Mathematica Bohemica

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Mathematica Bohemica, Vol. 119 (1994), No. 4, 381–384

Persistent URL: http://dml.cz/dmlcz/126112

ON INFINITE OUTERPLANAR GRAPHS

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(Received May 21, 1993)

Summary. In this Note, we study infinite graphs with locally finite outerplane embeddings, given a characterization by forbidden subgraphs

Keywords: infinite outerplanar graphs, end of an infinite graph

AMS classification: 05C10, 05C75

1. Introduction

Clearly, the classic result about outerplanar graphs is valid even if the graph is infinite and thus a planar graph is outerplanar (it admits an embedding where all its vertices lie on the same face) if and only if has no subvision of K_4 or $K_{2,3}$. But, as was pointed out by many authors (see [6]), when dealing with embeddings of finite graphs it is interesting to ensure embeddings without vertices or edges accumulation (p-embeddings) and in this case the previous result fails as Figure 1 shows.

The p-embeddings of the graphs pictured in Figure 1 are unique (see [5]) and so, since they are not outerplane, we have that those graphs are not p-outerplanar (on the other hand, if they are outerplanar, it is possible to shorten some of the infinite rays accumulating the vertices and edges of these rays to a point in the plane). We are going to prove that the two graphs given in Figure 2 are the two forbidden subgraphs for outerplane p-embeddings.

As it happens with other results on p-embeddings (see [3]), it can be remarked that the one-point compactifications of the underlying topological spaces to those graphs are homeomorphic to the two forbidden graphs for outerplane embeddings.

By an infinite graph we mean a connected graph such that its vertex set is countable and the degree at every vertex is finite (a locally finite countable graph). We

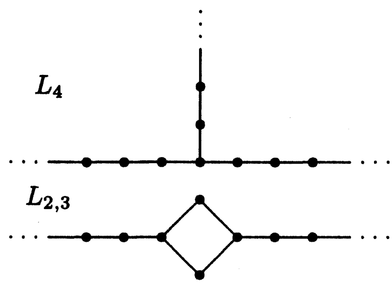

Figure 1: Two outerplanar graphs with no outerplane p-embeddings.

will use the notation and definitions of [4], except for using vertex instead of point and edge instead of line.

We will use an invariant of non-compact spaces, namely, the ends of Freudenthal. An infinite ray in a graph G is a morphism $\varphi\colon J \to G$ inducing an injection on both the vertex set and the edge set, where J represents a graph such that its underlying topological space is homeomorphic to the positive half-line R^+. Two rays in G define the same Freudenthal end if for any finite subgraph H of G, there exist vertices of both rays in $G - H$. The set of Freudenthal ends of G is denoted by $\mathcal{F}(X)$ and its cardinal by $e(X)$ (see [1] for details).

2. P-OUTERPLANAR GRAPHS

Definition 1. A graph G is said to be p-outerplanar if it can be embedded in the plane without vertices or edges accumulation so that all its vertices lie in the same face (a face is a connected component of the complement of the embedding with respect to the whole plane).

Observe that all blocks of the graphs given in Figure 1 are outerplanar (actually, the two graphs are outerplanar) but those graphs are not p-outerplanar, in fact it is possible to give graphs which are not p-outerplanar graphs and have a finite number of p-outerplanar blocks as is shown in Figure 2.

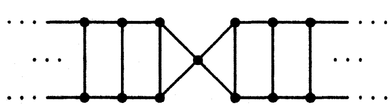

Figure 2: A outerplanar and non-p-outerplanar graph with two p-outerplanar blocks.

Nevertheless, we have

Lemma 2. *A graph G with only one end and infinitely many cut-vertices is p-outerplanar if and only if is outerplanar.*

Proof. G is outerplanar if and only if all its blocks are, and we can number those blocks in such a way that block B_i is joined in G with blocks B_{i-1} and B_{i+1} (block B_1 is only joined to block B_2). In this way we can glue each block to its adjacent blocks and we can ensure that all the vertices are in the unbounded face. □

Theorem 3. *A graph G is p-outerplanar if and only if it has no subdivision of K_4, $K_{2,3}$, L_4 or $L_{2,3}$ (see Figure 1).*

Proof. Obviously the four graphs given in the theorem are not p-outerplanar and thus it only remains to show that any p-outerplanar graph has no subdivision of K_4, $K_{2,3}$, L_4 or $L_{2,3}$.

As any p-outerplanar graph is outerplanar, it suffices to prove that an outerplanar graph is p-outerplanar if it has no subdivision of L_4 or $L_{2,3}$.

First, suppose that there exists a subgraph H of G homeomorphic to R^1 (the 1-dimensional Euclidean space). Then the restriction to H of any p-embedding of G if R^2 splits the plane in two half-planes. Thus it is straightforward to check that the two possible obstructions to be p-outerplanar are

- There exists a path in $G - H$ connecting two non-consecutive vertices of H. In this case, we have a subdivision of $L_{2,3}$.
- There exists an infinite ray in $G - H$. In this case, we have a subdivision of L_4.

If there is not such an H, then G has only a Freudenthal end and, as a consequence of Menger's theorem for infinite graphs (see [2]), in any H representative of that end (a representative of an end is a subgraph homoemorphic to the half-line $[0, +\infty)$ such that there exist its elements in the component of the complement of any compact defining that end), there exist infinitely many cut-vertices. Let $C = \{v_1, v_2, \ldots\}$ be that set of cut-vertices sorted as they appear in H and let G_i be the block of G which contains v_{i-1} and v_i (G_1 is the block of G which contains only v_1). Then, by Lemma 2, G is p-outerplanar if and only if G is outerplanar and, this happens if and only if all G_i's are outerplanar. □

Obviously if G has more that two ends, it is always possible to find a subgraph homeomorphic to L_4. Then G cannot be p-outerplanar and so we have

Corollary 4. *If G is p-outerplanar, then $e(G) \leqslant 2$.*

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