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Jaroslav Ivančo

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# NOTE ON INDEPENDENT SETS OF A GRAPH

JAROSLAV IVANČO, Košice

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*Summary.* Let the number of $k$-element sets of independent vertices and edges of a graph $G$ be denoted by $n(G,k)$ and $m(G,k)$, respectively. It is shown that the graphs whose every component is a circuit are the only graphs for which the equality $n(G,k)=m(G,k)$ is satisfied for all values of $k$.

*Keywords*: Independent sets of a graph, circuit

*AMS classification*: 05C75

In this note we consider only finite undirected graphs without loops or multiple edges. Concepts and notation not defined in the paper will be used as in [2].

Let $G$ be a graph; its number of vertices and edges will be denoted by $n$ and $m$, respectively. The number of distinct $k$-element sets of independent vertices (edges) of the graph $G$ is denoted by $n(G,k)$ ($m(G,k)$, respectively).

The main result of Gutman's paper [1] is the following theorem: *Let $G$ be a connected graph. Then the equalities $n(G,k)=m(G,k)$ are satisfied for all $k \geqslant 0$ if and only if $G$ is a circuit.* I. Gutman also conjectured the extension of this theorem which is proved below. We establish Gutman's conjecture with a very short proof.

**Theorem.** *The equalities $n(G,k)=m(G,k)$ are satisfied for all $k \geqslant 0$ if and only if every component of $G$ is a circuit.*

Proof. Let $G$ be a graph for which the equalities $n(G,k)=m(G,k)$ are satisfied for all values of $k$. As $n(G,1)=n$ and $m(G,1)=m$ we have $n=m$. If $\{u,v\}$ is an independent 2-element set of $G$ then $uv$ is an edge of $\overline{G}$ (i.e. the complement of $G$). Hence

$$n(G,2)=|E(\overline{G})|=\binom{n}{2}-m=\binom{n}{2}-\frac{1}{2}\sum_{v\in V}\deg v,$$

where $V = V(G)$. As $m(G,k) = n(L(G),k)$ (where $L(G)$ stands for the line graph of $G$) and $|E(L(G))| = -m + \frac{1}{2} \sum_{v \in V} (\deg v)^2$ (see [2]) we get

$$m(G,2) = \binom{m}{2} - |E(L(G))| = \binom{m}{2} + m - \frac{1}{2} \sum_{v \in V} (\deg v)^2$$
$$= \binom{n}{2} + \frac{1}{2} \sum_{v \in V} \deg v - \frac{1}{2} \sum_{v \in V} (\deg v)^2.$$

Since $n(G,2) = m(G,2)$ we have

$$\binom{n}{2} - \frac{1}{2} \sum_{v \in V} \deg v = \binom{n}{2} + \frac{1}{2} \sum_{v \in V} \deg v - \frac{1}{2} \sum_{v \in V} (\deg v)^2.$$

This implies

$$\sum_{v \in V} (2 - \deg v) \deg v = 0.$$

Thus

$$\sum_{v \in V} (2 - \deg v)^2 = 2 \sum_{v \in V} (2 - \deg v) - \sum_{v \in V} (2 - \deg v) \deg\ v$$
$$= 2\Big( \sum_{v \in V} 2 - \sum_{v \in V} \deg\ v \Big) - 0 = 2(2n - 2m) = 0.$$

Therefore every vertex of $G$ has the degree 2 and so every component of $G$ is a circuit.

On the other hand, if every component of $G$ is a circuit then $L(G)$ is isomorphic to $G$ and so the equalities

$$n(G,k) = n(L(G),k) = m(G,k)$$

are satisfied for all $k$. □

*References*


[1] *I. Gutman*: On independent vertices and edges of a graph. Topics in Combinatorics and Graph Theory (R. Bodendiek and R. Henn, eds.). Physica-Verlag, Heidelberg, 1990, pp. 291–296.

[2] *F. Harary*: Graph Theory. Addison-Wesley, Reading, MA, 1969.



*Author's address*: *Jaroslav Ivančo*, Department of Geometry and Algebra, P. J. Šafárik University, 041 54 Košice, Jesenná 5, Slovakia.