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A PERIODIC BOUNDARY VALUE PROBLEM IN HILBERT SPACE

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Summary. In the paper some existence results for periodic boundary value problems for the ordinary differential equation of the second order in a Hilbert space are given. Under some auxiliary assumptions the set of solutions is compact and connected or it is convex.

Keywords: periodic boundary value problem, Leray-Schauder theorem, convexity of set of solutions

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This paper deals with the problem

$$-x''(t) + \alpha^2 x(t) + f\big(t, x(t), x'(t)\big) = h(t), \tag{1}$$

$$x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi), \tag{2}$$

where $h\colon \langle -\pi, \pi\rangle \to H$, $f\colon \langle -\pi, \pi\rangle \times H \times H \to H$ and H is real Hilbert space with a norm $\|\cdot\|$, $\alpha \in \mathbb{R}$ is a positive constant.

We study the existence, uniqueness and some other properties of the set of solutions.

Similar problems concerning two point boundary value problems are solved in the papers by Schmitt and Thompson [ST], Mawhin [M] and Gupta [G]. This paper generalizes some results given in [R].

Preliminaries

We use the following function spaces:

$$L_1\big((-\pi,\pi),H\big) \text{ with the norm } \|u\|_1 = \int_{-\pi}^{\pi} \|u(t)\|^2\,\mathrm{d}t\,,$$

$$L_2\big((-\pi,\pi),H\big) \text{ with the norm } \|u\|_2 = \Big(\int_{-\pi}^{\pi} \|u(t)\|^2\,\mathrm{d}t\Big)^{\frac{1}{2}},$$

$$C\big(\langle-\pi,\pi\rangle\,,H\big) \text{ with the norm } \|u\|_0 = \sup_{t\in\langle-\pi,\pi\rangle} \|u(t)\|,$$

$$C_1\big(\langle-\pi,\pi\rangle\,,H\big) \text{ with the norm } \|u\|_{01} = \max\big\{\|u\|_0,\ \|u'\|_0\big\}.$$

Throughout the paper we denote these spaces as L_1, L_2, C, C_1, and assume that $h \in L_1$.

First we give an abstract formulation of the problem (1), (2).

Lemma 1 [ST, p. 281]. *A periodic boundary value problem* (1), (2) *is equivalent to the operator equation*

$$x = Tx,$$

where

$$Tx(t) = \int_{-\pi}^{\pi} G(t,s)\Big(h(s) - f\big(s,x(s),x'(s)\big)\Big)\,\mathrm{d}s\,, \tag{3}$$

and $G(t,s)$ is the Green function associated with the homogeneous problem $-x'' + \alpha^2 x = 0$, (2). (See [GŠŠ, p. 143], [R, Lemma 1].)

When $f(t,x,y)\colon \langle-\pi,\pi\rangle \times H \times H \to H$ is a completely continuous operator, then also the operator $T\colon C^1 \to C^1$ is completely continuous.

To obtain the existence of a solution to (1), (2) we use the following results.

Lemma 2 [R, Lemma 8]. *Let $y(t) \in C$, $y'(t) \in L_2$. Then there are such $a,b \in \mathbb{R}$ that*

$$\|y(t)\|_0 \leqslant a\|y'(t)\|_2 + b\|y(t)\|_2.$$

Lemma 3 (Nagumo type condition). *Let $R > 0$ be a constant, let $\Phi_R\colon \mathbb{R} \to \mathbb{R}$ be a positive nondecreasing continuous function such that*

$$\lim_{s\to\infty} \frac{s^2}{\Phi_R(s)} = \infty,$$

and let $\varphi(t) \in L_1$.

Then there is $M > 0$ such that, if $x(t) \in C^1$, $x''(t) \in L_1$, $\|x\|_0 < R$ and for almost every $t \in \langle -\pi, \pi \rangle$

$$\|x''(t)\| \leqslant \|\varphi(t)\| + \Phi_R(\|x'(t)\|),$$

then $\|x'\|_0 \leqslant M$.

Proof. Denote $q = \|x'\|_0 = \max \|x'(t)\| = \|x'(t_0)\|$. Let $\omega \in H$, $\|\omega\| = 1$ represent such a linear functional that $(x'(t_0), \omega) = \|x'(t_0)\|$. Denote $z(t) = (x(t), \omega)$. Let $\tau \in \mathbb{R}$, $|\tau| \leqslant \pi$, be such that $t_0 + \tau \in \langle -\pi, \pi \rangle$. Then there is $\xi \in \langle t_0, t_0 + \tau \rangle$, such that

$$z(t_0 + \tau) = z(t_0) + \tau\Big(z'(t_0) + \int_{t_0}^{\xi} z''(s)\,\mathrm{d}s\Big).$$

We denote $\delta = |\tau|$ and estimate

$$\delta\|x'(t_0)\| \leqslant 2R + \delta\Big|\int_{t_0}^{\xi} \|\varphi(s)\| + \Phi_R(\|x'(s)\|)\,\mathrm{d}s\Big| \leqslant 2R + \delta\|\varphi\|_1 + \delta^2\Phi_R(q).$$

Let $Q > 0$ be such that $\frac{q^2}{\Phi_2(q)} \geqslant 32R$ holds for $q > Q$.

Now for $q > 0$ we have

$$\delta q \leqslant 2R + \delta\|\varphi\|_1 + \delta^2 \frac{q^2}{32R},$$

i.e.

$$q \leqslant \frac{2R}{\delta} + \|\varphi\|_1 + \delta\frac{q^2}{32R}.$$

The right hand side function has its minimum at $\delta = \frac{8R}{q}$.

Now if $\frac{8R}{q} \geqslant \pi$, then $\frac{8R}{\pi} \geqslant q$. If $\frac{8R}{q} < \pi$, then choosing $\delta = \frac{8R}{q}$ we obtain $q \leqslant \frac{q}{2} + \|\varphi\|_1$ and $q \leqslant 2\|\varphi\|_1$.

That means we have obtained the estimate

$$\|x'\|_0 = q \leqslant \max\Big(Q, \frac{8R}{\pi}, 2\|\varphi\|_1\Big) = M.$$

□

Lemma 4 (Krasnosel'skij's theorem) [Z, Theorem 13.4]. *Let $T_n, T\colon \bar{\Omega} \subset X \to X$ be completely continuous operators for each $n \geqslant n_0$, let Ω be a nonempty open and bounded set in the Banach space X. Let*

$$\sup_{\bar{\Omega}} |Tx - T_n x|_x \to 0 \quad \textit{for } n \to \infty,$$

let the Leray-Schauder degree satisfy

$$d(I - T, \Omega, 0) \neq 0,$$

and let the equation

$$x = \bar{T}x = T_n x + \big(T(\bar{x}) - T_n(\bar{x})\big)$$

have for every $\bar{x} \in \Omega$ and every $n \geqslant n_0$ at most one solution.

Then the set of the fixed points of T is nonempty, compact and connected.

MAIN RESULTS

Theorem 1. *Let $f\colon \langle -\pi, \pi \rangle \times H \times H \to H$ be a completely continuous operator and*

(P1) *let there be constants $r > 0$ and a, b, c, $a + \frac{b^2}{4} < \alpha^2$, $b \geqslant 0$, $c \geqslant 0$, such that*

$$\big(f(t, x, y), x\big) \geqslant -a\|x\|^2 - b\|x\|\|y\| - c\|x\|$$

for every $(t, x, y) \in \langle -\pi, \pi \rangle \times H \times H$, $\|x\| > r$ or $\|y\| > r$,

(P2) *for each $R > 0$ let there exists a positive nondecreasing continuous function Φ_R satisfying*

$$\lim_{s \to \infty} \frac{s^2}{\Phi_R(s)} = \infty$$

such that if $\|x\| < R$ then

$$\|f(t, x, y)\| \leqslant \Phi_R(\|y\|).$$

Then there is a solution to the problem (1), (2) *for every $h(t) \in L_1$.*

Proof. We estimate the solution to the equation

$$x = \lambda T x \quad \text{for } \lambda \in (0, 1\rangle. \tag{4}$$

For $x(t)$ a solution to (4), we obtain

$$\int_{-\pi}^{\pi} (-x'', x)\,\mathrm{d}t + \int_{-\pi}^{\pi} \alpha^2 (x, x)\,\mathrm{d}t + \int_{M \cup N} \lambda\big(f(t, x, x'), x\big)\,\mathrm{d}t = \int_{-\pi}^{\pi} \lambda(h, x)\,\mathrm{d}t,$$

where $M = \{t,\ \|x(t)\| \leqslant r \text{ and } \|x'(t)\| \leqslant r\}$, $N = \langle -\pi, \pi \rangle - M$. Then

$$\|x'\|_2^2 + \alpha^2\|x\|_2^2 + \int_M \lambda\big(f(t, x, x'), x\big)\,\mathrm{d}t$$
$$- \lambda \int_N \big(a\|x\|^2 + b\|x\|\|x'\| + c\|x\|\big)\,\mathrm{d}t \leqslant \|h\|_1 \|x\|_0$$

and

$$\|x'\|_2^2 + \alpha^2\|x\|_2^2 - m - \lambda\Big(a\|x\|_2^2 + b\int_{-\pi}^{\pi}\|x\|\|x'\|\,\mathrm{d}t + c\int_{-\pi}^{\pi}\|x\|\,\mathrm{d}t\Big)$$
$$+\lambda\int_M \big(a\|x\|^2 + b\|x\|\|x'\| + c\|x\|\big)\,\mathrm{d}t \leqslant \|h\|_1\|x\|_0,$$

where $m = 2\pi \max\limits_{\|x\|\leqslant r,\ \|y\|\leqslant r} \big(f(t,x,y),x\big)$.

If $a\|x\|_2^2 + b\int_{-\pi}^{\pi}\|x\|\|x'\|\,\mathrm{d}t + c\int_{-\pi}^{\pi}\|x\|\,\mathrm{d}t \geqslant 0$, we estimate

$$b\|x\|\|x'\| \leqslant (1-\varepsilon)^2\|x'\|^2 + \frac{b^2}{4(1-\varepsilon)^2}\|x\|^2$$

and obtain

$$\big(1-(1-\varepsilon)^2\big)\|x'\|_2^2 + \left(\alpha^2 - a - \frac{b^2}{4(1-\varepsilon)^2}\right)\|x\|^2 - c\sqrt{2\pi}\|x\|_2 - m$$
$$-2\pi|a|r^2 - \|h\|_1\left(\sqrt{\frac{1}{2\pi}}\|x\|_2 + \sqrt{2\pi}\|x'\|_2\right) \leqslant 0,$$

that is

$$A_1\|x'\|_2^2 + A_2\|x\|_2^2 - A_3\|x\|_2 - A_4\|x'\|_2 - A_5 \leqslant 0,$$

where A_i are constants. Supposing $\varepsilon > 0$ is sufficiently small, A_1, A_2 are positive constants.

Then the last inequality implies $\|x\|_2 \leqslant C_1$, $\|x'\|_2 \leqslant C_2$ and by Lemma 2 we have $\|x\|_0 \leqslant \sqrt{\frac{1}{2\pi}}C_1 + \sqrt{2\pi}C_2 = C$.

In case that $a\|x\|_2^2 + b\int_{-\pi}^{\pi}\|x\|\|x'\|\,\mathrm{d}t + c\int_{-\pi}^{\pi}\|x\|\,\mathrm{d}t < 0$, we substitute this term by zero and obtain the a priori estimate $\|x\|_0 \leqslant C$ as well.

The assumption (P2) and Lemma 3 imply the estimate

$$\|x'\|_0 \leqslant M.$$

This means we obtain the a priori estimate of the solution of (4) in the space C^1. The Leray-Schauder theorem implies the existence of a solution to the problem (1), (2). □

Theorem 2. *Assume* (P2),

(P3) *H is a separable Hilbert space and $\{e_i\}$ is an orthonormal basis in H,*

(P4) *the operator $f\colon \langle -\pi,\pi\rangle \times H \times H \to H$ is continuous and bounded,*

(P5) *$h(t) \in L_2$,*

(P6) *there are constants a, b, $a + \frac{1}{4}b^2 < \alpha^2$, $b \geqslant 0$ such that*

$$(5) \qquad \big(f(t,x,y) - f(t,u,v), x - u\big) \geqslant -a\|x-u\|^2 - \frac{b^2}{4}\|x-u\|\|y-v\|$$

for every $x, y, u, v \in H$ and every $t \in \langle -\pi, \pi \rangle$.
Then there is a unique solution to the problem (1), (2).

Proof. Uniqueness.
Let x_1, x_2 be two solutions to (1), (2). Then

$$-x_1'' + x_2'' + \alpha^2(x_1 - x_2) + f(t, x_1, x_1') - f(t, x_2, x_2') = 0$$

and

$$\|x_1' - x_2'\|_2^2 + \alpha^2\|x_1 - x_2\|_2^2 + \int_{-\pi}^{\pi} \big(f(t,x_1,x_1') - f(t,x_2,x_2'), x_1 - x_2\big)\,\mathrm{d}t = 0.$$

The assumption (P6) implies that

$$\big(1 - (1-\varepsilon)^2\big)\|x_1' - x_2'\|_2^2 + \left(\alpha^2 - a - \frac{b^2}{4(1-\varepsilon)^2}\right)\|x_1 - x_2\|_2^2 \leqslant 0,$$

and then

$$\|x_1' - x_2'\|_2^2 = 0, \quad \|x_1 - x_2\|_2^2 = 0.$$

Hence $x_1(t) = x_2(t)$ for each $t \in \langle -\pi, \pi \rangle$. □

Existence. Let $E_n \subseteq H$ be a finite dimensional subspace $E_n = [e_1, \dots, e_n]$, let $P_n \colon H \to E_n$ be an orthogonal projection on E_n, $F_n \subset L_2$ a subspace $F_n = \{x(t) \in L_2,\ x(t) \colon \langle -\pi, \pi \rangle \to E_n\}$, let $Q_n \colon L_2\big(\langle -\pi, \pi \rangle, H\big) \to F_n$ be an orthogonal projection on F_n. Further denote $x_n = Q_n x$, $\operatorname{dom} L = \{x \in C^1,\ x'' \in L_2\}$, let $L \colon \operatorname{dom} L \to L_2$ be an operator

$$Lx = -x'' + \alpha^2 x$$

and $N \colon C^1 \to C$ an operator

$$Nx(t) = f\big(t, x(t), x'(t)\big).$$

We consider a system of finite dimensional problems

$$(6) \qquad -x_n'' + \alpha^2 x_n + P_n f(t, x_n, x_n') = P_n h(t),$$

$$(2) \qquad x_n(-\pi) = x_n(\pi), \quad x_n'(-\pi) = x_n'(\pi).$$

Obviously $P_n f\colon \langle -\pi, \pi\rangle \times E_n \times E_n \to E_n$ is a completely continuous operator.

The assumption (P6) implies by virtue of $u = v = 0$

$$(P_n f(t,x,y), x) \geqslant -a\|x\|^2 - b\|x\|\,\|y\| - c\|x\|, \tag{7}$$

where $c = \max_{t\in\langle -\pi,\pi\rangle} \|P_n f(t,0,0)\|$.

The preceding theorem implies the existence of a solution to the problem (6), (2). Moreover, the a priori estimates from the proof of this theorem hold. This means that there are constants independent on n such that

$$\|x_n\|_2 \leqslant C_1, \quad \|x_n'\|_2 \leqslant C_2, \quad \|x_n\|_0 \leqslant C \tag{8}$$

for every solution x_n to (6), (2).

A priori estimates (8) and the complete continuity of each of the operators

$$T_n x(t) = \int_{-\pi}^{\pi} G(t,s) P_n\Big(h(s) - f\big(s, x_n(s), x_n'(s)\big)\Big)\, \mathrm{d}s$$

imply that the set of solutions to each problem (6), (2) is compact both in C^1 and L_2.

As we have proved the uniqueness, this last statement is trivial. We use the idea of this proof also without uniqueness.

Denote the set of solutions to each problem (6), (2) as U_n and denote $V_n = \bigcup_{k=n}^{\infty} U_k$. Obviously $V_{n+1} \subset V_n$ and each V_n is a bounded set. Let $W_n = \overline{V}_n$ be a weak closure of the set V_n in the space L_2. Then W_n is weakly compact and $W_{n+1} \subset W_n$. Hence there is

$$x_0 \in \bigcap_{n=1}^{\infty} W_n$$

and a sequence $x_n \in V_n$ such that $x_n \rightharpoonup x_0$.

The equation (6) and the a priori estimates (8) imply that $\|x_n''\|_2 \leqslant c$, where c is a suitable constant. Now we choose a subsequence, we denote it again by $\{x_n\}$, such that

$$Lx_n = -x_n'' + \alpha^2 x_n \rightharpoonup v \quad \text{in } L_2.$$

As the graph of the linear operator L is a closed and convex set, it is also weakly closed and

$$v = Lx_0.$$

Thus $x_0 \in \operatorname{dom} L$.

Now we prove the inequality

$$\langle (L+N)u - h, u - x_0\rangle \geqslant 0. \tag{9}$$

First let $u \in \operatorname{dom} L \cap F_m$, $x_n \in F_n$ and $n \geqslant m$. We use the inequality (5) to obtain

$$\begin{aligned}\langle (L+N)x-(L+N)y, x-y\rangle &= \|x'-y'\|_2^2+\alpha^2\|x-y\|_2^2 \\ &\quad + \int_{-\pi}^{\pi} \big(f(t,x,x')-f(t,y,y'), x-y\big)\,dt \\ &\geqslant \big(1-(1-\varepsilon)^2\big)\|x'-y'\|_2^2 \\ &\quad + \left(\alpha^2 - a - \frac{b^2}{4(1-\varepsilon)^2}\right)\|x-y\|_2^2 \geqslant 0.\end{aligned}$$

Then

$$\begin{aligned}0 \leqslant \langle (L+N)u-(L+N)x_n, u-x_n\rangle &= \langle (L+N)u-h, u-x_n\rangle \\ &\quad - \langle (L+N)x_n-h, u-x_n\rangle .\end{aligned}$$

As $H = E_n \oplus E_n^{\perp}$, $u - x_n \in F_n$, $Q_n\big((L+N)x_n - h\big) \in F_n$ and x_n is a solution to (6), we have

$$\langle (L+N)x_n-h, u-x_n\rangle = \langle Q_n(L+N)x_n-h, u-x_n\rangle = 0.$$

Then

$$0 \leqslant \langle (L+N)u-h, u-x_n\rangle ,$$

and by $n \to \infty$ we obtain (9).

Now we prove that (9) holds for each $u \in \operatorname{dom} L$. From the Fourier series for $u(t)$ we obtain (cf. [R, Lemma 4, 5])

$$u(t) = \sum_{i=1}^{\infty} a_i(t)e_i, \quad u'(t) = \sum_{i=1}^{\infty} a_i'(t)e_i \quad \text{and} \quad u''(t) = \sum_{i=1}^{\infty} a_i''(t)e_i,$$

where $a_i(t) = \big(u(t), e_i\big) \in C^1$ and $a_i''(t) \in L_2$. Denote

$$u_n(t) = \sum_{i=1}^{n} a_i(t)e_i.$$

Then $u_n(t) \to u(t)$ in H for every $t \in \langle -\pi, \pi\rangle$. Since

$$\|u_n(s)-u_n(t)\| = \|P_n u(s) - P_n u(t)\| \leqslant \|u(s)-u(t)\|$$

and a similar inequality holds also for u_n', we have convergences

$$u_n \to u \text{ in } C, \quad u_n' \to u' \text{ in } C, \quad u_n'' \to u'' \text{ in } L_2.$$

The inequality

$$\langle (L+N)u_n - h, u_n - x_0\rangle \geqslant 0$$

for $u_n \in F_n$ and the convergences $Lu_n \to Lu$, $Nu_n \to Nu$ imply

$$\langle (L+N)u - h, u - x_0\rangle \geqslant 0$$

for every $u \in \operatorname{dom} L$.

Let now $v \in \operatorname{dom} L$, $\tau \geqslant 0$ and $u = x_0 + \tau v$. Then

$$\langle (L+N)(x_0+\tau v) - h, v\rangle \geqslant 0$$

and for $\tau \to 0$

$$\langle (L+N)x_0 - h, v\rangle \geqslant 0.$$

The density of $\operatorname{dom} L$ in L_2 implies

$$(L+N)x_0 - h = 0.$$

Theorem 3. *Let f be a completely continuous operator. Suppose the assumptions* (P1), (P2) *and*

(P6′) *there are constants a, b, $\bar{a} + \frac{1}{4}\bar{b}^2 = \alpha^2$, $\bar{b} \geqslant 0$ such that* (5) *holds for every $x, y, u, v \in H$ and every $t \in \langle -\pi, \pi\rangle$ are fulfilled.*

Then the set of solutions to the problem (1), (2) *is nonempty, compact and connected.*

Proof. We prove that the assumptions of Lemma 4 are fulfilled. The operator T is defined by (3).

We choose an open bounded set $\Omega = \{x(t) \in C^1,\ \|x(t)\|_{01} < C\}$, where C is the estimate of the norm of the solution of (4). The existence of such an estimate follows from Theorem 1.

The a priori estimate $\|x(t)\|_{01} < C$ for a solution $x(t)$ of the equation (4) implies that

$$d(I - \lambda T, \Omega, 0) = \text{const} \neq 0 \quad \text{for every } \lambda \in \langle 0, 1\rangle .$$

Denote $f_n(t,x,y) = \mu_n f(t,x,y)$, where $0 < \mu_n < 1$ and $\mu_n \to 1$ for $n \to \infty$. The sequence of operators T_n is given by

$$T_n x(t) = \int_{-\pi}^{\pi} G(t,s)\Big(h(s) - f_n\big(s, x(s), x'(s)\big)\Big)\,\mathrm{d}s .$$

The complete continuity of T implies that $T_n\colon C^1 \to C^1$ also is a completely continuous operator for every $n \in N$. Now we estimate

$$\sup_{x\in\Omega} \|T_n x(t) - Tx(t)\| = \sup_{x\in\Omega} \left\| \int_{-\pi}^{\pi} G(t,s)(1-\mu_n) f(s,x,x')\,\mathrm{d}s \right\| \leqslant 2\pi(1-\mu_n)G\cdot F,$$

$$\sup_{x\in\Omega} \|T_n x'(t) - Tx'(t)\| = \sup_{x\in\Omega} \left\| \int_{-\pi}^{\pi} \frac{\partial G(t,s)}{\partial t}(1-\mu) f(s,x,x')\,\mathrm{d}s \right\| \leqslant 2\pi(1-\mu_n)G_1\cdot F,$$

where F, G, G_1 are upper bounds of f, Green's function and its derivative.

The assumption (P6′) implies the inequality

$$\big(f_n(t,x,y) - f_n(t,u,v), x-u\big) \geqslant -\mu_n\bar{a}\|x-u\|^2 - \mu_n\bar{b}\|x-u\|\,\|y-v\|,$$

where $\mu_n\bar{a} + \mu_n\frac{1}{4}\bar{b}^2 < \alpha^2$.

Hence (P6) holds for f_n. In a similar way as in the proof of the preceding theorem the uniqueness of the solution to the problem

$$-x''(t) + \alpha^2 x(t) + f_n\big(t, x(t), x'(t)\big) = h(t) + g(t), \quad (2),$$

is proved for every $g, h \in L_1$.

Consequently, the operator equation

$$x = T_n x + (T\bar{x} - T_n\bar{x})$$

has a unique solution for every $\bar{x} \in \Omega$. □

Now Lemma 4 implies the statement of our theorem.

Theorem 4. *Suppose the assumptions* (P1)–(P5), (P6′) *hold.*
Then the set of solutions to the problem (1), (2) *is nonempty and convex.*

Proof. The proof is similar to that of Theorem 2. We consider the finite dimensional problem

$$-x_n'' + \alpha^2 x_n + P_n f(t, x_n, x_n') = P_n h + h_n, \tag{10}$$

$$x_n(-\pi) = x_n(\pi), \quad x_n'(-\pi) = x_n'(\pi), \tag{2}$$

where $\|h_n(t)\| \leqslant \frac{1}{n}$ for each $t \in \langle -\pi, \pi\rangle$, $h_n \in F_n$.

The assumption (P6) implies the inequality (7). Using Theorem 3 on the subspace E_n we obtain that the set of solutions to each boundary value problem (10), (2) is nonempty, compact and connected. Moreover, the proof of Theorem 1 implies the a priori estimates (8) for each solution x_n to the problem (10), (2).

Denote by U_n the set of solutions to (10), (2), where h_n satisfies $\|h_n\|_2 \leqslant k_n$, where k_n is a given sequence with $k_n \to 0$ for $n \to \infty$. Denote $V_n = \bigcup_{k=n}^{\infty} U_k$ and let $W_n = \overline{\operatorname{conv} V_n}$ be the weak closure of the convex hull of the set V_n in L_2. Then V_n, W_n are bounded sets, W_n is weakly compact, $W_{n+1} \subset W_n$ and there is

$$x_0 \in \bigcap_{n=1}^{\infty} W_n$$

and a subsequence $x_n \in \operatorname{conv} V_n$ such that $x_n \rightharpoonup x_0$ in L_2. Moreover, $x_n = \sum_{i=1}^{k} \lambda_i y_i$, where $y_i \in V_n$, $\lambda_i \in \langle 0, 1\rangle$, $\sum_{i=1}^{k} \lambda_i = 1$. The equation (10) implies that there is c such that $\|y_n''\| < c$, hence also $\|x_n''\| < c$. From this estimate we obtain similarly as in the proof of Theorem 2 that $x_0 \in \operatorname{dom} L$. We again prove the inequality

$$\langle (L+N)u - h, u - x_0 \rangle \geqslant 0. \tag{11}$$

By the same method as in the proof of Theorem 2 we obtain that

$$0 \leqslant \langle (L+N)u - h - h_{ni}, u - y_i \rangle .$$

Then

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{k} \lambda_i \langle (L+N)u - h - h_{ni}, u - y_i \rangle \\ &\leqslant \langle (L+N)u - h, u - x_n \rangle - \sum_{i=1}^{k} \lambda_i \langle h_{ni}, u - y_i \rangle \\ &\leqslant \langle (L+N)u - h, u - x_n \rangle + k_n \sum_{i=1}^{k} \lambda_i \|u - y_i\|_2. \end{aligned}$$

For $n \to \infty$ we obtain for each u, $\|u\|_2 < 2C_1$ the inequality (11). Now again similarly as in the proof of Theorem 2 we choose $u = x_0 + \tau v$ and derive the inequality

$$\langle (L+N)(x_0 + \tau v) - h, v \rangle \geqslant 0$$

which, with respect to $\|x_0\| < C_1$, holds for each v, $\|v\| < C_1$ and $\tau \in \langle 0, 1\rangle$. For $\tau \to 0$ we obtain

$$\langle (L+N)x_0 - h, v \rangle \geqslant 0$$

for every $v \in \operatorname{dom} L$, $\|v\| < C_1$, and then $(L+N)x_0 - h = 0$. Hence for every $x_0 \in \bigcap_{n=1}^{\infty} W_n$, x_0 is a solution to (1), (2). Moreover, $\bigcap_{n=1}^{\infty} W_n$ is the intersection of a decreasing sequence of convex sets.

Now let x_1, x_2 be two solutions to (1), (2). To prove the convexity of the set of solutions we show that there is a sequence k_n such that $x_i \in \bigcap_{n=1}^{\infty} W_n$ for $i = 1, 2$. Denote $x_{ni} = P_n x_i$, $i = 1, 2$. Then

$$-x''_{ni} + \alpha^2 x_{ni} + P_n f\big(t, x_i(t), x'_i(t)\big) = P_n h(t),$$
$$-x''_{ni} + \alpha^2 x_{ni} + P_n f\big(t, x_{ni}(t), x'_{ni}(t)\big) = P_n h(t) + h_{ni}(t),$$

where

$$h_{ni}(t) = P_n f\big(t, x_{ni}(t), x'_{ni}(t)\big) - P_n f\big(t, x_i(t), x'_i(t)\big),$$

and

$$\|h_{ni}(t)\| \leqslant \big\| f\big(t, x_{ni}(t), x'_{ni}(t)\big) - f\big(t, x_i(t), x'_i(t)\big)\big\| = k_{ni}$$

for every $t \in \langle -\pi, \pi \rangle$. Obviously $k_{ni} \to 0$ for $n \to \infty$. Now if we choose $k_n = \max(k_{n1}, k_{n2})$, then $x_i \in \bigcap_{n=1}^{\infty} W_n$ for $i = 1, 2$. This means that the set of solutions is convex. □

References

[D] *K. Deimling*: Ordinary differential equations in Banach spaces. Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[GŠŠ] *M. Greguš, M. Švec, V. Šeda*: Ordinary differential equations. Alfa, Bratislava, 1985. (In Slovak.)

[G] *Chaitan P. Gupta*: Boundary value problems for differential equations in Hilbert spaces involving reflection of the argument. JMAA *128* (1987), 375–388.

[M] *J. Mawhin*: Two point boundary value problems for nonlinear second order differential equations in Hilbert spaces. Tohoku Math. J. *32* (1980), 225–233.

[R] *B. Rudolf*: Periodic boundary value problem in Hilbert space for differential equation of second order with reflection of the argument. Mathematica Slovaca *42* (1992), 65–84.

[ST] *K. Schmitt, R. Thompson*: Boundary value problems for infinite systems of second-order differential equations. J. Differential Equations *18* (1975), 277–295.

[Z] *E. Zeidler*: Functional analysis and its applications I. Springer-Verlag, 1986.

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