$ onto $C^{1+\delta}_{2\pi}(X)$.

The local uniqueness given by the Implicit Function Theorem together with the fact that u_θ: $u_\theta(t) = u(t + \theta)$ is a solution of (3.1) iff u is a solution ensures the uniqueness assertion in the following theorem:

Theorem. *Let the assumptions* (H1)–(H7) *be fulfilled. Then there are* $\varepsilon > 0$, $\eta > 0$, $0 < \delta < 1$ *nad continuously differentiable functions* $(\varrho, \mu, u) : (-\eta, \eta) \to \to \mathbf{R} \times \mathbf{R} \times (C^{2+\delta}_{2\pi}(X) \cap C^{1+\delta}_{2\pi}(X^1))$ *with the following properties:*

(a) $F(\varrho(s), \mu(s), u(s)) = 0$ for $|s| < \eta$,

(b) $\mu(0) = 0$, $u(0) = 0$, $\varrho(0) = 1$ and $u(s) \neq 0$ if $0 < |s| < \eta$.

(c) *If* $(\mu_1, u_1) \in \mathbf{R} \times Z$ *is a solution of* (3.1) *of period* $2\pi\varrho_1$, *where* $|\varrho_1 - 1| < \varepsilon$, $|\mu_1| < \varepsilon$ *and* $|u_1|_Z < \varepsilon$, *then there exist numbers* $s \in \langle 0, \eta)$ *and* $\theta \in \langle 0, 2\pi)$ *such that* $u_1(\varrho_1 \tau) = u(s)(\tau + \theta)$ *for* $\tau \in \mathbf{R}$.

4. AN EXAMPLE

We shall give here examples of memory operators L with singular kernels satisfying the conditions (H2), (H3), and an application of the previous results to the problem

$$u_t = a u_{xx} + bu + c \int_{-\infty}^{t} k(t - s)\, u_{xx}(s)\, ds + \tag{4.1}$$
$$+ \int_{-\infty}^{t} k_1(t - s)\, (g(\mu, u(s)_x)_x)\, ds$$
$$u(t, 0) = u(t, \pi) = 0\,, \quad x \in \langle 0, \pi \rangle\,, \quad t \in \mathbf{R}\,.$$

Lemma 4. *Let* $k(t) = t^{-\gamma} e^{-pt}$ *with* $\gamma < (2 - \alpha)/3$, $p > 0$. *Then the operator* L: $Y^\alpha \to X$, $L\varphi = \int_0^\infty k(s)\, A^\alpha \varphi(-s)\, ds$ *satisfies* (H2), (H3).

Proof. The continuity of L is obvious.

$$\|B(ik)\,x\| = \left\|\int_0^\infty s^{-\gamma} e^{-(p+ik)s} A^\alpha x \,ds\right\| = \left|\frac{\Gamma(1-\gamma)}{(p+ik)^{1-\gamma}}\right| \|x\|_\alpha \leqq \frac{C}{|k|^{1-\gamma}} \|x\|_\alpha ,$$

so the condition (H3) (i) holds with $\beta = 1 - \gamma$. To prove (H3) (ii), let $u \in C_{2\pi}^{\delta+\beta-\alpha}(X^\alpha)$, and write $Lu_t - Lu_s$ as follows:

$$Lu_t - Lu_s = \int_{-\infty}^{t} (t-\tau)^{-\gamma} e^{-p(t-\tau)} A^\alpha u(\tau)\,d\tau -$$

$$- \int_{-\infty}^{s} (s-\tau)^{-\gamma} e^{-p(s-\tau)} A^\alpha u(\tau)\,d\tau =$$

$$= \int_{-\infty}^{s} \left[(t-\tau)^{-\gamma} e^{-p(-\tau)} - (s-\tau)^{-\gamma} e^{-p(s-\tau)}\right] A^\alpha(u(\tau) - u(s))\,ds +$$

$$+ A^\alpha u(s) \int_{-\infty}^{s} \left[(t-\tau)^{-\gamma} e^{-p(t-\tau)} - (s-\tau)^{-\gamma} e^{-p(s-\tau)}\right] d\tau +$$

$$+ \int_s^t (t-\tau)^{-\gamma} e^{-p(t-\tau)} A^\alpha u(\tau)\,d\tau\,.$$

If we choose $\delta \in (2\gamma + \alpha - 1,\, 1 - \gamma)$, we can estimate the first integral with help of the Mean Value Theorem by $C(t-s)\,\|u\|_{C_{2\pi}^{\delta+\beta-\alpha}(X^\alpha)}$, the second integral is equal to

$$\int_{t-s}^\infty e^{-p\sigma}\sigma^{-\gamma}\,d\sigma - \int_0^\infty e^{-p\sigma}\sigma^{-\gamma}\,d\sigma = - \int_0^{t-s} e^{-p\sigma}\sigma^{-\gamma}\,d\sigma\,,$$

so that the second term as well as the third are estimated by $C(t-s)^{1-\gamma}\,\|u\|_{C_{2\pi}(X^\alpha)}$, which implies that $Lu_t \in C_{2\pi}^{\delta}(X)$.

Now, we turn our attention to the equation (4.1). We take $X = L^2(0, \pi)$, $\mathscr{D}(A) = W^{2,2}(0, \pi) \cap W^{2,1}(0, \pi)$, $\alpha = 1$, $0 < \gamma < \frac{1}{3}$.

$$Ax = ax'' + bx\,, \quad L\varphi = c\int_0^\infty s^{-\gamma} e^{-ps} D_2^2 \varphi(-s, \cdot)\,ds\,.$$

Then (H1)–(H3) are satisfied and

$$Ax + B(ik)\,x = \left(a + \frac{c\Gamma(1-\gamma)}{(p+ik)^{1-\gamma}}\right) x'' + bx\,.$$

The eigenvalues of A are the numbers $b - an^2$, $n = 1, 2, \dots$, and it is easily seen that there are no eigenvalues of $A + B(ik)$ with nonnegative real parts when $b \leq 0$. In that case the zero solution is asymptotically stable. On the other hand, we can choose constants a, b, c such that (H4), (H5) are fulfilled. g is supposed to be smooth with $g_u(0, 0) = 0$, $k_1 \in L^1$. If $\operatorname{Re} g_{\mu u}(0, 0) \hat{k}_1(\mathrm{i}) \neq 0$, then the transversality condition (H7) holds, so all assumptions of Theorem 1 are fulfilled and consequently there exists a nontrivial branch of periodic solutions of (4.1).

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Souhrn

HOPFOVA BIFURKACE PRO PARABOLICKÉ ROVNICE S NEKONEČNÝM ZPOŽDĚNÍM

HANA PETZELTOVÁ

V práci je dokázána existence Hopfovy bifurkace pro funkcionální diferenciální rovnice parabolického typu $u(t) = A\,u(t) + Lu_t + f(\mu, u(t), u_t)$, kde $u_t(s) = u(t + s)$, $s \in \mathbf{R}^-$, přičemž operátory L a f jsou definovány na prostorech funkcí s hodnotami v $\mathscr{D}(A)$. Výsledek je aplikován na integrodiferenciální rovnici se singulárním jádrem.

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