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AN ALGEBRA OF QUASIORDERED LOGIC

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By q-lattice (see [1]) we mean an algebra $(A; \vee, \wedge)$ satisfying the following identities:

associativity:	$a \vee (b \vee c) = (a \vee b) \vee c,$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c,$
commutativity:	$a \vee b = b \vee a,$	$a \wedge b = b \wedge a,$
weak absorption:	$a \vee (a \wedge b) = a \vee a,$	$a \wedge (a \vee b) = a \wedge a,$
weak idempotence:	$a \vee (b \vee b) = a \vee b,$	$a \wedge (b \wedge b) = a \wedge b,$
equalization:	$a \vee a = a \wedge a.$	

Some elementary results on q-lattices are presented in [1]. A *quasiorder* on $A \neq \emptyset$ is a reflexive and transitive binary relation on A. Let Q be a quasiorder on A. It is well known (see e.g. [2]) that the relation $E_Q = Q \cap Q^{-1}$ (where Q^{-1} is an inverse relation of Q) is an equivalence on A, and the factor set A/E_Q is ordered by the order $\leqslant_Q$ defined as follows:

$$(*) \qquad B, C \in A/E_Q, \quad B \leqslant_Q C \quad \text{iff} \quad \langle b, c\rangle \in Q \quad \text{for each } b \in B,\ c \in C.$$

We call $\leqslant_Q$ *an order induced by* Q.

The following two theorems were proved in [1]:

Theorem 1. *Let $(A; \vee, \wedge)$ be a q-lattice. The binary relation Q on A defined by*

$$\langle a, b\rangle \in Q \quad \textit{iff} \quad a \vee b = b \vee b \quad \textit{(or, equivalently } a \wedge b = a \wedge a)$$

is a quasiorder such that the induced ordered set $(A/E_Q, \leqslant_Q)$ is a lattice.

We call $(A/E_Q, \leqslant_Q)$ the *lattice induced by* $\leqslant_Q$ and Q the *quasiorder induced* by a q-lattice $(Q, \vee, \wedge)$.

In [1], also the converse theorem is proved:

Theorem 2. *Let Q be a quasiorder on a set $A \neq \emptyset$ such that $(A/E_Q, \leqslant_Q)$ is a lattice. If κ is a choice function $\kappa\colon \operatorname{Exp} A \to A$ such that $\kappa([a]_{E_Q}) \in [a]_{E_Q}$ for each $a \in A$, then $(A; \vee, \wedge)$ is a q-lattice, where $\vee$, $\wedge$ are defined as follows:*

$$(**) \qquad \begin{aligned} a \vee b &= \kappa\big(\sup \leqslant_Q ([a]_{E_Q}, [b]_{E_Q})\big), \\ a \wedge b &= \kappa\big(\inf \leqslant_Q ([a]_{E_Q}, [b]_{E_Q})\big). \end{aligned}$$

Hence, the theory of q-lattices as quasiordered sets is an analogue of the theory of lattices as ordered sets. Moreover, we can prove

Theorem 3. *Let $(A; \vee, \wedge)$ be a q-lattice and Q an induced quasiorder on A. Then E_Q is the least congruence on $(A; \vee, \wedge)$ such that the factor q-lattice by E_Q is a lattice, i.e. $(A/E_Q, \leqslant_Q)$ is the modification of $(A; \vee, \wedge)$ in the variety of all lattices.*

Proof. Let $x \in [a]_{E_Q}$, $y \in [b]_{E_Q}$. Then $\langle x, a\rangle \in Q$, $\langle a, x\rangle \in Q$, $\langle y, b\rangle \in Q$, $\langle b, y\rangle \in Q$, i.e. $x \vee a = a \vee a = x \vee x$, $y \vee b = b \vee b = y \vee y$ by Theorem 1. Hence, by associativity and commutativity, also $(x\vee y)\vee(a\vee b) = (x\vee a)\vee(y\vee b) = (a\vee a)\vee(b\vee b)$, i.e. $\langle x \vee y, a \vee b\rangle \in Q$. Analogously, $(x\vee y)\vee(a\vee b) = (x\vee a)\vee(y\vee b) = (x\vee x)\vee(y\vee y)$, i.e. $\langle a \vee b, x \vee y\rangle \in Q$, thus $\langle a \vee b, x \vee y\rangle \in E_Q$. Hence $x \vee y \in [a \vee b]_{E_Q}$. Dually we can prove that $x \wedge y \in [a \wedge b]_{E_Q}$. Thus E_Q is a congruence on $(A; \vee, \wedge)$.

It is easy to see that the factor q-lattice $(A/E_Q; \vee, \wedge)$ is a lattice and that E_Q is the least congruence on $(A; \vee, \wedge)$ with the required property. Indeed, if E is a congruence on $(A; \vee, \wedge)$ such that $E \subseteq E_Q$ and $E \neq E_Q$, then there is a pair x, $y \in A$ such that

$$\langle x, y\rangle \in Q \quad \text{but } \langle x, y\rangle \notin E.$$

The factor q-lattice $(A/E; \vee, \wedge)$ cannot be a lattice since the induced relation Q/E on A/E is not antisymmetrical. □

Example 1. Let $A = \{a, b, c, d, p, q, r, s, t, u, v, w\}$ be a set. The quasiorder Q on A is visualized in Fig. 1 (here $\langle x, y\rangle \in Q$ iff the point x is connected with y by a path composed of arrows with the same orientation).

The induced lattice is visualized in Fig. 2, where $\{a, b, c, d\}$, $\{r, s, t\}$, $\{u\}$, $\{p, q\}$, $\{v, w\}$ are all classes of the congruence E_Q. Choose a choice function by Theorem 2, e.g.

$$\begin{aligned} &\kappa(\{v, w\}) = w, \quad \kappa(\{u\}) = u, \quad \kappa(\{r, s, t\}) = t, \\ &\kappa(\{p, q\}) = p, \quad \kappa(\{a, b, c, d\}) = a, \end{aligned}$$

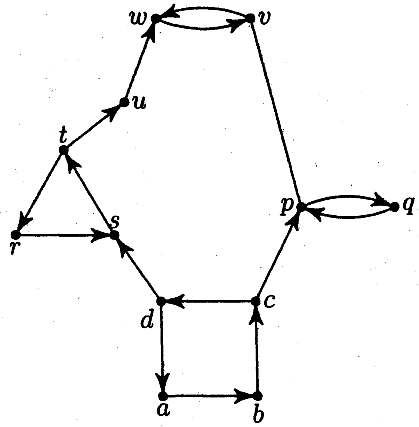

Fig. 1

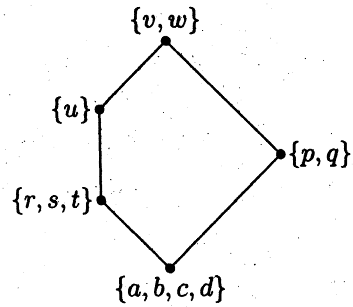

Fig. 2

and introduce $\vee$ and $\wedge$ by (**). Then, by Theorem 2, $(A; \vee, \wedge)$ is a q-lattice, where e.g.

$$b \vee b = c \vee d = a, \quad d \vee r = t, \quad u \wedge q = a,$$
$$s \vee p = w, \quad c \wedge c = a, \quad q \vee q = q \wedge q = p, \quad \text{etc.}$$

Definition. A q-lattice $(A; \vee, \wedge)$ is *distributive* if it satisfies the identity

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Lemma 1. *A q-lattice $(A; \vee, \wedge)$ is distributive iff the induced lattice is distributive.*

The proof is an easy consequence of the foregoing Theorems 1, 2, 3.

Hence, a q-lattice is distributive if and only if it does not contain a q-lattice isomorphic to one of those in Fig. 3.

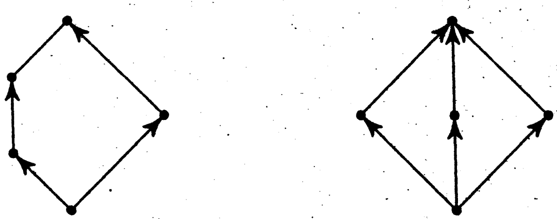

Fig. 3

Definition. A q-lattice $(A; \vee, \wedge)$ is *bounded* if there exist elements 0 and 1 in A such that

$$x \wedge 0 = 0 \quad \text{and} \quad x \vee 1 = 1 \quad \text{for each } x \in A.$$

The element 0 is called *zero* and 1 is called *unit* of $(A; \vee, \wedge)$.

E x a m p l e 2. The q-lattice in Fig. 1 is bounded, the element a being its zero and w its unit.

Lemma 2. *If $(A; \vee, \wedge)$ is a bounded q-lattice, then $\langle 0, x\rangle \in Q$ and $\langle x, 1\rangle \in Q$ for each $x \in A$, where Q is the induced quasiorder, i.e. $x \vee 0 = x \vee x$ and $x \wedge 1 = x \wedge x$.*

P r o o f. By definition, $0 \wedge 0 = 0$ and $1 \vee 1 = 1$, thus also $x \wedge 0 = 0$, $x \vee 1 = 1$, i.e. $\langle 0, x\rangle \in Q$ and $\langle x, 1\rangle \in Q$. □

R e m a r k. Contrary to the case of lattices it can happen that also $\langle y, 0\rangle \in Q$ or $\langle 1, z\rangle \in Q$ for some elements y, z of a q-lattice $(A; \vee, \wedge)$. For example, in the q-lattice in Fig. 1 we have $\langle b, 0\rangle \in Q$, $\langle 0, b\rangle \in Q$ or $\langle c, 0\rangle \in Q$ or $\langle d, 0\rangle \in Q$, where $0 = a$, and $\langle 1, v\rangle \in Q$, $\langle v, 1\rangle \in Q$, where $1 = w$.

Definition. Let $(A; \vee, \wedge)$ be a bounded q-lattice. An element $b \in A$ is called a *complement of* $a \in A$ if $a \vee b = 1$ and $a \wedge b = 0$. If each $a \in A$ has a complement, $(A; \vee, \wedge)$ is called a *complemented q-lattice.*

E x a m p l e 3. The q-lattice $(A; \vee, \wedge)$ in Fig. 1 is complemented. For example, u is a complement of p, r is a complement of p, v is a complement of b, etc.

Definition. Elements x, y of a q-lattice $(A; \vee, \wedge)$ are *neighbours* if

$$x \vee x = y \vee y \qquad \text{or, equivalently, } x \wedge x = y \wedge y.$$

It is clear that x and y are neighbours iff $[x]_{E_Q} = [y]_{E_Q}$, where Q is an induced quasiorder.

Lemma 3. *Let $(A; \vee, \wedge)$ be a distributive q-lattice and let x, y, $z \in A$. If the elements y, z are complements of x, then y, z are neighbours.*

P r o o f. Since

$$y \vee 0 = y \vee (x \wedge z) = (y \vee x) \wedge (y \vee z) = 1 \wedge (y \vee z),$$
$$z \vee 0 = z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) = 1 \wedge (y \vee z),$$

thus $y \vee 0 = z \vee 0$. By Lemma 2, we have

$$y \vee y = y \vee 0 = z \vee 0 = z \vee z.$$

□

Let $(A; \vee, \wedge)$ be a distributive complemented q-lattice. Introduced the unary operation $*$ on A by the rule

$$(***) \qquad a^* = b \vee b,$$

where b is a complement of a. By Lemma 3 this definition is correct.

Lemma 4. *Let $(A; \vee, \wedge)$ be a complemented distributive q-lattice. Then a^* is a complement of $a \in A$ and $0^* = 1$ and $1^* = 0$.*

Proof. Let $a \in A$. Then $a^* = b \vee b$ for a complement b of a. By weak idempotence, we have $a \vee a^* = a \vee (b \vee b) = a \vee b = 1$. By equalization and weak idempotence, we obtain

$$a \wedge a^* = a \wedge (b \vee b) = a \wedge (b \wedge b) = (a \wedge b) = 0.$$

Thus, a^* is a complement of a. By the definition of 0 and 1, clearly $1 \wedge 0 = 0$ and $0 \vee 1 = 1$, thus 0 and 1 are complemented elements. Moreover, $0 \wedge 0 = 0$, by equalization also $0 \vee 0 = 0$, thus $1^* = 0$. Analogously it can be shown that $0^* = 1$.

□

Theorem 4. *Let $(A; \vee, \wedge)$ be a distributive complemented q-lattice. Then for each a, $b \in A$ we have*

(1) $(a \vee b)^* = a^* \wedge b^*$, $(a \wedge b)^* = a^* \vee b^*$ *(De Morgan laws)*

(2) *if $\langle a, b\rangle \in Q$, then $\langle b^*, a^*\rangle \in Q$, where Q is an induced quasiorder.*

Proof. (1) Evidently, $(a \wedge b) \vee (a^* \vee b^*) = 1$ and $(a \wedge b) \wedge (a^* \vee b^*) = 0$. Thus $a^* \vee b^*$ is a complement of $a \wedge b$. By Lemma 4, $(a \wedge b)^*$ is a complement of $a \wedge b$. Hence by Lemma 3, these elements are neighbours. It $(***)$, (1) is evident. The second statement can be proved dually.

(2) If $\langle a, b\rangle \in Q$, then $a \wedge b = a \wedge a$. Hence

$$a^* \vee b^* = (a \wedge b)^* = (a \wedge a)^* = a^* \vee a^*, \text{ which gives } \langle b^*, a^*\rangle \in Q.$$

□

Let $(A; \vee, \wedge)$ be a distributive complemented q-lattice. Consider 0 and 1 as nullary operations on A. We have just proved that $*$ is a unary operation on A. Hence, we are ready to introduce

Definition. An algebra $(A; \vee, \wedge, *, 0, 1)$ is called an *algebra of quasiordered logic* if its reduct $(A; \vee, \wedge)$ is a complemented distributive q-lattice, 0 and 1 are a zero and a unit of $(A; \vee, \wedge)$, respectively, and $*$ is a unary operation defined by $(***)$.

As a consequence of Theorems 3 and 4 we have the following

Corollary. *Let $(A; \vee, \wedge, *, 0, 1)$ be an algebra of quasiordered logic. The following conditions are equivalent:*

(1) *$(A; \vee, \wedge, *, 0, 1)$ is a Boolean algebra;*

(2) *the q-lattice $(A; \vee, \wedge)$ is a lattice;*

(3) *the induced quasiorder Q is an order.*

Hence, the concept of algebra of quasiordered logic is a generalization of Boolean algebra. Since Boolean algebras form a basis of two-valued propositional calculus, we are interested in what "logic" is connected with these algebras. In the next part we will show that it gives an "almost two-valued" logic which is more natural than that based on two-element Boolean algebra.

Consider the four element algebra of quasiordered logic A_2^2 whose diagram is in Fig. 4. Its elements are 0, 1, F, T and the operations $\vee$, $\wedge$, $*$ are defined as follows:

$\vee$	0	F	1	T
0	0	0	1	1
F	0	0	1	1
1	1	1	1	1
T	1	1	1	1

$\wedge$	0	F	1	T
0	0	0	0	0
F	0	0	0	0
1	0	0	1	1
T	0	0	1	1

$0^* = 1$
$\mathrm{F}^* = 1$
$1^* = 0$
$\mathrm{T}^* = 0$

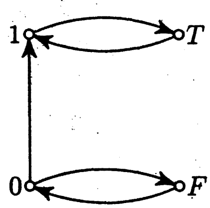

Fig. 4

It is easy to check that $(A_2^2; \vee, \wedge)$ is a complemented distributive q-lattice (which is not a lattice), and 0 and 1 are the zero and the unit, respectively.

A proposition P is called *atomic* if it does not contain any logical connective (conjunction, disjunction, negation). If P, Q are propositions, the propositions (P *and* Q), (P *or* Q), (*non* P), will be called the *composed propositions.*

In the two-valued logic, every proposition has just one of the logical values TRUE or FALSE.

However, we can make a difference between propositions whose logical values should be verified empirically and those whose logical values can be evaluated by the rules of propositional calculus. For example, the atomic proposition "this apple is red" cannot be analyzed by logical rules but by some empirical investment. In such a case, let us denote its logic value by T if the proposition is true and by F in the opposite case. It looks rather natural that the logical value of such an atomic proposition is different from that of the composed proposition "$2+3=5$ *implies that this apple is red*". Therefore, we assign the value 1 or 0 to the composed proposition if it is true or false, respectively. Denote by $v(P)$ the logical value of the proposition P. In accordance with two-valued logic, we use the same rules for evaluation of logic values of composed propositions, i.e.

$$v(P \text{ and } Q) = v(P) \wedge v(Q), \quad v(P \text{ or } Q) = v(P) \vee v(Q),$$
$$v(\text{non } P) = v(P^*),$$

where $\vee$, $\wedge$, $*$ are operations of the algebra A_2^2. It is clear that all common logical principles are preserved in this "logic" but we can make differences between the given true or false of atomic propositions and the syntactical true or false of logical constructions.

Connections between the algebra of quasiordered logic and the Boolean algebra are expressed in the following theorems:

Theorem 5. *Let $(A; \vee, \wedge, *, 0, 1)$ be an algebra of quasiordered logic and let Q be the induced quasiorder. Then the factor algebra*

$$(A/E_Q; \vee, \wedge, *, [0]_{E_Q}, [1]_{E_Q})$$

is a Boolean algebra which is a modification of A in the variety of all Boolean algebras.

Theorem 6. *Let Q be a quasiorder on a set $A \neq \emptyset$ such that $(A/E_Q, \leqslant_Q)$ is an at least two element distributive complemented lattice with zero 0 and unit 1. Let κ be a choice function $\kappa\colon \operatorname{Exp} A \to A$ such that*

$$\kappa([a]_{E_Q}) \in [a]_{E_Q} \quad \textit{for each } a \in A.$$

Introduce operations $\vee$, $\wedge$ by $(**)$ and put $x^* = a^c$ for each $x \in [a]_{E_Q}$, where a^c is a complement of a in the lattice $(A/E_Q, \leqslant_Q)$. Then $(A; \vee, \wedge, *, 0, 1)$ is an algebra of quasiordered logic.

The proofs follow from the foregoing results.

References

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