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DOMINATION IN GRAPHS WITH FEW EDGES

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Summary. The domination number $\gamma(G)$ of a graph G and two its variants are considered, namely the signed domination number $\gamma_s(G)$ and the minus domination number $\gamma^-(G)$. These numerical invariants are compared for graphs in which the degrees of vertices do not exceed 3.

Keywords: domination number, signed domination number, minus domination number.

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1. INTRODUCTION

In this paper we will consider finite undirected graphs without loops and multiple edges. We will study three numerical invariants of graphs which concern the domination.

If x is a vertex of a graph G, then $N[x]$ denotes the closed neighbourhood of x, i.e. the set consisting of x and of all vertices which are adjacent to x in G. If f is a function which maps the vertex set $V(G)$ of G into a set of numbers and $S \subseteq V(G)$, then $f(S) = \sum_{x \in S} f(x)$.

The concept of the domination number of a graph is well-known. A subset D of $V(G)$ is called dominating in G, if for each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to x. The minimum number of vertices of a dominating set in G is called the domination number of G and denoted by $\gamma(G)$.

There is still another definition of $\gamma(G)$. A function $f\colon V(G) \to \{0,1\}$ is called a domination function, if $f(N[x]) \geqslant 1$ for each $x \in V(G)$. The minimum of $f(V(G))$ taken over all domination functions f of G is called the domination number $\gamma(G)$ of G.

Both definitions are equivalent. If a dominating set D is given, we may take the function f such that $f(x) = 1$ for $x \in D$ and $f(x) = 0$ for $x \in V(G) - D$; then f

is a dominating function and $f(V(G)) = |D|$. On the other hand, if a dominating function f is given, we may put $D = \{x \in V(G) \mid f(x) = 1\}$; then D is a dominating set and $|D| = f(V(G))$.

In [1] the signed domination number and in [2] the minus domination number were introduced. A function $f\colon V(G) \to \{-1, 1\}$ (or $f\colon V(G) \to \{-1, 0, 1\}$) is called a signed (or minus, respectively) dominating function of G, if $f(N[x]) \geqslant 1$ for each $x \in V(G)$. The minimum of $f(V(G))$ taken over all signed (or minus) dominating functions f of G is called the signed (or minus, respectively) domination number of G. The signed domination number of G is denoted by $\gamma_s(G)$, the minus domination number of G by $\gamma^-(G)$.

The dominating function and the signed dominating function are particular cases of the minus dominating function. Hence $\gamma^-(G) \leqslant \gamma(G)$, $\gamma^-(G) \leqslant \gamma_s(G)$ for every graph G.

By G^2 we denote the graph whose vertex set is $V(G)$ and in which two vertices are adjacent if and only if their distance in G is 1 or 2. The independence number $\alpha(G)$ is the maximum cardinality of an independent set in G, i.e. of a set of vertices which are pairwise non-adjacent. The symbol $\delta(G)$ (or $\Delta(G)$) denotes the minimum (or maximum, respectively) degree of a vertex in G. In what follows we will study graphs G with $\Delta(G) \leqslant 3$.

2. Minus domination number

We prove two theorems comparing $\gamma^-(G)$ with $\gamma(G)$.

Theorem 1. *Let G be a graph, let $\Delta(G) = 2$. Then*

$$\gamma^-(G) = \gamma(G).$$

Proof. Let f be a minus dominating function of G such that $f(V(G)) = \gamma^-(G)$. If $f(x) \neq -1$ for all $x \in V(G)$, then f is a dominating function of G. Therefore $\gamma^-(G) = f(V(G)) \geqslant \gamma(G)$. Since $\gamma^-(G) \leqslant \gamma(G)$ as well, we have $\gamma^-(G) = \gamma(G)$. Thus we suppose that there exists a vertex $u_3 \in V(G)$ with $f(u_3) = -1$. Then u_3 is adjacent to two vertices u_2, u_4 such that $f(u_2) = f(u_4) = 1$; otherwise $f(N[u_3]) \leqslant 0$ would hold. The vertex u_2 (or u_4) must be adjacent to a vertex u_1 (or u_5) such that $f(u_1) = 1$ (or $f(u_5) = 1$, respectively). We will change the values of f in u_3 and u_4 to 0. If u_5 is not adjacent to a vertex with the value -1 or if $u_5 = u_1$ or $u_5 = u_2$, then the function obtained from f in this way is also a minus dominating function of G. Thus suppose that u_5 is adjacent to a vertex u_6 with the value -1 (even after the change). Then u_6 is adjacent to u_7 and u_7 is adjacent to u_8; both u_7 and u_8 have the value 1. We consider u_8 instead of u_5 and proceed in the same way. After a finite number of steps we obtain a vertex u_{3k+2} for a positive integer k such that either $u_{3k+2} = u_1$ or $u_{3k+2} = u_2$ or u_{3k+2} has degree 1 or u_{3k+2} is adjacent to a vertex u_{3k+3} with the value 0 or 1. Then we may change the values of f for all u_i with $i \equiv 0 \pmod 3$ from -1 to 0 and for all u_i with $i \equiv 1 \pmod 3$, $i \geqslant 4$ from 1 to 0. We obtain a new minus dominating function f_1 of G such that $f_1(V(G)) = f(V(G))$ and f_1 assigns the value -1 to less vertices than f does. If f_1 assigns -1 to at least one vertex, we repeat this procedure and proceed in this way until we obtain a function g such that $g(x) \neq -1$ for all $x \in V(G)$ and $g\big(V(G)\big) = f\big(V(G)\big) = \gamma^-(G)$. The function g is a dominating function of G and $\gamma(G) \leqslant \gamma^-(G)$, hence $\gamma^-(G) = \gamma(G)$. □

Theorem 2. *For each positive integer k there exists a connected graph G_k with $8k$ vertices such that $\Delta(G_k) = 3$, $\gamma^-(G) = 2k$, $\gamma(G) = \lceil \frac{5}{2}k \rceil$.*

Proof. First we construct a graph H. We put $V(H) = \{u_0, u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2\}$. The edges of H are u_0u_1, u_1u_2, u_2u_3, u_3v_1, v_1v_2, v_2v_3, u_1w_1, v_1w_1, u_2w_2, v_2w_2. If we identify the vertices u_0, v_3, we obtain a graph G_1. Now for $k \geqslant 2$ let $H_1, \ldots, H_k$ be disjoint copies of H. For $i = 1, \ldots, k-1$ we identify v_3 in H_i with u_0 in H_{i+1} and, moreover, v_3 in H_k with u_0 in H_1. The graph thus obtained will be G_k. Now we construct a function $f_0\colon V(H) \to \{-1, 0, 1\}$ in the following way. We put $f_0(u_0) = f_0(u_3) = f_0(v_3) = 0$, $F_0(u_1) = f_0(u_2) = f_0(v_1) = f_0(v_2) = 1$, $f_0(w_1) = f_0(w_2) = -1$. Further we define $f\colon V(G_k) \to \{-1, 0, 1\}$. In G_1 we may simply say that $f \equiv f_0$. For $k \geqslant 2$ each vertex $x \in V(G_k)$ is contained in H_i for some i and corresponds uniquely to a vertex $x_0 \in V(H)$; we may put $f(x) = f_0(x_0)$. We have $f(V(G)) = 2k$ and thus $\gamma^-(G_k) \leqslant 2k$. Now for $i = 1, \ldots, k$ denote by H_i' the graph obtained from H_i by deleting the vertex corresponding to u_0. The graphs $H_1', \ldots, H_k'$ are pairwise vertex-disjoint. Suppose that $\gamma^-(G_k) \leqslant 2k - 1$ and let g be the minus dominating function such that $g(V(G_k)) = \gamma^-(G_k)$. Then $\gamma^-(G_k) = \sum_{i=1}^{k} g(V(H_i'))$ and there exists i such that $g(V(H_i')) \leqslant 1$. If no vertex in H_i' is labelled by -1 in g, then at most one is labelled by 1 and all others by 0 and evidently $g(N[x]) \leqslant 0$ for some $x \in V(H_i')$, which is a contradiction. If exactly one vertex in H_i' is labelled by -1, then two vertices adjacent to it are labelled by 1 in order that its closed neighbourhood might have the sum of values of g at least 1. As $g(V(H_i')) \leqslant 1$, all other vertices must be labelled by 0 and again $g(N[x]) \leqslant 0$ for some $x \in V(H_i')$, which is a contradiction. If there are at least two vertices labelled by -1 in $V(H_i')$, then their closed neighbourhoods must be pairwise disjoint (as $\Delta(G) = 3$ no vertex may be adjacent to two vertices labelled by -1) and each of those neighbourhoods must contain at least two vertices labelled by 1. This implies that $g(V(H_i')) \geqslant 2$, which is again a contradiction. We have proved that $\gamma^-(G_k) = 2k$.

Evidently $\gamma(G_1) = 3 = \lceil \frac{5}{2} \cdot 1 \rceil$. Suppose that $\gamma(G_k) < \lceil \frac{5}{2}k \rceil$ for some $k \geqslant 2$. If k is even, then $\gamma(G_k) \leqslant \frac{5}{2}k - 1$. Let D be a dominating set in G_k with $\gamma(G_k)$ vertices. Consider the (pairwise disjoint) subgraphs $H_1' \cup H_2, H_3' \cup H_4, \ldots, H_{k-1}' \cup H_k$ of G_k. Then at least one of these graphs contains less than 5 vertices of D; without loss of generality let it be $H_1' \cup H_2$ and let $D_0 = D \cap V(H_1' \cup H_2)$. Only the vertex corresponding to u_1 in H_1 and the vertex corresponding to v_3 in H_2 may be dominated by a vertex of $D - D_0$. Thus D_0 is a subset of $V(H_1' \cup H_2)$ such that $|D_0| \leqslant 4$ and each vertex of $V(H_1' \cup H_2)$ different from u_1 in H_1' and v_3 in H_2 is dominated by a vertex of D_0. By exhausting all cases we can show that such a set D_0 does not exist, which is a contradiction. Hence $\gamma(G_k) \geqslant \frac{5}{2}k$ for k even. Now we can construct a dominating set D with $|D| = \frac{5}{2}k$ in such a way that in each H_i for i odd we take the vertices corresponding to u_1, u_3, v_2 and in each H_i for i even we take the vertices corresponding to u_2, v_1. We have proved that $\gamma(G) = \lceil \frac{5}{2}k \rceil$ for k even. For k odd the proof is analogous. □

Conjecture. Let G be a regular graph of degree 3. Then

$$\gamma^-(G) = \gamma(G).$$

3. Signed domination number

Here we will compare $\gamma_s(G)$ with $\alpha(G^2)$ and $\gamma(G)$.

Theorem 3. *Let G be a graph with n vertices, let $\Delta(G) \leqslant 3$. Let V_0 be the set of all vertices of G of degrees 0 and 1 and of all vertices which are adjacent to vertices of degree 1 in G. Let a be the maximum number of vertices of a subset of $V(G) - V_0$ which is independent in G^2. Then*

$$\gamma_s(G) = n - 2a.$$

Proof. Let f be a signed dominating function of G such that $f(V(G)) = \gamma_s(G)$. Let $V^+ = \{x \in V(G) \mid f(x) = 1\}$, $V^- = \{x \in V(G) \mid f(x) = -1\}$. Each vertex of V^- must be adjacent to at least two vertices of V^+; therefore it can have degree neither 0 nor 1. A vertex x which is adjacent to a vertex y of degree 1 cannot be in V^-; otherwise $f(N[y]) = f(x) + f(y) = f(y) - 1 \leqslant 0$. Therefore $V^- \subseteq V(G) - V_0$. Suppose that two vertices x, y of V^- are adjacent in G^2. Then either they are adjacent in G, or there exists a vertex z adjacent in G with both x and y. As $\Delta(G) \leqslant 3$, we have $f(N[x]) \leqslant f(x) + f(y) + 2 = 0$ in the former case, $f(N[z]) \leqslant f(x) + f(y) + 2 = 0$ in the latter, which is a contradiction with the fact that f is a signed dominating function. Therefore V^- is an independent set in G^2. We have $f(V(G)) = |V^+| - |V^-| = n - 2|V^-|$. On the other hand, let A be an independent set in G^2 such that $A \subseteq V(G) - V_0$. Let $g\colon V(G) \to \{-1, 1\}$ be such that $g(x) = -1$ for all $x \in A$ and $g(x) = 1$ for all $x \in V(G) - A$. If $x \in A$, then $x \notin V_0$ and x is adjacent to at least two vertices; let y, z be such two vertices. As A is independent in G^2, the vertices y, z are not in A and $g(y) = g(z) = 1$. Therefore $g(N[x]) \geqslant g(x) + g(y) + g(z) = -1 + 1 + 1 = 1$. If $x \notin A$ and x is adjacent to a vertex $y \in A$, then the degree of x is at least 2 and x is adjacent to a vertex $z \in A$ and to no vertex of A different from y. Then $g(N[x]) \geqslant g(x) + g(y) + g(z) = 1 + (-1) + 1 = 1$. The function g is a signed dominating function of G. We have proved that a subset M of $V(G)$ is the set of vertices in which some signed dominating function has the value -1 if and only if M is a subset of $V(G) - V_0$ which is independent in G^2. This implies the assertion. □

Corollary 1. *Let G be a graph with n vertices, let $\delta(G) \geqslant 2, \Delta(G) \leqslant 3$. Then*

$$\gamma_s(G) = n - 2\alpha(G^2).$$

Theorem 4. *Let G be a graph, let c be the number of its connected components. Then*

$$\gamma_s(G) - \gamma(G) \leqslant 2c.$$

Proof. Each connected component of G is a path or a circuit. Consider a path P_m with m vertices; let its vertices be $u_1, \ldots, u_m$ and let its edges be $u_i u_{i+1}$ for $i = 1, \ldots, m-1$. Evidently $\gamma(P_m) \geqslant \lceil \frac{1}{3} m \rceil$. If $m \equiv 0 \pmod 3$ or $m \equiv 2 \pmod 3$, then the set D of all u_i for $i \equiv 2 \pmod 3$ is a dominating set in P_m with $\lceil \frac{1}{3} m \rceil$ vertices. If $m \equiv 1 \pmod 3$, then $D \cup \{u_m\}$ is such a set. We have $\gamma(P_m) = \lceil \frac{1}{3} m \rceil$. Now if f is a signed dominating function of P_m, we have $f(u_1) = f(u_2) = f(u_{m-1}) = f(u_m) = 1$ and if $f(u_i) = f(u_j) = -1$, $i \neq j$, then $|i - j| \geqslant 3$ (see the proof of Theorem 3). We can choose the function f such that $f(u_i) = -1$ if and only if $i \equiv 0 \pmod 3$ and $i \leqslant m-2$; otherwise $f(u_i) = 1$. The function f is a signed dominating function of P_m. Denote $V^+ = \{x \in V(P_m) \mid f(x) = 1\}$, $V^- = \{x \in V(P_m) \mid f(x) = -1\}$. Evidently V^- has the maximum number of vertices among the subsets of $V(P_m)$ which are independent in P_m^2 and contain no vertex of degree 1 and no vertex adjacent to a vertex of degree 1; we have $f(V(P_m)) = m - 2|V^-| = \gamma_s(P_m)$. If $m \equiv 2 \pmod 3$, then $|V^-| = \frac{1}{3}(m-2)$ and $\gamma_s(P_m) = \frac{1}{3}(m+1)+1$, $\gamma_s(P_m) - \gamma(P_m) = 1$. If $m \equiv 0 \pmod 3$, then $|V^-| = \frac{1}{3}m - 1$ and $\gamma_s(P_m) = \frac{1}{3}m + 2$, $\gamma_s(P_m) - \gamma(P_m) = 2$. If $m \equiv 1 \pmod 3$, $m \geqslant 4$, then $|V^-| = \frac{1}{3}(m-1) - 1$ and $\gamma_s(P_m) = \frac{1}{3}(m+2)+2$, $\gamma_s(P_m) - \gamma(P_m) = 2$. Trivially, for $m = 1$ we have $\gamma_s(P_1) = 1$, $\gamma_s(P_1) - \gamma(P_1) = 0$. Now consider the circuit C_m with m vertices. We have $\gamma(C_m) = \lceil \frac{1}{3}m \rceil$. We choose the function f such that $f(u_i) = -1$ if and only if $i \equiv 0 \pmod 3$; this is evidently again a signed dominating function such that $f(V(C_m)) = \gamma_s(C_m)$. If we again denote $V^- = \{x \in V(C_m) \mid f(x) = -1\}$, then $|V^-| = \lfloor \frac{1}{3}m \rfloor$. Therefore for $m \equiv 0 \pmod 3$ we have $\gamma_s(C_m) = \frac{1}{3}m$, $\gamma_s(C_m) - \gamma(C_m) = 0$. For $m \equiv 1 \pmod 3$ we have $\gamma_s(C_m) = \frac{1}{3}(m+2)$, $\gamma_s(C_m) - \gamma(C_m) = 0$. For $m \equiv 2 \pmod 3$ we have $\gamma_s(C_m) = \frac{1}{3}(m+1)+1$, $\gamma_s(C_m) = \gamma(C_m) = 1$. The domination number of a graph is the sum of domination numbers of its connected components and the same holds also for the signed domination number. This implies the assertion. □

Corollary 2. *Let G be a regular graph of degree 2, let c be the number of its connected components. Then*

$$\gamma_s(G) - \gamma(G) \leqslant c.$$

Theorem 5. *Let G be a regular graph of degree 3, let its number n of vertices be divisible by 4, let $\alpha(G^2) = \frac{1}{4}n$. Then $\gamma(G) = \frac{1}{4}n$, $\gamma_s(G) = \frac{1}{2}n$, i.e.*

$$\gamma_s(G) = 2\gamma(G).$$

Proof. Let A be an independent set in G^2 such that $|A| = \frac{1}{4}n$. If $x \in A$, $y \in A$, $x \neq y$, then the distance between x and y in G is at least 3 and thus $N[x] \cap N[y] = \emptyset$. As G is 3-regular, $|N[x]| = 4$ for each $x \in V(G)$. We have $\left|\bigcup_{x \in A} N[x]\right| = \frac{1}{4}n \cdot 4 = n$ and thus $\bigcup_{x \in A} N[x] = V(G)$. The sets $N[x]$ for $x \in A$ form a partition of $V(G)$. This implies that A is a dominating set in G and $\gamma(G) \leqslant |A| = \frac{1}{4}n$. The domination number of a 3-regular graph cannot be less than $\frac{1}{4}n$, therefore $\gamma(G) = \frac{1}{4}n$. By Theorem 3 we have $\gamma_s(G) = n - 2\alpha(G^2) = \frac{1}{4}n$. □

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