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SOLUTION OF THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

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Summary. Given a family of curves constituting the general solution of a system of ordinary differential equations, the natural question occurs whether the family is identical with the totality of all extremals of an appropriate variational problem. Assuming the regularity of the latter problem, effective approaches are available but they fail in the non-regular case. However, a rather unusual variant of the calculus of variations based on infinitely prolonged differential equations and systematic use of Poincaré-Cartan forms makes it possible to include even all constrained variational problems. The new method avoids the use of Lagrange multiplitiers. For this reason, it is of independent interest especially in regard to the 23rd Hilbert's problem.

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Half-century ago, the well-known paper [Do] appeared under the same title as above. Only regular first order variational integrals were discussed there but it seems that lengthy calculations producing a somewhat depressive final results have negatively affected the following progress in this area. We believe that our geometrical method based on *infinitely prolonged Monge systems* (*diffieties*) together with a far going generalization of *Poincaré-Cartan* ($\mathcal{PC}$) *forms* will be more successful.

Let us briefly recall the main principles paraphrasing a little [Do] and thus dealing with the *variational integral*

$$(1)\qquad \int f(x,y^1,\dots,y^m,z^1,\dots,z^m)\,\mathrm{d}x \to \text{extremum},\quad z^i \equiv \mathrm{d}y^i\,/\,\mathrm{d}x$$

in the *regular case* $\det(f_{ij}) \neq 0$ ($f_{ij} \equiv \partial^2 f/\partial z^i\partial z^j$). If $e^i \equiv \partial f/\partial y^i - d(\partial f/\partial z^i)/\,\mathrm{d}x$ are the familiar *Euler-Lagrange* ($\mathcal{EL}$) *operators*, the $\mathcal{EL}$ system $\mathrm{e}^i \equiv 0$ can be uniquely brought into the shape

$$\text{(2)} \qquad \mathrm{d}z^i\,/\,\mathrm{d}x \equiv g^i(x, y^1, \ldots, y^m, z^1, \ldots, z^m), \quad \mathrm{d}y^i/\mathrm{d}x \equiv z^i$$

with derivatives separated on the left. Then the *inverse problem* concerns the reverse determination of (1) if (2) is known. It should not be confused with the *weak inverse problem* when the $\mathcal{EL}$ operators are given. The latter problem is much easier since the $\mathcal{EL}$ operators can be characterized by *Helmholz identities* and if these are satisfied then $f = \int \sum y^i e^i \,\mathrm{d}t$ by a quadrature (at least symbolically), cf. [An, Sa, Ol]. In the regular case, the two inverse problems are closely related: certain linear combinations (with unknown coefficients equal to f_{ij}) of the first group of equations (2) provide the $\mathcal{EL}$ operators (by using the Helmholz identities as a criterion). This is the $\mathcal{VIF}$ *(variations integrating factors)* method, cf. [An]. The direct approach to the inverse problem is also possible. Conditions

$$\partial f/\partial y^i - (\partial/\partial x + \sum z^j \partial/\partial y^j + \sum g^j \partial/\partial z^j)\partial f/\partial z^i \equiv 0$$

for the sought function f easily result from the expanded transcription of the $\mathcal{EL}$ operators, however, the latter system proves to be of poor quality and obscures the nature proper of the task, see the rather ingenious and artificial adaptations invented in this connection in [Do].

The non-regular case lies without the scope of all available methods. But we should like to deal even with the inverse problem for all *constrained variational integrals*

$$\text{(3)} \qquad \int p^*\lambda \to \text{ extremum}, \quad p^*\omega \equiv 0 \quad (\omega \in \Omega)$$

where $p = p(t)$ are curves (mappings of an interval $a \leqslant t \leqslant b$ into the underlying space), λ is a given 1-form (the *Lagrange density*) and Ω is a given set of 1-forms realizing the differential constraints. (Clearly (1) arises as a very particular case of (3) with $\lambda = f\,\mathrm{d}x$ and Ω consisting of all *contact forms* $\vartheta^i \equiv \mathrm{d}y^i - z^i\,\mathrm{d}x$.) This is possible within the framework of a little strange calculus of variations based on systematic use of generalized Poincaré-Cartan forms. The final result is as follows. In one direction, given a variational integral (3), we are able to derive the $\mathcal{EL}$ system not involving the Lagrange multiplitiers for the relevant extremals. In the reverse direction, given a system of differential equations, we are able to write down a complete collection of requirements for the sought Lagrange density λ (or better, for the relevant $\mathcal{PC}$ form of a kind prescribed in advance: of given rank, variables, order, and so on). The crucial part of these requirements consists (as a rule) of an overdetermined system of differential equations and then the compatibility is regarded as quite an other task here so that only few comments to this point will be occasionally adjoined. For the convenience of the reader, the main body of the article is devoted to preparatory examples which are of independent interest. They are chosen successively more and more complicated and naturally enter the exposition of the general theory at the very end.

In more detail, we begin with a geometrical transcription of [Do] to derive a slightly simpler resolving equations than in [Do] by means of quite other and less technical arguments. We continue with a still simpler approach to [Do] by employing the family of first integrals. In particular, it will be seen that the inverse problem can be expressed in geometrical terms: to determine a symplectical structure if a one-parameter family of its Lagrange subspaces is given in advance. The next part is devoted to non-regular integrals (1) with $m = 2$. Besides a rather complete discussion of the relevant inverse problem (which cannot be resolved by the $\mathcal{VIF}$-method since, as we shall see, the given $\mathcal{EL}$ system is not *algebraically* generated by the $\mathcal{EL}$ operators but appears only after a prolongation), we also deal with the peculiar subcase when the $\mathcal{EL}$ system is constituted by a single equation for two unknown functions. Then, passing to higher order variational integrals, the underlying spaces become not quite clear (the $\mathcal{EL}$ system and $\mathcal{PC}$ forms depend on some higher derivatives which cannot be specified beforehand) and so we take a vigorous measure, the infinite prolongations, and briefly discuss a few typical examples. In this manner, the common concepts are adapted to the concluding part which deals with the constrained variational integrals (3). In order to make our expression self-contained, a somewhat unusual concept of a *standard critical point* of a functional [Ch] is recalled which immediately gives the same extremals as [Gr] without any effort. But using an axiomatic approach to Monge systems, we are also able to eliminate the Lagrange multipliers (the auxiliary variables λ_α in [Gr]). This enables us to deal with the inverse problem in full generality. At last, the concluding part is concerned with various topics, in particular we mention the geodesics field theory and the 23rd Hilbert's problem.

Our reasonings are carried out in the real C^∞-smooth category. The definition domains are not specified and, following the common convenience, we tacitly deal with *generic situations* unless otherwise stated. (So the ranks are locally constant, certain functions do not change sign, certain modules over the ring of C^∞-smooth functions possess free bases which turn into bases of $\mathbf{R}$-linear spaces after taking their value at a point, and so on.) We shall not use any advanced tools omitting all needless formalisms of the jet theory. But it may happen that some concepts will look somewhat strange (especially in the concluding parts) and ought to be followed with a certain care. At this place we should like to recall the ***Lie derivative*** $\mathcal{L}_Z = Z\rfloor \mathrm{d} + \mathrm{d}Z\rfloor$ along the vector field Z, and the *modified* Lie derivative $\mathcal{M}_Z = \partial/\partial x + \mathcal{L}_Z$ acting on differential forms which depend on a parameter x (cf. Section 3 below). Let Φ be the module of all differential 1-forms. A submodule $\Xi \subset \Phi$ is called *flat* if $\mathrm{d}\Xi \cong 0 \pmod{\Xi}$. Alternatively, if $\Xi^\perp$ denotes the module of all vector fields Z satisfying $\Xi(Z) = 0$, the last congruence is equivalent to $\mathcal{L}_Z\Xi \subset \Xi$ ($Z \in \Xi^\perp$), or to $[\Xi^\perp, \Xi^\perp] \subset \Xi^\perp$. Then, if $\Xi = \{\xi^1, \ldots, \xi^m\}$ is generated by a finite family of forms $\xi^1, \ldots, \xi^m$, the Frobenius theorem may be applied (recall the tacitly assumed genericity) and we conclude that Ξ has a basis consisting of total differentials: $\Xi = \{\mathrm{d}h^1, \ldots, \mathrm{d}h^n\}$, $\mathrm{d}h^1 \wedge \ldots \wedge \mathrm{d}h^n \neq 0$. The functions $h^1, \ldots, h^n$ (and alternatively, any composed function $h = h(h^1, \ldots, h^n)$) are the *first integrals* (of Ξ). In particular, for any 1-form ξ, the module $\mathrm{Adj}\,\mathrm{d}\xi$ consisting of all forms of the kind $Z \rfloor \mathrm{d}\xi$ (Z is ranging through all vector fields) is flat and the form $\mathrm{d}\xi$ can be expressed in terms of the relevant first integrals, see [Ca, Br]. Analogously, if $\Psi \subset \Phi$ is a submodule then the module $\mathrm{Adj}\,\Psi$ generated by all forms from Ψ together with all forms of the kind $Z \rfloor \mathrm{d}\psi$ ($\psi \in \Psi$, $Z \in \Psi^\perp$) is also flat [Ca, Br]. This will be needed in Section 23 below together with certain identities between Lie derivatives and Lie brackets. (Incidentally, there is a basis of Ψ expressible in terms of the first integrals of $\mathrm{Adj}\,\Psi$.) Of course, it is not much necessity to recall such well-known results here, and we do so only to specify a little the notation and terminology which seem to be rather diverse in current literature.

THE FIRST ORDER REGULAR PROBLEM

1. Fundamental concepts. Our reasonings will be carried out in the space of variables $x, y^1, \ldots, y^m, z^1, \ldots, z^m$ ($m \geqslant 1$) endowed with the contact forms $\vartheta^i \equiv \mathrm{d}y^i - z^i\,\mathrm{d}x$ and the Lagrange density $\lambda = f\,\mathrm{d}x$ ($f = f(x, y^1, \ldots, y^m, z^1, \ldots, z^m)$). We shall deal with the variational integral (1). One can verify that there is a unique form ξ satisfying

(4) $$\xi \cong \lambda, \mathrm{d}\xi \cong 0 \quad (\mathrm{mod}\ \vartheta^1, \ldots, \vartheta^m),$$

namely the famous $\mathcal{PC}$ form $\xi = \lambda + \sum f_i\vartheta^i$ ($f_i \equiv \partial f/\partial z^i$). Clearly

(5) $$\mathrm{d}\xi = \sum a^i\vartheta^i \wedge \mathrm{d}x + \sum a^{ij}\vartheta^i \wedge \vartheta^j + \sum f_{ij}\,\mathrm{d}z^i \wedge \vartheta^j$$

where $a^i \equiv \partial f/\partial y^i - \partial f_i/\partial x - \sum z^i \partial f_i/\partial y^j$, $a^{ij} \equiv \frac{1}{2}(\partial f_j/\partial y^i - \partial f_i/\partial y^j)$. The module $\mathrm{Adj}\,\mathrm{d}\xi$ is generated by the forms

(6) $$\sum a^i\vartheta^i,\ \sum f_{ij}\vartheta^j,\ a^i\,\mathrm{d}x + \sum a^{ij}\vartheta^j - \sum f_{ij}\,\mathrm{d}z^j \quad (i = 1, \ldots, m).$$

Let us suppose the regularity $\det(f_{ij}) \neq 0$ from now on. Then the curves which satisfy the Pfaff's system $\varphi \equiv 0$($\varphi \in \mathrm{Adj}\,\mathrm{d}\xi$) are exactly the extremals. (Indeed, they satisfy the contact conditions $\vartheta^i \equiv 0$ owing to the regularity, and the $\mathcal{EL}$ system $e^i \equiv 0$ owing to the equations $a^i\,\mathrm{d}x - \sum f_{ij}\,\mathrm{d}z^j \equiv 0$ which follow from (6).) One can also see that all vector fields lying in $\mathrm{Adj}\,\mathrm{d}\xi^\perp$ are multiples of $F = \partial/\partial x + \sum z^i\partial/\partial y^i + \sum g^i\partial/\partial z^i$

(where $g^1, \dots, g^m$ are the same functions as in (2)). This follows from the fact that the forms ϑ^i, $a^i\,\mathrm{d}x - \sum f_{ij}\,\mathrm{d}z^j$ $(i = 1, \dots, m)$ constitute a basis of $\operatorname{Adj}\mathrm{d}\xi$ in the regular case. As a result, the vector field F is tangent to the extremals.

On the other hand, $\operatorname{Adj}\mathrm{d}\xi$ is flat. Let $\operatorname{Adj}\mathrm{d}\xi = \{\mathrm{d}h^1, \dots, \mathrm{d}h^{2m}\}$ in terms of the relevant first integrals. Then $\mathrm{d}\xi$ can be expressed as

$$(7) \qquad \mathrm{d}\xi = \sum H^{ij}\,\mathrm{d}h^i \wedge \mathrm{d}h^j \quad (H^{ij} \equiv H^{ij}(h^1, \dots, h^{2m}))$$

with appropriate functions H^{ij} or, alternatively, ξ can be expressed as

$$(8) \quad x = \sum H^i\,\mathrm{d}h^i + \mathrm{d}V \ (H^i \equiv H^i(h^1, \dots, h^{2m})),\ V = V(x, y^1, \dots, y^m, z^1, \dots, z^m),$$

which follows from the Poincaré lemma. The regularity means that $\det(H^{ij}) = \det(\partial H^j/\partial h^i - \partial H^i/\partial h^j) \neq 0$.

2. The inverse problem. We suppose that the extremals are given in advance and search for the relevant density $\lambda = f\,\mathrm{d}x$ or, which is equivalent, for the relevant $\mathcal{PC}$ form ξ. In more detail, the vector field F (tangent to the extremals) is given and we wish to determine a form ξ of the special kind $\mathcal{A}$: $\xi = f\,\mathrm{d}x + \sum f_i\vartheta^i$ (with an appropriate f and $f_i \equiv \partial f/\partial z^i$) satisfying moreover $\mathcal{B}$: $\det(f_{ij}) \neq 0$ (the regularity, $f_{ij} \equiv \partial^2 f/\partial z^i \partial z^j$) and $\mathcal{C}$: $F \rfloor \mathrm{d}\xi = 0$ (equivalent to the inclusion $F \in \operatorname{Adj}\mathrm{d}\xi^\perp$, that is, ensuring that the given extremals belong to the $\mathcal{PC}$ form ξ).

Alternatively, in terms of first integrals, if functions $h^i \equiv h^i(x, y^1, \dots, y^m, z^1, \dots, z^m)$ satisfy $Fh^i \equiv 0$ $(i = 1, \dots, 2m)$ and $\mathrm{d}h^1 \wedge \dots \wedge \mathrm{d}h^{2m} \neq 0$, then $\mathcal{C}$ means that $\mathrm{d}\xi$ can be expressed as (7) with appropriate $H^{ij} \equiv H^{ij}(h^1, \dots, h^{2m}) = K^{ij}(x, y^1, \dots, y^m, z^1, \dots \dots, z^m)$. It follows that (owing to $\mathcal{C}$) the form $\mathrm{d}\xi$ can be easily determined if its restriction

$$\mathrm{d}\tilde{\xi} = \sum K^{ij}(\text{const.}, y^1, \dots, z^m)\,\mathrm{d}\tilde{h}^i \wedge \mathrm{d}\tilde{h}^j,\ \tilde{h}^k \equiv h^k(\text{const.}, y^1, \dots, z^m),$$

on a fixed hyperplane $x =$ const. is known. (Indeed, in this case we know the restrictions $H^{ij}(\tilde{h}^1, \dots, \tilde{h}^{2m}) \equiv K^{ij}(\text{const.}, y^1, \dots, z^m)$, hence the original functions $H^{ij}(h^1, \dots, h^{2m})$ by the substitution $\tilde{h}^i \to h^i$. Thus the form (7) on the total space is determined.) The requirement $\mathcal{B}$ can be interpreted by saying that $\mathrm{d}\tilde{\xi}$ provides a symplectical structure on this hyperplane. Our next aim is to "reduce" the remaining requirement $\mathcal{A}$ to the hyperplane, too.

3. The reduction. (i) If a form $\psi = f\,\mathrm{d}x + \sum u^i\vartheta^i$ satisfies $F \rfloor \mathrm{d}\psi = 0$ then $u^i \equiv f_i$ (easy direct verification). It follows that $\mathcal{A}$ can be replaced by the congruence $\mathcal{A}'$: $\xi \cong 0 \pmod{\mathrm{d}x, \vartheta^1, \dots, \vartheta^m}$.

(ii) Let ψ be a 1-form satisfying $\mathrm{d}\psi \cong 0 \pmod{\mathrm{d}x, \mathrm{d}y^1, \dots, \mathrm{d}y^m}$. Then ψ turns into a complete differential if $x, y^1, \dots, y^m$ are kept constant, hence

$$\psi = u\,\mathrm{d}x + \sum u^i\,\mathrm{d}y^i + \mathrm{d}V = f\,\mathrm{d}x + \sum u^i\vartheta^i + \mathrm{d}V \quad (f = u + \sum u^i z^i).$$

So we obtain a form $\xi = f\,\mathrm{d}x + \sum u^i\vartheta^i$ satisfying $\mathcal{A}'$ and $\mathrm{d}\xi = \mathrm{d}\psi$. It follows that $\mathcal{A}'$ can be replaced by the congruence $\mathrm{d}\xi \cong 0 \pmod{\mathrm{d}x, \mathrm{d}y^1, \dots, \mathrm{d}y^m}$ and denoting $\eta = \mathrm{d}x \wedge \mathrm{d}y^1 \wedge \dots \wedge \mathrm{d}y^m$, this congruence is expressed by the equation $\mathcal{A}''\colon \eta \wedge \mathrm{d}\xi = 0$.

(iii) For $K \geqslant 1$ large enough, there surely exists a linear dependence of the kind $\mathcal{L}_F^K\eta = \sum w^k\mathcal{L}_F^k\eta$ (sum over $k = 0, \dots, K-1$, with appropriate coefficients $w^1, \dots, w^{K-1}$. Assuming $\mathcal{C}$, we have $\mathcal{L}_F\,\mathrm{d}\xi = dF\rfloor\,\mathrm{d}\xi = 0$ and thus

$$\mathcal{L}_F^K(\eta \wedge \mathrm{d}\xi) = \mathcal{L}_F^k\eta \wedge \mathrm{d}\xi = \sum w^k\mathcal{L}_F^k\eta \wedge \mathrm{d}\xi = \sum w^k\mathcal{L}_F^k(\eta \wedge \mathrm{d}\xi).$$

This may be regarded as a linear K-th order differential equation for the form $\eta \wedge \mathrm{d}\xi$. (In more elementary terms of temporary coordinates $t, t^1, \dots, t^{2m}$ such that $F = \partial/\partial t$, we obtain a classical linear system of ordinary differential equations for the coefficients of the form $\eta \wedge \mathrm{d}\xi$ expressed in terms of these coordinates.) It follows that if the Cauchy data at a fixed hyperplane $x =$ const. are vanishing, the solution $\eta \wedge \mathrm{d}\xi$ vanishes in the total space (and $\mathcal{A}''$ is satisfied). More explicitly, if the forms $\mathcal{L}_F^k(\eta \wedge \mathrm{d}\xi) = \mathcal{L}_F^k\eta \wedge \mathrm{d}\xi$ vanish when $x =$ const. is kept fixed but $\mathrm{d}x \neq 0$ is retained for $k = 0, 1, \dots, K-1$, then $\eta \wedge \mathrm{d}\xi = 0$ in the total space.

(iv) Let $\overline{\mathrm{d}\xi}$ be the form which arises if $\mathrm{d}x = 0$ is inserted into $\mathrm{d}\xi$. (If moreover $x =$ const. is kept fixed in $\overline{\mathrm{d}\xi}$, we obtain the restriction $\tilde{\mathrm{d}\xi}$.) The form $\overline{\mathrm{d}\xi}$ will be regarded as a differential form depending on the parameter x in the *reduced space* of variables $y^1, \dots, y^m, z^1, \dots, z^m$. Let us introduce the vector field $G = \sum z^i\partial/\partial y^i + \sum g^i\partial/\partial z^i$ (depending on the parameter x) and the form $\mu = \mathrm{d}y^1 \wedge \dots \wedge \mathrm{d}y^m$ in the reduced space. Recall moreover the modified Lie derivative $\mathcal{M}_G = \partial/\partial x + \mathcal{L}_G$. (A little formally, $F = \partial/\partial x + G$, $\mathcal{L}_F = \mathcal{M}_G$, $\eta = \mathrm{d}x \wedge \mu$, $\mathcal{M}_G u = \mathcal{L}_F u = Fu$ for any function $u = u(x, y^1, \dots, y^m, z^1, \dots, z^m)$ which is regarded either as a function on the original space or as a function depending on a parameter on the reduced space.) Then

$$\mathcal{L}_F^k\eta \wedge \mathrm{d}\xi = \mathrm{d}x \wedge \mathcal{L}_F^k\mu \wedge \overline{\mathrm{d}\xi} = \mathrm{d}x \wedge \mathcal{M}_G^k\mu \wedge \overline{\mathrm{d}\xi}$$

and (inserting here $x =$ const. but $\mathrm{d}x \neq 0$) it follows that the vanishing of the initial values at $x =$ const. (discussed in (iii)) can be expressed by

$$(\mathcal{M}_G^k\mu)^\sim \wedge \tilde{\mathrm{d}\xi} \equiv 0 \quad (k = 0, \dots, K-1). \tag{9}$$

The tilde means the restriction (we put $x =$ const. and $\mathrm{d}x = 0$). Altogether, $\mathcal{C}$ and (10) imply $\mathcal{A}''$ and the reduction is achieved.

4. Summary. Let F (hence $\operatorname{Adj} d\xi$ determined by $F \in \operatorname{Adj} d\xi^{\perp}$, and also the first integrals $h^1, \ldots, h^{2m}$ of $\operatorname{Adj} d\xi$) be given and let us search for the relevant $\mathcal{PC}$ form ξ. If $d\xi$ is already known then ξ is determined up to a total differential (the Poincaré lemma) and the Lagrange density $\lambda = f\, dx$ up to a divergence (cf. (4) and (ii) above). The form $d\xi$ can be determined if the restriction $d\tilde{\xi}$ on a fixed hyperplane $x = \text{const.}$ is known (cf. Section 2). In order to determine $d\tilde{\xi}$, the system (9) with K large enough is to be resolved. We are interested only in solutions $d\tilde{\xi}$ of the maximal possible rank (i.e., in symplectic structures $d\tilde{\xi}$ on the reduced space).

5. Technical remarks. (i) The formula

$$\mathcal{M}_G^k \mu = \sum \tfrac{k!}{k_1! \ldots ! k_m!}\, d\mathcal{M}_G^{k_1} y^1 \wedge \ldots \wedge d\mathcal{M}_G^{k_m} y^m \quad (k = k_1 + \ldots + k_m) \tag{10}$$

follows from the Leibniz rule. Inserting here

$$\mathcal{M}_G y^i \equiv z^i, \quad \mathcal{M}_G^{2+k} y^i \equiv F^k g^i,$$

one can obtain $\mathcal{M}_G^k \mu$ in explicit terms.

(ii) Owing to $\mathcal{A}$, one may assume $d\tilde{\xi} = \sum dv^i \wedge dy^i$ where $v_i \equiv \partial v / \partial z^i$, $v = v(y^1, \ldots, y^m, z^1, \ldots, z^m) = \tilde{f}$ (the restriction of f). Inserting such a $d\tilde{\xi}$ with $dv_i \equiv \sum v_i^j\, dy^j + \sum v_{ij}\, dz^j$ where $v_i^j \equiv \partial^2 v / \partial z^i \partial y^j$, $v_{ij} \equiv \partial^2 v / \partial z^i \partial z^j$ into (9), a system of second order differential equations for the function $v = \tilde{f}$ appears and we search for solutions with $\det(v_{ij}) \neq 0$.

(iii) Alternatively, owing to $\mathcal{A}'$, we may put $d\tilde{\xi} = \sum dv_i \wedge dy^i$ with unknown functions $v_i \equiv v_i(y^1, \ldots, y^m, z^1, \ldots, z^m)$. In this case, we obtain a system of first order differential equations for the unknowns $v_1, \ldots, v_n$.

(iv) Also the substitution $d\tilde{\xi} = \sum \xi_i \wedge dy^i$ with unknown differential forms $\xi_i \equiv \sum v_i^j\, dy^j + \sum v_{ij}\, dz^j$ is possible. The family of linear relations for the unknown coefficients v_i^j, v_{ij} arising from (9) must be completed by the closedness requirement $d^2 \tilde{\xi} = \sum dv_i^j \wedge dy^j \wedge dy^i + \sum dv_{ij} \wedge dz^j \wedge dz^i = 0$.

(v) For the constant K in (9) we may take the least positive integer such that the relations $\mathcal{M}_G^k \mu \wedge \nu \equiv 0$ $(k = 0, \ldots, K-1)$ imply $\mathcal{M}_G^K \mu \wedge \nu = 0$ for any 2-form ν on the reduced space. Such a K depends on the given vector field G (the form $\mu = dy^1 \wedge \ldots \wedge dy^m$ is fixed). One can then see that $\mathcal{M}_G^{\ell} \mu \wedge \nu \equiv 0$ for any $\ell \geqslant K$.

6. The Darboux case [Da]. We should like to briefly demonstrate the procedure outlined in Sections 4, 5 on few examples. Let us first look at the case $m = 1$. Then $\mu = dy^1$, $d\tilde{\xi} = dv^1 \wedge dy^1 = v_1^1\, dz^1 \wedge dy^1$ (cf. (ii) or (iii) of Section 5) so that (9) is identically satisfied, an arbitrary nonvanishing form (requirement $\mathcal{B}$) may be taken for $d\tilde{\xi}$. It follows that $d\xi = H^{12}\, dh^1 \wedge dh^2$ where h^1, h^2 are fixed first integrals with $dh^1 \wedge dh^2 \neq 0$ and $H^{12} = H^{12}(h^1, h^2) \neq 0$ may be an arbitrary nonvanishing function. Alternatively we may also put $\mathrm{d}\xi = \mathrm{d}h^1 \wedge \mathrm{d}h^2$ with h^2 fixed and h^1 an arbitrary first integral with $\mathrm{d}h^1 \wedge \mathrm{d}h^2 \neq 0$. So we have $\xi = h^1\,\mathrm{d}h^2 + \mathrm{d}V$, hence

$$\xi \cong \left(h^1\left(\frac{\partial h^2}{\partial x} + z^1\frac{\partial h^2}{\partial y^1}\right) + \frac{\partial V}{\partial x} + z^1\frac{\partial V}{\partial y^1}\right)\mathrm{d}x + \left(h^1\frac{\partial h^2}{\partial z^1} + \frac{\partial V}{\partial z^1}\right)\mathrm{d}z^1 \quad (\mathrm{mod}\ \vartheta^1).$$

Since $\mathcal{B}, \mathcal{C}$ (and even $\mathcal{A}''$) are already satisfied, the remaining requirement $\mathcal{A}$ $h^1\partial h^2/\partial z^1 + \partial V/\partial z^1 = 0$ permits to determine the function $V = -\int h^1\partial h^2/\partial z^1 \cdot \mathrm{d}z^1$ and thus the general solution

$$f = h^1\left(\frac{\partial h^2}{\partial x} + z^1\frac{\partial h^2}{\partial y^1}\right) + \frac{\partial V}{\partial x} + z^1\frac{\partial V}{\partial y}.$$

An alternative formula involving fixed first integrals h^1, h^2 and an arbitrary nonvanishing composed function $H^{12} = H^{12}(h^1, h^2)$ can be derived analogously.

7. The Douglas case. Passing to the much more difficult case $m = 2$, we shall be able to slightly simplify the resolving equations [Do] owing to a very adaptable choice of the sought $\mathrm{d}\tilde{\xi}$ (cf. points (ii)–(iv) in Section 5) and to complete the elimination of the parameter x. Then the compatibility problem slightly simplifies, too, but the excellent exposition of the Riquier compatibility test [Do 109–115, 116–125] cannot be improved and so we omit this topic here.

Assuming $m = 2$, we have $\mu = \mathrm{d}y^1 \wedge \mathrm{d}y^2$, $\mathrm{d}\tilde{\xi} = w\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 + \sum v_{ij}\,\mathrm{d}z^i \wedge \mathrm{d}y^j$ ($w = v_2^1 - v_1^2$). Denoting

$$\mathcal{M}_G^k \mu = a^k\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 + \sum b_i^{kj}\,\mathrm{d}z^i \wedge \mathrm{d}y^j + \sum c^k\,\mathrm{d}z^1 \wedge \mathrm{d}z^2,$$

the requirements (9) are expressed by

$$(11^k) \qquad c^k w - b_2^{k2}v_{11} + b_2^{k1}v_{12} + b_1^{k2}v_{21} - b_1^{k1}v_{22} = 0.$$

According to (v) Section 5, one can see that $K \leqslant 6$ so that only the relations (11^0)–(11^5) are important. Moreover, a^k are not needed and the transposition $y^1 \longleftrightarrow y^2$, $z^1 \longleftrightarrow z^2$ turns μ into $-\mu$ and thus b_1^{k1}, b_1^{k2} into $-b_2^{k2}$, $-b_2^{k1}$. So the shortened table (following from (10) by lengthy but easy calculation) is quite sufficient:

$$\begin{aligned}
&c^0 = b_i^{0j} = c^1 = b_1^{11} = 0,\ b_1^{12} = 1,\ c^2 = 2,\ b_1^{21} = -g_{01}^2,\ b_1^{22} = g_{01}^1,\\
&c^3 = 3(g_{01}^1 + g_{02}^2),\ b_1^{31} = -g_{11}^2 + 3g_0^{21},\ b_1^{32} = g_{11}^1 + 3g_0^{22},\\
&c^4 = 4(g_{11}^1 + g_{22}^2) + 6g_{01}^{[i}g_{02}^{j]},\ b_1^{41} = -g_{21}^2 + 4g_1^{21} - 6g_0^{[i1}g_{01}^{j]},\\
&b_1^{42} = g_{21}^1 + 4g_1^{22} - 6g_0^{[i2}g_{10}^{j]},\ c^5 = 5(g_{21}^1 + g_{22}^2) + 10(g_{11}^{[i}g_{02}^{j]} - g_{12}^{[i}g_{01}^{j]}),\\
&b_1^{51} = -g_{31}^2 + 5g_2^{21} + 10(g_{11}^{[i}g_0^{j]1} - g_{01}^{[i}g_0^{j]1}),\\
&b_1^{52} = g_{31}^1 + 5g_2^{22} + 10(g_{11}^{[i}g_2^{j]2} - g_{01}^{[i}g_0^{j]2}).
\end{aligned}$$

Here we abbreviate

$$g_{k\,j_1\dots j_q}^{ii_1\dots i_p} = \partial^{p+q}F^k g^i/\partial y^{i_1}\dots\partial y^{i_p}\partial z^{j_1}\dots\partial z^{j_q}$$

and use the square bracket to denote the alternation, e.g., $g_a^{[i}g_c^{j]b} = g_a^1 g_c^{2b} - g_a^2 g_c^{1b}$.

As the relations (11) are concerned, (11^0) is identity and (11^1) means that $v_{12} = v_{21}$. At this place, it is suitable to recall Douglas' notation $L = v_{11}$, $M = v_{12} = v_{21}$, $N = v_{22}$. Then (11^2) reads

$$w = \frac{1}{2}(g_{02}^1 L - g_{01}^1 M + g_{02}^2 M - g_{01}^2 N) \tag{12}$$

and permits to get rid of w from (11^{3-5}) to obtain certain requirements

$$AL + BM + CN = A_1 L + B_1 M + C_1 N = A_2 L + B_2 M + C_2 N = 0 \tag{13}$$

for the unknowns L, M, N. We state only the values

$$A = \frac{3}{2}(g_{01}^1 + g_{02}^2)g_{02}^1 - g_{12}^1 + 3g_0^{12},\ C = -\frac{3}{2}(g_{01}^1 + g_{02}^2)g_{01}^2 + g_{11}^2 - 3g_0^{11},$$
$$B = -\frac{3}{2}(g_{01}^1 + g_{02}^2)(g_{01}^1 - g_{02}^2) + g_{11}^1 - g_{12}^2 - 3(g_0^{11} - g_0^{22})$$

which will appear most frequently (and note aside that $A, \dots, C_2$ differ from [Do (7.6)–(7.9)] by a mere constant factor). The following development will depend on the rank of the matrix Δ of the linear system (13).

8. Continuation. If rank $\Delta = 3$ then (13) admits only the trivial solution $L = M = N = 0$, the regularity $\det(v_{ij}) = LN - M^2 \neq 0$ is not satisfied and the inverse problem is not solvable.

If rank $\Delta = 0$ (thus $A = \dots = C_2 = 0$) then we have the sole condition (12) which turns into the equation

$$\frac{\partial^2 v}{\partial y^1 \partial z^2} - \frac{\partial^2 v}{\partial y^2 \partial z^1} = \frac{1}{2}\left(\frac{\partial g^1}{\partial z^2}\frac{\partial^2 v}{(\partial z^1)^2} + \left(\frac{\partial g^2}{\partial z^2} - \frac{\partial g^1}{\partial z^1}\right)\frac{\partial^2 v}{\partial z^1 \partial z^2} - \frac{\partial g^2}{\partial z^1}\frac{\partial^2 v}{(\partial z^2)^2}\right)$$

for the unknown function $v = \tilde{f}$ if one applies (ii) Section 5. (As a particular case including the inverse problem with extremal straight lines in the three-dimensional space, the equation $\partial^2 v/\partial y^1 \partial z^2 - \partial^2 v/\partial y^2 \partial z^1 = 0$ satisfying the Asgeirsson mean value formula appears!) No further comments are needed.

If rank $\Delta = 2$ then the last equation (13) may be omitted and we obtain

$$L = (BC_1 - CB_1)u,\ M = (CA_1 - AC_1)u,\ N = (AB_1 - BA_1)u \tag{14}$$

where $u = u(y^1, y^2, z^1, z^2)$ is a new unknown function. We search for a nonvanishing solution u in the domain where the regularity

$$(BC_1 - CB_1)(AB_1 - BA_1) - (AC_1 - CA_1)^2 \neq 0$$

is satisfied. Applying (iv) Section 5 and substituting (12), (14) into the relevant closedness requirement $\mathrm{d}^2 \tilde{\xi} = 0$ (cf. Section 5), this inequality guarantees that the resulting system for the unknown function u can be represented as

$$\partial u/\partial y^1 = uP,\ \partial u/\partial y^2 = uQ,\ \partial u/\partial z^1 = uR,\ \partial u/\partial z^2 = uS, \tag{15}$$

where $P, \ldots, S$ are certain known functions. It follows that the solution exists if and only if $P\,\mathrm{d}y^1 + Q\,\mathrm{d}y^2 + R\,\mathrm{d}z^1 + S\,\mathrm{d}z^2$ is a total differential. We shall not give more comments but it is to be noted that all "generic" variational problems are involved in the case rank $\Delta = 2$ as follows by a simple perturbation argument applied to the function f. One can also see that this function is determined up to the substitutions $f \to cf + \partial g/\partial x + \sum z^i \partial g/\partial y^i$ ($c \neq 0$ a constant $g = g(x, y^1, y^2)$).

The remaining case rank $\Delta = 1$ is the most difficult one. We shall outline a shorter method than [Do] to obtain the resolving equations. In this case, (13) reduces to the single equation $AL + BM + CN = 0$. We shall assume $A \neq 0$ for brevity and take care of the equation $At^2 + Bt + C = 0$ with certain roots t, $\bar{t}$. The subcases $t \neq \bar{t}$, $t = \bar{t}$ will be discussed separately.

9. Inequal roots. We introduce the coframe $\alpha = t\,\mathrm{d}y^1 + \mathrm{d}y^2$, $\bar{\alpha} = \bar{t}\,\mathrm{d}y^1 + \mathrm{d}y^2$, $\beta = t\,\mathrm{d}z^1 + \mathrm{d}z^2$, $\bar{\beta} = \bar{t}\,\mathrm{d}z^1 + \mathrm{d}z^2$ and the "nearly dual" frame

$$Y = \frac{\partial}{\partial y^1} - \bar{t}\frac{\partial}{\partial y^2},\ \overline{Y} = \frac{\partial}{\partial y^1} - t\frac{\partial}{\partial y^2},\ Z = \frac{\partial}{\partial z^1} - \bar{t}\frac{\partial}{\partial z^2},\ \overline{Z} = \frac{\partial}{\partial z^1} - t\frac{\partial}{\partial z^2}$$

in the sense that $(t - \bar{t})\,\mathrm{d}g = Yg \cdot \alpha + \overline{Y}g \cdot \bar{\alpha} + Zg \cdot \beta + \overline{Z}g \cdot \bar{\beta}$ is satisfied for any function g. One can then verify the crucial formula

$$\mathrm{d}\tilde{\xi} = R\beta \wedge \alpha + \overline{R}\bar{\beta} \wedge \bar{\alpha} + S\alpha \wedge \bar{\alpha}$$

where $R = (M - \bar{t}N)/(t - \bar{t})$, $\overline{R} = (M - tN)/(t - \bar{t})$, $S = w/(t - \bar{t})$. Using (12) we obtain $S = aR + \bar{a}\overline{R}$ where

$$\begin{aligned} a = \frac{1}{2(t - \bar{t})A} \{ & g^1_{12} g^0_{01} - g^2_{11} g^1_{02} - 3(g_0^{12} g^2_{01} + g_0^{11} + g^2_{02}) \\ & + t\big(g^1_{0[i} g^1_{1j]} - g^{[i}_{12} g^{j]}_{02} + 3\big(g_0^{1[i} g^1_{0j]} + g_0^{[i2} g^{j]}_{02}\big)\big)\} \end{aligned} \tag{16}$$

and $\bar{a}$ arises by the transposition $t \longleftrightarrow \bar{t}$. Applying (iv) Section 5, the closedness requirement

$$\mathrm{d}^2\,\tilde{\xi} = \mathrm{d}R \wedge \beta \wedge \alpha + \mathrm{d}\bar{R} \wedge \bar{\beta} \wedge \bar{\alpha} + (a\,\mathrm{d}R + \bar{a}\,\mathrm{d}\bar{R}) \wedge \alpha \wedge \bar{\alpha} + R\gamma + \bar{R}\bar{\gamma} = 0$$

appears where γ, $\bar{\gamma}$ are certain 3-forms not depending on R, $\bar{R}$. Using the above frame Y, $\bar{Y}$, Z, $\bar{Z}$, this requirement can be represented by an equivalent and rather special system of the kind

$$\bar{Y}R = (\ldots),\ (\bar{X} + aY)R = (\ldots),\ Y\bar{R} = (\ldots),\ (X + \bar{a}\bar{Y}) = (\ldots)$$

where $(\ldots)$ are certain expressions linear in R, $\bar{R}$. The self-evident identity

$$M^2 - LN = M^2 + (BM/A + CN/A)N = (M - tN)(M - \bar{t}N) = (t - \bar{t})R\bar{R}$$

means that we are interested in nonvanishing solutions $R \neq 0$, $\bar{R} \neq 0$. Then $M = tR + \bar{t}\bar{R}$, $N = R + \bar{R}$, $L = -BM/A - CN/A$ and w given by (12) determine the sought form $\mathrm{d}\tilde{\xi}$. (If the roots are complex conjugate, the complexification of the tangent space appears as temporary tool.)

10. Equal roots. If $t = \bar{t} = -B/2A$, then

$$\mathrm{d}\tilde{\xi} = N\beta \wedge \alpha + P(\mathrm{d}z^1 \wedge \alpha + \beta \wedge \mathrm{d}y^1) + Q\alpha \wedge \mathrm{d}y^1$$

with the same α, β as above and $P = M - tN$, $Q = -w = aN + bP$ where

$$\begin{aligned} a &= \{\text{the same as in}(16)\}/2A, \\ b &= (g^1_{0[i}g^1_{1j]} + g^{[i}_{12}g^{j]}_{02} + g_0^{1[i}g^1_{0j]} + g_0^{[i2}g^{j]}_{02})/2A. \end{aligned}$$

Applying (iv) Section 5, the closedness requirement $\mathrm{d}^2\,\tilde{\xi} = 0$ yields a system of the kind

$$\bar{Z}N - \frac{\partial P}{\partial y^2} = (\ldots),\ \bar{Z}P = (\ldots),\ \Big(\bar{Y} + a\frac{\partial}{\partial z^2} + b\bar{Z}\Big)N = (\ldots),\ \Big(\bar{Y} + a\frac{\partial}{\partial z^2}\Big)P = (\ldots)$$

where $(\ldots)$ are certain expressions linear in N, P. The identity $M^2 - LN = (M - tN)^2 = P^2$ means that we search for a solution N, P with $P \neq 0$. At this place, we conclude the geometric exposition of the inverse problem closely related to the point of view of [Do]. The conception of the subsequent chapters will successively become more and more different.

11. Digression. Before passing to the theme proper, we shall once more derive the crucial resolving requirement (9) by using first integrals for new coordinates. To this aim, let $y^i \equiv \bar{y}^i(x, h^1, \dots, h^{2m})$, $z^i \equiv \bar{z}^i(x, y^1, \dots, h^{2m})$ $(= \partial\bar{y}^i/\partial x)$ be the equations of extremals (the integration constants $h^1, \dots, h^{2m}$ are kept fixed for a moment). Then $x, h^1, \dots, h^{2m}$ can be used for alternative coordinates in the total space and in particular

$$\eta = \mathrm{d}x \wedge \mathrm{d}\bar{y}^1 \wedge \dots \wedge \mathrm{d}\bar{y}^m = \mathrm{d}x \wedge \sum h^{i_1 \dots i_m}\, \mathrm{d}h^{i_1} \wedge \dots \wedge \mathrm{d}h^{i_m} = \mathrm{d}x \wedge \mu,$$

where the coefficients are known functions of $x, h^1, \dots, h^{2m}$. Assuming (7) with unknown functions H^{ij} (which ensures $\mathcal{C}$), the requirement $\mu \wedge \mathrm{d}\xi = 0$ (ensuring $\mathcal{A}''$) can be expressed by a family of certain linear relations $\sum c_\ell^{ij} H^{ij} = 0$ $(\ell = 1, \dots, L)$ between the functions H^{ij}. Here the coefficients c_ℓ^{ij} are well-known, they can be expressed as linear combinations of the above functions $h^{i_1 \dots i_m}$. But unlike $H^{ij} \equiv H^{ij}(h^1, \dots, h^{2m})$, they may depend on the coordinate x. It follows that necessarily also

$$\sum \partial^k c_\ell^{ij}/\partial x^k \cdot H^{ij} \equiv 0 \quad (\ell = 1, \dots, L;\ k = 0, 1, \dots)$$

must be satisfied. But in reality, only a finite number $k = 0, \dots, K-1$ of these relations is enough (where $K \leqslant$ number of all $H^{ij} = m(m-1)/2$), even at a fixed value $x = \text{const}$. This is a mere reformulation of the result (iii) Section 5 in terms of new coordinates, of course. The present derivative $\partial/\partial x$ exactly corresponds to the previous $\mathcal{L}_Z$.

It is to be noted that the equation $\mu \wedge \mathrm{d}\xi = 0$ means that $\mathrm{d}\xi \cong 0$ (mod $\mathrm{d}\bar{y}^1, \dots, \mathrm{d}\bar{y}^m$), i.e., the system $\mathrm{d}\bar{y}^1 = \dots = \mathrm{d}\bar{y}^m = 0$ determines a one-parameter family of Lagrangian subspaces (given in advance) of the sought symplectical structure $\mathrm{d}\tilde{\xi}$ ($= \mathrm{d}\xi$ in the new coordinates). This provides a very nice geometrical interpretation of the inverse problem.

12. An alternative approach to the inverse problem employing the first integrals $h^1, \dots, h^{2m}$ in a different and more direct manner can be explained as follows. The requirement $\mathcal{C}$ is satisfied if we assume the formula (7) with certain (as yet unknown) functions H^{ij} (cf. Section 2). Then $\mathcal{B}$ is equivalent to the condition $\det(H^{ij}) \neq 0$. The remaining requirement $\mathcal{A}''$, i.e., the congruence $\mathrm{d}\xi \cong 0$ (mod $\mathrm{d}x, \mathrm{d}y^1, \dots, \mathrm{d}y^m$) is expressed by

$$\sum c_{pq}^{ij} H^{ij} \equiv 0 \quad \left(c_{pq}^{ij} \equiv \frac{\partial h^1}{\partial z^p}\frac{\partial h^j}{\partial z^q} - \frac{\partial h^i}{\partial z^q}\frac{\partial h^j}{\partial z^p} \right)$$

where the sum is over $i, j = 1, \dots, 2m$, for every $p, q = 1, \dots, m$. (One can also verify that for given functions $H^i \equiv H^i(h^1, \dots, h^{2m})$, the same conditions with

$H^{ij} \equiv \partial H^j/\partial h^i - \partial H^i/\partial h^j$ ensure the existence of a function V such that the form (8) satisfies the first congruence (4).) Since the coefficients c^{ij}_{pq} may depend on x, also the requirements

$$(17) \qquad \sum \partial^k c^{ij}_{pq}/\partial x^k \cdot H^{ij} \equiv 0 \quad (p, q = 1, \ldots, m; k = 0, 1, \ldots)$$

make a good sense and should be satisfied. But in reality, only a finite number of the conditions (17) at a fixed value $x =$ const. is enough. (The reason is quite analogous as in (iii) Section 5 and need not be repeated here.)

If $H^{ij} \equiv \partial H^j/\partial h^i - \partial H^i/\partial h^j$ is inserted into (17) with $x =$ const. kept fixed, a very interesting overdetermined system of differential equations for the unknown functions $H^k \equiv H^k(h^1, \ldots, h^{2m})$ arises. Unlike the previous chapter where a special choice of coordinates and frames was preferred (which is a typical feature of the Riquier method), the present choice of variables $h^1, \ldots, h^{2m}$ is apriori free which is suitable if the compatibility tests based on the involutiveness are applied (cf. [Br, Ca]). Alternatively (17) can be regarded as a system of mere linear algebraic relations between the unknown functions H^{ij} but in this case the closedness requirement $\mathrm{d}^2 \xi = \sum dH^{ij} \wedge \mathrm{d}h^i \wedge \mathrm{d}h^j = 0$ must be adjoined (cf. (iv) Section 5) and we obtain an exterior linear system of the kind thoroughly discussed in [Br].

13. The Darboux case. If $m = 1$ then $p = q = 1$ and (17) is identically satisfied. It follows that (for given first integrals h^1, h^2 with $\mathrm{d}h^1 \wedge \mathrm{d}h^2 \neq 0$) we may choose $\xi = H^1\,\mathrm{d}h^1 + H^2\,\mathrm{d}h^2$ quite arbitrarily provided the necessary regularity condition $H^{12} = \partial H^2/\partial h^1 - \partial H^1/\partial h^2 \neq 0$ be satisfied. Then the function V in (8) is to be determined from the equation $H^1 \partial h^1/\partial z^1 + H^2 \partial h^2/\partial z^1 + \partial V/\partial z^1 = 0$ in order to ensure the first congruence (3). As a result, the formula

$$(18) \quad f = \sum H^i \left(\frac{\partial h^i}{\partial x} + z^1 \frac{\partial H^i}{\partial y^1} \right) + \frac{\partial V}{\partial x} + z^1 \frac{\partial V}{\partial y^1} \quad \left(V = -\int \sum H^i \frac{\partial h^i}{\partial z^1}\,\mathrm{d}z^1\right)$$

provides the general solution of the inverse problem.

14. The Douglas case. Assuming $m = 2$, the unknown functions H^{ij} ($i < j$; $i, j = 1, \ldots, 4$) are subjected to the relevant requirements (17), i.e.

$$(19) \qquad \sum (\partial^k c^{ij}_{12}/\partial x^k)^{\sim} H^{ij} \equiv 0 \quad (k = 0, \ldots, K - 1)$$

($x =$ const. in the coefficients) and we need $\det(H^{ij}) \neq 0$. Since there are 6 unknowns H^{ij}, the number of linearly independent equations (17) is $K \leqslant 6$ (and it depends on the nature of the coefficients c^{ij}_{12}, i.e., on the nature of the given first integrals $h^1, \ldots, h^4$).

If $K = 6$ then $H^{ij} \equiv 0$ is the only solution of (19) and the inverse problem is unsolvable. If $K = 5$ then all solutions of (19) are proportional, i.e., $H^{ij} \equiv uK^{ij}$ where $K^{ij} \equiv K^{ij}(h^1, \ldots, h^4)$ are known functions. One can then see that the unknown factor u is subjected to a system of the kind (15) with the independent variables $h^1, \ldots, h^4$ (instead of y^1, y^2, z^1, z^2) and the resulting conclusion is analogous as in Section 8. The cases $K \leqslant 3$ lead to a mere Cauchy-Kowalewska system and will not be discussed here. The remaining case $K = 4$ is the most interesting one and exactly corresponds to Sections 9, 10.

15. Continuation. Assume $K = 4$. Then (19) consists of 4 independent equations and may be represented (if necessary, after a sufficiently general change of variables $h^1, \ldots, h^4$) in the special and equivalent shape

$$H^{4i} \equiv L^i \quad (i = 1, 2, 3), \; L = 0$$

where $L^1, \ldots, L^3$, L are linearly dependent on H^{12}, H^{13}, H^{23} and do not involve any function H^{4i}. (In fact, such a state is achieved if the condition (19) produces only one relation between the variables H^{12}, H^{13}, H^{23}. In equivalent terms, if $H^{ij} = uK^{ij} + vL^{ij}$ is a general solution of (19) with u, v arbitrary parameters and K^{ij}, L^{ij} known, then the pencil of forms $\sum(uK^{ij} + vL^{ij})\, dh^i \wedge dh^j$ when restricted to $dh^4 = 0$ should still be a pencil involving two essential parameters u, v. But such a state can be achieved by a sufficiently general choice of the variable h^4.) Inserting $H^{ij} \equiv \partial H^j/\partial h^i - \partial H^i/\partial h^j$, we obtain the resolving system

$$\partial H^i/\partial h^4 \equiv \partial H^4/\partial h^i + L^i \quad (i = 1, 2, 3), \quad L = 0, \tag{20}$$

where the absence of $\partial H^4/\partial h^4$ prevents its bringing into the Cauchy-Kowalewska shape.

Turning to the problem of compatibility of (20), we shall abbreviate $\partial H^i/\partial h^k \equiv H^i_k$, $\partial H^{ij}/\partial h^k \equiv H^{ij}_k$ and analogously for higher derivatives. Moreover let us denote $L = R^0$.

We should like to propose a compatibility test advantageous for the discussion of "nearly Cauchy-Kowalewska" systems of differential equations. The test is of inductive nature and consists of the reduction of a given system to another system of equations for admissible Cauchy data on a hyperplane. So, unlike in the involutiveness theory, it proceeds from higher- to lower-dimensional compatibility problems and the most important objects (e.g., the characteristics of codimension one) are (at least implicitly) indicated at the very beginning of calculations.

As the system (20) involving unknown functions $H^1, \ldots, H^4$ of independent variables $h^1, \ldots, h^4$ is concerned, it clearly implies $H^i_{4k} \equiv H^4_{ik} + \partial L^i/\partial h^k$. It follows

$$H^{ij}_4 = H^j_{4i} - H^i_{4j} = H^4_{ji} + \partial L^j/\partial h^i - H^4_{ij} - \partial L^i/\partial h^j = \partial L^j/\partial h^i - \partial L^i/\partial h^j$$

which are clearly functions of H^{ij}_k $(i,j,k = 1,2,3)$. As a result, the expressions H^{ij}_4 can be eliminated from the equation $(\partial L/\partial h^4 =)$ $\partial R^0/\partial h^4 = 0$ and we obtain a linear relation $R^1 = 0$ between H^{12}, H^{13}, H^{23} without derivatives $\partial/\partial h^4$, that is,

$$R^1(\ldots, H^{ij}_k, \cdots, H^{ij}, \ldots) = 0 \quad (i,j,k = 1,2,3).$$

Quite analogously, owing to $H^i_{4k\ell} \equiv H^4_{ik\ell} + \partial^2 L/\partial h^k \partial h^\ell$, the equation $\partial R^1/\partial h^4 = 0$ (equivalent to $\partial^2 L/(\partial h^4)^2 = 0$) can be replaced by an equivalent linear relation of the kind

$$R^2(\ldots, H^{ij}_{k\ell}, \ldots, H^{ij}_k, \ldots, H^{ij}, \ldots) = 0 \quad (i,j,k,\ell = 1,2,3).$$

Continuing in this way, we obtain a series of relations $R^s \equiv 0$ $(s = 0,1,\ldots)$ between derivatives of H^{12}, H^{13}, H^{23} with respect to h^1, h^2, h^3. One can observe that if a certain relation $R^S = 0$ is a linear combination of the preceding $R^0 = \ldots = R^{S-1} = 0$ possibly derived with respect to h^1, h^2, h^3, then also all the following relations $R^{S+k} = 0$ do not bring anything new (being analogous linear combinations, too; this follows by a simple analysis of the above recurrent construction of the sequence $R^0, R^1, \ldots$). Such an index $S < \infty$ does exist (as follows from the general finiteness principles of the compatibility theory).

At this place, the compatibility of the original system (20) is converted into the compatibility of the system $R^0 = \ldots = R^{S-1} = 0$ with $x^4 =$ const. kept fixed. In fact, solutions of this system provide the Cauchy data for (the Cauchy-Kowalewska system constituted by) the first group of equations (20) with unknown functions H^1, H^2, H^3 and H^4 quite arbitrarily chosen in advance. Then the remaining equation $L = 0$ of (20) is satisfied (at least in the formal sense, i.e., compatible) since all derivatives $\partial^s L/(\partial h^4)^s = 0$ vanish at the hyperplane $x^4 =$ const. (being equivalent to $R^s \equiv 0$, that is to $R^1 = \ldots = R^{S-1} = 0$ in virtue of the first group (20)).

So we have to deal with the compatibility of the system $R^0 = \ldots = R^{S-1} = 0$ with unknown functions $K^i \equiv K^i(h^1, h^2, h^3) = H^i(h^1, h^2, h^3, \text{const.})$ replacing the previous H^i in the expressions $H^{ij} = \partial H^j/\partial h^i - \partial H^i/\partial h^j$ $(i,j = 1,2,3)$ which occur in R^s. We propose still onother simplification as follows. Denoting $K^{ij} \equiv \partial K^j/\partial h^i - \partial K^i/\partial h^j$ for clarity, assume that $R^0 = AK^{12} + BK^{13} + CK^{23}$ with $A \neq 0$. Then K^{12} may be eliminated from the remaining relations $R^1 = \ldots = R^{S-1} = 0$ (if $S > 1$) which turn into a system of linear differential equations for K^{13}, K^{23} regarded as unknown function. However, then the closedness requirement

$$\mathrm{d}\sum K^{ij}\,\mathrm{d}h^i \wedge \mathrm{d}h^j = \mathrm{d}\left((K^{13}\,\mathrm{d}h^1 + K^{23}\,\mathrm{d}h^2) \wedge \left(\tfrac{C}{A}\,\mathrm{d}h^1 - \tfrac{B}{A}\,\mathrm{d}h^2 + \mathrm{d}h^3\right)\right) = 0$$

must be taken into account (compare with (iv) Section 5). It yields the condition

$$\left(\frac{\partial}{\partial h^2} + \frac{B}{A}\frac{\partial}{\partial h^3} - \frac{\partial B/A}{\partial h^3}\right) K^{13} = \left(\frac{\partial}{\partial h^1} - \frac{C}{A}\frac{\partial}{\partial h^3} + \frac{\partial C/A}{\partial h^3}\right) K^{13}$$

for the remaining unknowns K^{13}, K^{23} which is to be adjoined to the previous $R^1 = \ldots = R^{S-1} = 0$ (with K^{12} already eliminated). In principle, this is a better result than in Sections 9, 10 since we have only three independent variables h^1, h^2, h^3. Note at least that by an appropriate change of variables h^1, h^2, h^3, the factor $\frac{C}{A}\,\mathrm{d}h^1 - \frac{B}{A}\,\mathrm{d}h^2 + \mathrm{d}h^3$ could be transformed either to $h^2\,\mathrm{d}h^1 + \mathrm{d}h^3$ or to $\mathrm{d}h^3$ (the Darboux theorem). Then the above condition essentially simplifies but we do not continue these reasonings.

THE NON-REGULAR INVERSE PROBLEMS

16. Fundamental structural results. Leaving the regular variational problems, we enter a rather dangerous realm involving many striking matters. The common methods (especially the Dirac theory of constraints) seem to be of little purpose for our aim and we are compelled to follow another way. For simplicity, we will deal only with the case $m = 2$. Our methods can be applied for the general m as well but it is a toilsome task which deserves a separate book.

So retaining the previous notation, we will thoroughly deal with the variational integral

$$\text{(21)} \qquad \int f(x, y^1, y^2, z^1, z^2)\,\mathrm{d}x \to \text{ extremum} \quad (f_{11} \neq 0,\ f_{11}f_{12} = (f_{12})^2).$$

One can then see that (4) essentially simplifies:

$$\text{(22)} \qquad \mathrm{d}\xi = (a^1\vartheta^1 + a^2\vartheta^2) \wedge \mathrm{d}x + a\vartheta^1 \wedge \vartheta^2 + f_{11}\zeta \wedge \vartheta$$

where $a^i \equiv \partial f/\partial y^i - \partial f_i/\partial x - \sum z^j \partial f_i/\partial y^j$, $a = \frac{1}{2}(\partial f_2/\partial y^1 - \partial f_1/\partial y^2)$. In particular, we introduce the forms $\zeta = \mathrm{d}z^1 + c\,\mathrm{d}z^2$, $\vartheta = \vartheta^1 + c\vartheta^2$ with $c = f_{12}/f_{11}$ which will be of the highest importance. Clearly the forms

$$a^1\vartheta^1 + a^2\vartheta^2,\ a^1\,\mathrm{d}x + a\vartheta^2 - f_{11}\zeta,\ a^2\,\mathrm{d}x - a\vartheta^1 - cf_{11}\zeta, \vartheta$$

generate the module $\mathrm{Adj}\,\mathrm{d}\xi$ and it follows that all extremals satisfy Pfaff's system $\varphi \equiv 0$ ($\varphi \in \mathrm{Adj}\,\mathrm{d}\xi$). (The converse is not true but the curves satisfying the system *and* the contact conditions $\vartheta^1 = \vartheta^2 = 0$ already are the extremals.) One can then see that the forms

$$\text{(23)} \qquad e\,\mathrm{d}x,\ e\vartheta^2,\ \vartheta,\ a^1\,\mathrm{d}x + a\vartheta^2 - f_{11}\zeta \quad (e = a^2 - ca^1)$$

may be used for generators, too. It follows that $e = 0$ on every extremal. Assuming that the function e effectively depends on z^1, z^2 (which can be expressed by $\partial e/\partial z^1 \neq$

0, see below), we shall prove that there exists another function $\bar{e}$ vanishing on all extremals. In other terms, the $\mathcal{EL}$ system consists of *two* differential equations $e = \bar{e} = 0$ of the first order.

For the aim mentioned, look at the module $\operatorname{Adj} d\xi$ near the points where $e \neq 0$. It follows from (23) that near these points, $\operatorname{Adj} d\xi$ is generated by dx, dy^1, dy^2 together with a certain differential dz where $z = z(x, y^1, y^2, z^1, z^2)$ is a function satisfying the congruence

$$dz \cong \frac{\partial z}{\partial z^1} dz^1 + \frac{\partial z}{\partial z^2} dz^2 \cong \frac{\partial z}{\partial z^1}\zeta \quad (\text{mod } dx, dy^1, dy^2).$$

(In particular $\partial z/\partial z^2 = c\,\partial z/\partial z^1$ so that $\partial z/\partial z^1 = 0$ implies also $\partial z/\partial z^2 = 0$. We shall soon see that e can be expressed in terms of x, y^1, y^2, z, thus $\partial e/\partial z^1 = 0$ implies $\partial e/\partial z^2 = 0$, too.) The congruence clearly implies

$$\zeta = \Big(dz - \frac{\partial z}{\partial x} dx - \frac{\partial z}{\partial y^1} dy^1 - \frac{\partial z}{\partial y^2} dy^2 \Big) \Big/ \frac{\partial z}{\partial z^1}. \tag{24}$$

The form $d\xi$ can be expressed in terms of the variables x, y^1, y^2, z (even at the points where $e = 0$). Consequently, if we insert $\vartheta^i \equiv dy^i - z^i dx$, $\vartheta = dy^1 + c\,dy^2 - (z^1 + cz^2)\,dx$ and ζ given by (24) into the formula (22), then all coefficients of the exterior products like $dy^i \wedge dy^j, \dots, dx \wedge dz$ necessarily depend only on the variables x, y^1, y^2, z. Omitting these easy calculations, we state only the most important part of the final result: *the functions e, c, $z^1 + cz^2$ can be expressed in terms of the above variables.* (In particular, the function e might play the role of the function z, as well.) Hence

$$de \cong M\,dx + \frac{\partial e}{\partial z^1} dz^1 + \frac{\partial e}{\partial z^2} dz^2 = M\,dx + \frac{\partial e}{\partial z^1}\zeta \quad (\text{mod } \vartheta^1, \vartheta^2) \tag{25}$$

where we denote $M = \partial e/\partial x + z^1 \partial e/\partial y^1 + z^2 \partial e/\partial y^2$. (In particular we obtain the self-evident but highly important identity $\partial e/\partial z^2 = c\,\partial e/\partial z^1$.) Every extremal satisfies $e = 0$ (and thus $de = 0$) and $a^1\,dx - f_{11}\zeta = 0$ (see (23)). Consequently

$$de + \frac{1}{f_{11}} \frac{\partial e}{\partial z^1}(a^1\,dx - f_{11}\zeta) = \bar{e}\,dx = 0 \quad \Big(\bar{e} = M + \frac{1}{f_{11}} \frac{\partial e}{\partial z^1} a^1\Big)$$

on every extremal. It follows that $\bar{e} = 0$ on every extremal, and the sought function $\bar{e}$ is explicitly found.

It is to be moreover proved that the functions $e, \bar{e}$ are functionally independent. Instead, we shall see that the equations $e = \bar{e} = 0$ determine a submanifold of codimension at least 2. For this goal observe that the restriction of the module $\operatorname{Adj} d\xi$ to this submanifold is generated by the forms ϑ and $a\vartheta^2$ (cf. (23), (25) with

$e = \bar{e} = de = 0$), that is, either by the forms ϑ^1, ϑ^2 (if $a \neq 0$), or by the single form ϑ (if $a = 0$). On the other hand, the restriction (of the *flat* module Adj $\mathrm{d}\xi$) is a flat module and it clearly follows that the dimension of the submanifold cannot exceed 3 (look either at the congruences $\mathrm{d}\vartheta^1 = \mathrm{d}\vartheta^2 \cong 0 \pmod{\vartheta^1, \vartheta^2}$, or at the congruence $\mathrm{d}\vartheta \cong 0 \pmod{\vartheta}$), and this concludes the proof.

We shall not analyse all events which in principle may occur. For certainty, let us assume that $e = \bar{e} = 0$ is exactly a first order system, that is, it may be represented in the shape $z^1 = g^1(x, y^1, y^2)$, $z^2 = g^2(x, y^1, y^2)$ with derivatives separated on the left.

17. The inverse problem. Every first order system $\mathrm{d}y^i / \mathrm{d}x \equiv g^i(x, y^1, y^2)$ $(i = 1, 2)$ may be regarded as the $\mathcal{EL}$ system for an appropriate variational integral

$$\int (z^1 f_1 + z^2 f_2)\, \mathrm{d}x \to \text{ extremum } \quad (f_i \equiv f_i(x, y^1, y^2))$$

linear in variables z^1, z^2, as one can easily find. But we are interested in the reconstruction of quite other variational integrals (21) to obtain *all* possible solutions of the inverse problem.

Passing to calculations, let the mentioned $\mathcal{EL}$ system $z^i \equiv g^i$ be given. Choose a function $e = e(x, y^1, y^2, z^1, z^2)$ with $\partial e / \partial z^1 \neq 0$. Since $e = 0$ should belong to the $\mathcal{EL}$ system, we further assume that the function e vanishes on the submanifold $z^i - g^i \equiv 0$. Moreover, in virtue of the above results, we require that

$$\frac{\partial e}{\partial z^2} \Big/ \frac{\partial e}{\partial z^1} = c(x, y^1, y^2, e) \tag{26}$$

is a function merely of x, y^1, y^2, e. (Note at this place the geometrical sense of (26): if the variables x, y^1, y^2 are kept fixed, then the level sets $e(\cdot, \cdot, \cdot, z^1, z^2) = \text{const.}$ consist of a family of straight lines in the plane z^1, z^2. It immediately follows that a lot of such functions e does exist.) Assuming (26), one can verify that also $z^1 + cz^2 = d(x, y^1, y^2, e)$ is a function merely of the variables x, y^1, y^2, e (look at the Jacobian of e, d).

With the function e already chosen, our next aim is to determine the $\mathcal{PC}$ form ξ. Since $\mathrm{d}\xi$ can be expressed in terms of x, y^1, y^2, e, it follows that $\xi = P\,\mathrm{d}x + Q\,\mathrm{d}y^1 + R\,\mathrm{d}y^2 + S\,\mathrm{d}e + \mathrm{d}V$, hence

$$\mathrm{d}\xi = \mathrm{d}P \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y^1 + \mathrm{d}R \wedge \mathrm{d}y^2 + \mathrm{d}S \wedge \mathrm{d}e \tag{27}$$

where $P, \ldots, S$ are functions of x, y^1, y^2, e. The form $\mathrm{d}\xi$ should satisfy the second congruence (4), that is,

$$\frac{\partial P}{\partial e} - \frac{\partial S}{\partial x} = z^1 \left(\frac{\partial S}{\partial y^1} - \frac{\partial Q}{\partial e} \right) + z^2 \left(\frac{\partial S}{\partial y^2} - \frac{\partial R}{\partial e} \right).$$

Since the left hand side is a function of x, y^1, y^2, e, it follows that

$$(28) \qquad \frac{\partial S}{\partial y^2} - \frac{\partial R}{\partial e} = c\left(\frac{\partial S}{\partial y^1} - \frac{\partial Q}{\partial e}\right), \ \frac{\partial P}{\partial e} - \frac{\partial S}{\partial x} = d\left(\frac{\partial S}{\partial y^1} - \frac{\partial Q}{\partial e}\right)$$

where $d = z^1 + cz^2$. One can then express a^1, a^2 (cf. (22)) in terms of the coefficients $P, \ldots, S$ (cf. (27)) and then obtain the requirement

$$(29) \qquad a^2 - ca^1 = e = c\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y^1}\right) - \frac{\partial R}{\partial x} + \frac{\partial P}{\partial y^2} - d\left(\frac{\partial R}{\partial y^1} - \frac{\partial Q}{\partial y^2}\right)$$

by easy calculations. At the stage already achieved, if (28), (29) are valid and $\partial S/\partial y^1 \neq \partial Q/\partial e$, then the module Adj $\mathrm{d}\xi$ is of dimension 2 at all points where $e = 0$ and of dimension 4 otherwise (direct verification). So it follows that the relevant ξ is a $\mathcal{PC}$ form to the integral (21) with

$$(30) \quad f = P + z^1 Q + z^2 R + \partial V/\partial x + z^1 \partial V/\partial y^1 + z^2 \partial V/\partial y^2 \ \left(V = -\int S \,\mathrm{d}e\right),$$

as follows by virtue of the first congruence (4). Moreover, we have already ensured that the equation $e = 0$ is involved in the $\mathcal{EL}$ system.

At last, we pass to the complete control over the $\mathcal{EL}$ system under consideration. Owing to (28), the module Adj $\mathrm{d}\xi$ to the form (27) is generated by ϑ^1, ϑ^2, $\mathrm{d}e$ exactly at those points where the equations

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y^1} = z^2\left(\frac{\partial R}{\partial y^1} - \frac{\partial Q}{\partial y^2}\right), \quad \frac{\partial P}{\partial y^2} - \frac{\partial R}{\partial x} = z^1\left(\frac{\partial R}{\partial y^1} - \frac{\partial Q}{\partial y^2}\right)$$

are satisfied (direct verification). The equations should be equivalent to the $\mathcal{EL}$ system $z^i - g^i \equiv 0$, that is, to the equivalent system $e = z^2 - g^2 = 0$ (we employ the inequality $\partial e/\partial z^1 \neq 0$). This equivalence takes place if the sole condition

$$(31) \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y^1} = g^2\left(\frac{\partial R}{\partial y^1} - \frac{\partial Q}{\partial y^2}\right) \text{ on the hyperplane } e = 0$$

is satisfied. Conversely, if $\partial R/\partial y^1 \neq \partial Q/\partial e$ then the condition (31) clearly implies the identity $z^2 - g^2 = 0$ on the hyperplane $e = 0$.

18. Summary. If an $\mathcal{EL}$ system $z^1 - g^1(x, y^1, y^2) = z^2 - g^2(x, y^1, y^2) = 0$ is given and we search for the relevant variational integral (21), then a function e is to be chosen as above, which uniquely determines the other auxiliary functions c (cf. (26)) and $d = z^1 + cz^2$. Then the system (28), (29) with the boundary condition (31) for the unknown functions P, Q, R, S of independent variables x, y^1, y^2, e is to be solved under the inequalities $\partial S/\partial y^1 \neq \partial Q/\partial e$, $\partial R/\partial y^1 \neq \partial Q/\partial y^2$. With such P, Q, R, S already known, the sought variational integral is determined by (30).

19. Particular example. Assume $g^1 = g^2 = 0$ for a little diversion. Then we may choose $e = z^1$ (which is *not* the most general possibility), hence $c = 0$, $d = e$. The functions

$$R = \int_0^e \frac{\partial S}{\partial y^2}\,\mathrm{d}e + \Phi,\quad P = \int_0^e \left(\frac{\partial S}{\partial x} + e\frac{\partial S}{\partial y^1} + Q\right)\mathrm{d}e - eQ + \Psi$$

with $\Phi = \Phi(x, y^1, y^2)$, $\Psi = \Psi(x, y^1, y^2)$ arbitrary provide the general solution of (28). By inserting this into (29), we obtain

$$e = \int_0^e \left(\frac{\partial Q}{\partial y^2} - \int_0^e \frac{\partial^2 S}{\partial y^1 \partial y^2}\,\mathrm{d}e\right)\mathrm{d}e - \frac{\partial \Phi}{\partial x} - e\frac{\partial \Phi}{\partial y^1} + \frac{\partial \Psi}{\partial y^2}$$

so that Φ, Ψ cannot be quite arbitrary but subjected to the condition $\partial\Phi/\partial x = \partial\Psi/\partial y^2$ (as follows by inserting $e = 0$ in the last equation). It moreover follows that

$$1 = \frac{\partial Q}{\partial y^2} - \int_0^e \frac{\partial^2 S}{\partial y^1 \partial y^2}\,\mathrm{d}e - \frac{\partial \Phi}{\partial y^1}$$

by derivation, whence

$$Q = \int_0^e \frac{\partial S}{\partial y^1}\,\mathrm{d}e + y^2 + \int_0^{y^2} \frac{\partial \Phi}{\partial y^2}\,\mathrm{d}y^2 + \Theta \quad (\Theta = \Theta(x, y^1, e)).$$

The inequality $\partial S/\partial y^1 \neq \partial Q/\partial e$ is satisfied if $\partial\Theta/\partial e \neq 0$. The inequality $\partial R/\partial y^1 \neq \partial Q/\partial y^2$ is always true. The boundary condition (31) means that $\partial\Theta/\partial x = \partial\Psi/\partial y^1$ at the hyperplane $e = 0$ and permits to specify Ψ (and even Φ) in terms of Θ. We will not state the relevant explicit formula here and mention only the choice $S = \Phi = \Psi = 0$, $\Theta = e$ which yields the solution $f = (y^2 + (z^2)^2/2)z^1$ by using (30). But the point is that we are in principle able to determine *all* the relevant variational integrals (21).

20. The underdetermined case. On this occasion, let us mention the case of the underdetermined $\mathcal{EL}$ system, that is, the case of such variational integral (21) for which the function e is identically vanishing (and thus the extremals do not depend merely on constants but on arbitrary functions). One might expect that such a kind of "degeneration" can occur only if f is in reality depending on one variable function, that is, if

$$f = g(x, h, \mathrm{d}h\,/\,\mathrm{d}x) \text{ where } h = h(x, y^1, y^2).$$

However, we shall see that it is far from being true. We shall even resolve the relevant inverse problem.

So let us deal with the variational integral (21) and assume $e = 0$. Then the $\mathcal{EL}$ system consists of (prolongations of) the differential equation $\mathrm{d}z^1 / \mathrm{d}x + c\, \mathrm{d}z^2 / \mathrm{d}x = a^1/f_{11}$ arising from the last form in (23) linear in the second derivatives $\mathrm{d}z^i / \mathrm{d}x \equiv \mathrm{d}^2 y^i / \mathrm{d}x^2$. On the other hand, assuming $e = 0$, the module $\mathrm{Adj}\, \mathrm{d}\xi$ is generated by merely two forms (see (23)), hence $\mathrm{d}\xi = du \wedge \mathrm{d}v$ and $\xi = u\, \mathrm{d}v + \mathrm{d}w$ for appropriate functions u, v, w. But alternatively $\xi = f\, \mathrm{d}x + f_1\vartheta^1 + f_2\vartheta^2$ whence

$$(32) \qquad u(v_x + z^1v^1 + z^2v^2) + w_x + z^1w^1 + z^2w^2 = f,$$

$$(33) \qquad uv^1 + w^1 = f_1, \quad uv^2 + w^2 = f_2,$$

$$(34) \qquad uv_1 + w_1 = uv_2 + w_2 = 0,$$

where the abbreviations like $u_x = \partial u/\partial x$, $u^i \equiv \partial u/\partial y^i$, $u_i \equiv \partial u/\partial z^i$ are used. Owing to (34) necessarily $v = \bar{v}(x, y^1, y^2, z)$ and $w = \bar{w}(x, y^1, y^2, z)$ are composed functions of the above mentioned kind with a certain argument $z = \bar{z}(x, y^1, y^2, z^1, z^2)$. Then (34) implies that $u = -\bar{w}_z/\bar{v}_z = \bar{u}(x, y^1, y^2, z)$ is also a composed function of the kind mentioned. Inserting this into (33), we obtain

$$\bar{u}(\bar{v}^i + \bar{z}^i\bar{v}_z) + \bar{w}^i + \bar{z}^i\bar{w}_z = \bar{u}\bar{v}^i + \bar{w}^i \equiv f_i \quad (i = 1, 2).$$

It follows that $f_1 = r(x, y^1, y^2, z)$, $f_2 = s(x, y^1, y^2, z)$ are composed functions of the kind mentioned. Then (32) yields

$$(35) \qquad M + z^1r + z^2s = f \quad (M = M(x, y^1, y^2, z) = \bar{u}\bar{v}_x + \bar{w}_x).$$

Let us denote $N = M_z + z^1r_z + z^2s_z$. Then (35) implies $\bar{z}_1N = \bar{z}_2N = 0$ by derivation with respect to z^1, z^2 and it follows that $N = 0$.

By looking at the identity $N = 0$ one can observe that if the variables x, y^1, y^2 are kept fixed, then z^1 and z^2 are affinely related, that is, $z = Az^1 + Bz^2 + C$ where A, B, C are functions of x, y^1, y^2. Looking at the identity $N = 0$ again, one can conclude that

$$r_z = AD, \; s_z = BD, \; M_z = (C - z)D$$

where $D = D(x, y^1, y^2, z)$ is a certain nonvanishing factor. Our calculations are coming to the end. Regarding x, y^1, y^2, z^1, z^2, z as independent variables for a moment, we recall (35) and write

$$
\begin{aligned}
f &= \int \partial(M + z^1 r + z^2 s)/\partial z \cdot \mathrm{d}z = \int (C - z + z^1 A + z^2 B) D \,\mathrm{d}z \\
&= (C + z^1 A + z^2 B) \int D \,\mathrm{d}z - \int z D \,\mathrm{d}z = z \int D \,\mathrm{d}z - \int z D \,\mathrm{d}z \\
&= \int \Big(\int D \,\mathrm{d}z \Big) \mathrm{d}z = \bar{f}(x, y^1, y^2, z).
\end{aligned}
$$

So it follows that $f = \bar{f}(x, y^1, y^2, Az^1 + Bz^2 + C)$ is a very special composed function. With this result, the original identity $e = a^2 - ca^1 = 0$ is expressed by the equation

$$
\begin{aligned}
A\frac{\partial \bar{f}}{\partial y^2} - B\frac{\partial \bar{f}}{\partial y^1} + \Big(A\Big(\frac{\partial C}{\partial y^2} - \frac{\partial B}{\partial x}\Big) - B\Big(\frac{\partial C}{\partial y^1} - \frac{\partial A}{\partial x}\Big) \\
+ (z - c)\Big(\frac{\partial A}{\partial y^2} - \frac{\partial B}{\partial y^1}\Big)\Big) \frac{\partial \bar{f}}{\partial z} = 0
\end{aligned}
$$

as follows by direct substitution. Here A ($A \neq 0$ since $c = f_{12}/f_{11} = B/A$), B, C may be arbitrarily chosen in advance and then the last equation has a lot of solutions $\bar{f}$. (In particular if $A = \partial h/\partial y^1$, $B = \partial h/\partial y^2$, $C = \partial h/\partial x$ where $h = h(x, y^1, y^2)$ is arbitrary, we obtain the variational integrals depending in reality on merely one variable function.)

21. Continuation. The results hitherto obtained seem to be interesting but of little importance for the inverse problem. So let us again return to the original $\mathcal{PC}$ form $\xi = u \,\mathrm{d}v + \mathrm{d}w$ from the very beginning. The second congruence (4) then reads

$$(u_x + \sum z^i u^i)/(v_x + \sum z^i v^i) = u_1/v_1 = u_2/v_2 \tag{36}$$

after some simple calculations. Conversely, if (36) is satisfied for certain functions u and v, then there exists a function w such that the form $\xi = u \,\mathrm{d}v + \mathrm{d}w$ satisfies the second congruence (4) and thus ξ is a $\mathcal{PC}$ form. (Explicitly, $f = u(v_x + \sum z^i v^i) + w_x + \sum z^i w^i$ where w is determined to satisfy $uv_i \equiv w_i$. Such a function does exist as follows from the second and third equation (36).) Note at last that the module $\mathrm{Adj}\,\mathrm{d}\xi$ is generated (mod ϑ^1, ϑ^2) by the forms

$$\mathrm{d}u \cong (u_x + \sum z^i u^i)\,\mathrm{d}x + \sum u_i \,\mathrm{d}z^i, \mathrm{d}v \cong (v_x + \sum z^i v^i)\,\mathrm{d}x + \sum v_i \,\mathrm{d}z^i \tag{37}$$

that are proportional owing to (36). So omitting the latter form in (37), the generating equation of the $\mathcal{EL}$ system is given by the former:

$$u_1 \,\mathrm{d}z^1 / \mathrm{d}x + u_2 \,\mathrm{d}z^2 / \mathrm{d}x + (u_x + z^1 u^1 + z^2 u^2) = 0. \tag{38}$$

After these preliminaries, we may turn to the subject proper.

22. The inverse problem. Given is a multiple of the equation (38), that is, an equation of the kind $\mathrm{d}z^1 / \mathrm{d}x + U\,\mathrm{d}z^2 / \mathrm{d}x + V = 0$ (we assume the coefficient of $\mathrm{d}z^1 / \mathrm{d}x$ in (38) nonvanishing, for certainty). We search for certain functions u, v satisfying (36) and such that the corresponding $\mathcal{EL}$ equation (38) differs from the given one by a mere factor. (We have already seen that the knowledge of these functions u, v permits to determine the corresponding $\mathcal{PC}$ form $\xi = u\,\mathrm{d}v + \mathrm{d}w$ and even the relevant variational integral (21) by the first congruence (4).) Denote $\vartheta = \mathrm{d}z^1 + U\,\mathrm{d}z^2 + V\,\mathrm{d}x$ for a moment and consider the module $\Theta = \{\vartheta^1, \vartheta^2, \vartheta\}$ generated by the forms $\vartheta^1, \vartheta^2, \vartheta$. The module Θ is not flat (easy) and contains both differentials $\mathrm{d}u$ and $\mathrm{d}v$ (look at the congruences (37) and observe that the right hand sides are multiples of ϑ), hence it contains the flat submodule $\{\mathrm{d}u, \mathrm{d}v\}$.

On the other hand, one can directly find all flat submodules generated by two independent forms of the module Θ. Using the common methods [Br, Gr], one can find that only the submodule generated by the forms

$$\vartheta^1 + Uv^2,\ \vartheta + W\vartheta^2 \quad \left(W = V\frac{\partial U}{\partial z^1} - U\frac{\partial V}{\partial z^1} + \frac{\partial V}{\partial z^2} - \sum z^i \tfrac{\partial U}{\partial y^i} - \tfrac{\partial U}{\partial x}\right)$$

is flat provided the functions U, V satisfy the family of identities

$$\begin{aligned} U\frac{\partial U}{\partial z^1} - \frac{\partial U}{\partial z^2} &= U\frac{\partial V}{\partial y^1} - \frac{\partial V}{\partial y^2} + W\frac{\partial V}{\partial z^1} - V\frac{\partial W}{\partial z^1} + \sum z^i \tfrac{\partial W}{\partial y^i} + \tfrac{\partial W}{\partial x} \\ &= U\frac{\partial V}{\partial z^1} - \frac{\partial V}{\partial z^2} + 2W \\ &= U\frac{\partial U}{\partial y^1} - \frac{\partial U}{\partial y^2} + W\frac{\partial U}{\partial z^1} - U\frac{\partial W}{\partial z^1} + \frac{\partial W}{\partial z^2} = 0. \end{aligned}$$

So these identities yield necessary conditions for the equation $\mathrm{d}z^1 / \mathrm{d}x + U\,\mathrm{d}z^2 / \mathrm{d}x + V = 0$ to constitute the (underdetermined) $\mathcal{EL}$ system for a certain variational integral (21).

Conversely, let the coefficients U, V of the equation satisfy the above identities. Then we have the flat module $\{\vartheta^1 + U\vartheta^2, \vartheta + W\vartheta^2\} = \{\mathrm{d}u, \mathrm{d}v\}$, where u, v are appropriate functions. The form $\mathrm{d}\xi = \mathrm{d}u \wedge \mathrm{d}v$ satisfies the second congruence (4) (being a multiple of the form $(\vartheta^1 + U\vartheta^2) \wedge (\vartheta + W\vartheta^2)$). Since this congruence is equivalent to (36), there is a function w such that $\xi = u\,\mathrm{d}v + \mathrm{d}w$ is a $\mathcal{PC}$ form (cf. Section 21). Clearly $\vartheta + W\vartheta^2 \in \operatorname{Adj}\mathrm{d}\xi$, hence $\vartheta \in \operatorname{Adj}\mathrm{d}\xi$ modulo the contact forms. But the extremals satisfy Pfaff's system $\varphi \equiv 0$ ($\varphi \in \operatorname{Adj}\mathrm{d}\xi$) and the contact conditions $\vartheta^i \equiv 0$, thus in particular the equation $\vartheta = 0$ which is however equivalent to the differential equation given in advance. No other $\mathcal{EL}$ equations may appear since the module $\operatorname{Adj} d\xi = \{du, dv\} = \{\vartheta^1 + U\vartheta^2, \vartheta + W\vartheta^2\}$ is generated by the single form ϑ modulo the contact forms.

We have seen that even the inverse problem for non-regular variational integrals can be effectively investigated. However it is to be noted that the general concept of regularity seems to be not yet well understood in current literature, cf. Section 27.

Higher order variational problems

23. The scalar case. Modifying a little the notation, we pass to the inverse problem for the variational integrals

$$\int f(x, y_0, \ldots, y_n)\, dx \to \text{extremum}, \quad y_s \equiv d^s y / dx^s\,. \tag{39}$$

Assuming $n > 1$, the family of variables $x, y_0, \ldots, y_n$ appearing in (39) proves to be too narrow at a very early stage of investigations since the $\mathcal{PC}$ forms and $\mathcal{EL}$ systems involve some derivatives of higher orders that cannot be easily specified in advance. An analogous difficulty will appear again and again in future, so we take a radical measure from now on: the infinite prolongations employing the derivatives of all orders. Since the functions and differential forms will always depend on a finite number of variables as before, the common rules of calculations may be accepted without any change. The vector fields will be represented by infinite series but it does not cause any trouble.

As the variational integral (39) is concerned, we introduce the space of variables $x, y_0, y_1, \ldots$ endowed with the contact forms $\vartheta_s \equiv dy_s - y_{s+1}\, dx$ $(s = 0, 1, \ldots)$ and the vector field $D = \partial/\partial x + \sum y_{s+1}\partial/\partial y_s$ (infinite series) usually called the *total* or *formal derivative*. Since every function $g = g(x, y_0, \ldots, y_{m(g)})$ under consideration will depend on a finite number of variables, clearly $dg = Dg \cdot dx + \sum \partial g/\partial y_s \cdot \vartheta^s$ makes good sense. Let us define the $\mathcal{PC}$ form ξ by the congruences

$$\xi \cong f\, dx \pmod{\text{all } \vartheta_s}, \quad D \rfloor d\xi \cong 0 \pmod{\vartheta_0}. \tag{40}$$

In a certain sense, (40) is a generalization of (4). The $\mathcal{PC}$ form is uniquely determined. In explicit terms

$$\xi = f\, dx + \sum a^s \vartheta_s \quad \left(s = 0, \ldots, n-1;\; a^{n-1} = \frac{\partial f}{\partial y_n},\; a^{s-1} = \frac{\partial f}{\partial y_s} - Da^s\right), \tag{41}$$

$$d\xi = ev_0 \wedge dx + \sum a^s_r \vartheta_r \wedge \vartheta_s \quad \left(e = \sum (-D)^s \tfrac{\partial f}{\partial y_s},\; a^s_r \equiv \tfrac{\partial a^s}{\partial y^r}\right). \tag{42}$$

Here e is the $\mathcal{EL}$ *operator*. The (infinitely prolonged) extremals $y_s \equiv \bar{y}_s(x)$ ($s = 0, 1, \ldots$) satisfy the *contact conditions* $\vartheta_s \equiv 0$ (clearly equivalent to the recurrence $\bar{y}_{s+1}(x) \equiv \mathrm{d}\,\bar{y}_s(x)/\,\mathrm{d}x$) and the $\mathcal{EL}$ equation $e = 0$ (and thus the prolongation $De = D^2e = \ldots = 0$).

We are interested in the *inverse problem*, i.e., we wish to determine the variational integral (39) if the extremals are given in advance. It is sufficient to deal only with the *regular case* defined by $\partial^2 f/\partial y_n^2 \neq 0$. The argument for this assertion is as follows.

First recall that the $\mathcal{EL}$ operator e is identically vanishing if and only if $f = Dg$ for an appropriate function g. (We omit the proof and refer to a far going generalization in Section 39.) If (39) is a non-regular variational integral then clearly $f = A + By_n$ where A and B do not depend on y_n. So, instead of (39), we may deal with the variational integral

$$\int (f - Dg)\,\mathrm{d}x \to \text{ extremum } \quad \left(g = \int B\,\mathrm{d}y_{n-1}\right)$$

with the same $\mathcal{EL}$ operator (and thus the same extremals) but the new kernel function $f - Dg$ *not involving* y_n (easy verification). If the new variational integral (of order $n-1$) again would be a non-regular one, we may repeat the construction. So it follows that, as the inverse problem is concerned, we may suppose regularity of the sought integral (39) and thus the given $\mathcal{EL}$ equation of the kind $y_{2n} = g(x, y_0, \ldots, y_{2n-1})$ with the highest derivative of order $2n$ separated on the left.

24. The inverse problem can be resolved both by the direct and by the $\mathcal{VIF}$ method but we will outline a geometrical approach more adaptable for our future aims. So let us introduce the subspace E consisting of all points which satisfy all equations $D^k e \equiv 0$ ($k = 0, 1, \ldots$), explicitly $y_{2n+k} \equiv D^k g$. Denoting by ι the natural inclusion of E into the total space of all variables $x, y_0, y_1, \ldots$, the pull-backs $x = \iota^* x$, $y_0 = \iota^* y_0$, ..., $y_{2n-1} = \iota^* y_{2n-1}$ may be used for coordinates on E. (We accept the common convention of omitting the pull-backs whenever possible.) In terms of these coordinates

$$F = \partial/\partial x + y_1 \partial/\partial y_0 + \ldots + y_{2n-1}\partial/\partial y_{2n-2} + g\partial/\partial y_{2n-1}$$

is a vector field on E, the restriction of the vector field D (which is tangent to E). The curves lying in E and tangent to F are just the extremals. Concerning the $\mathcal{PC}$ form ξ, it is expressed in terms of the variables $x, y_0, \ldots, y_{2n-1}$ (direct verification), so we may identify $\xi = \iota^*\xi$. Owing to regularity, the module $\operatorname{Adj} \mathrm{d}\xi$ is generated by the forms $\vartheta_0 = \iota^*\vartheta_0$, ..., $\vartheta_{2n-2} = \iota^*\vartheta_{2n-2}$, $\mathrm{d}y_{2n-1} - g\,\mathrm{d}x$ (as follows from (42) by easy calculations). So $\mathrm{d}\xi$ is of the maximal possible rank $2n$ and moreover $F \in \operatorname{Adj} \mathrm{d}\xi^{\perp}$ or, equivalently, $F \rfloor \mathrm{d}\xi = 0$ is satisfied.

Let us turn to the inverse problem. Then E (equipped with the above coordinates and contact forms) and F are given in advance and we search for the relevant $\mathcal{PC}$ form ξ. It is sufficient to ensure the properties (40), rank $\mathrm{d}\xi = 2n$, $F \rfloor \mathrm{d}\xi = 0$. They are analogous to the requirements $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of Section 2. (In fact, the second congruence (40) means that ξ is a $\mathcal{PC}$ form to a variational integral (39) determined by the first congruence (40). Then the rank condition on $\mathrm{d}\xi$ ensures regularity, and the extremals are guaranteed by the last condition.) Slightly adapted arguments of Sections 2 and 3 imply that it is sufficient to determine the restriction $\mathrm{d}\tilde{\xi}$ of $\mathrm{d}\xi$ on a fixed hyperplane $x = \text{const}$. The form $\mathrm{d}\tilde{\xi}$ should be a symplectical structure in the reduced space of variables $y_0, \ldots, y_{2n-1}$ satisfying the requirement (9) where

$$\mu = \mathrm{d}y_0 \wedge \ldots \wedge \mathrm{d}y_{n-1}, \quad G = y_1 \partial/\partial y_0 + \ldots + y_{2n-1} \partial/\partial y_{2n} + g \partial/\partial y_{2n-1}.$$

We omit the proof and note that an alternative approach through the first integrals following the lines of Section 12 can be realized, too.

25. Example. Let $n = 2$ and let $y_4 = g(x, y_0, \ldots, y_3)$ be the given $\mathcal{EL}$ equation. Assuming $\mathrm{d}\tilde{\xi} = \sum a^{rs} \mathrm{d}y_r \wedge \mathrm{d}y_s$ and

$$\mathcal{M}_G^k \mu = \sum \binom{k}{i} \mathrm{d}y_i \wedge \mathrm{d}y_{k-i-1} = \sum b_k^{rs} \mathrm{d}y_r \wedge \mathrm{d}y_s \quad (y_{4+k} \equiv F^k g)$$

(both sums with $r < s$ and $r, s = 0, \ldots, 3$), the requirements (9) are expressed by

$$a^{01} b_k^{23} - a^{02} b_k^{13} + a^{03} b_k^{12} + a^{12} b_k^{03} - a^{13} b_k^{02} + a^{23} b_k^{01} = 0.$$

We state only those coefficients which are nonvanishing and really needed:

$$b_0^{01} = b_1^{02} = b_2^{03} = b_2^{12} = 1, \quad b_3^{03} = \frac{\partial g}{\partial y_3}, \quad b_3^{13} = 2,$$

$$b_4^{03} = \frac{\partial F g}{\partial y_3}, \quad b_4^{12} = 3\frac{\partial g}{\partial y_2}, \quad b_4^{13} = 3\frac{\partial g}{\partial y_3}, \quad b_4^{23} = 2,$$

$$b_5^{03} = \frac{\partial F^2 g}{\partial y_3}, \quad b_5^{12} = 4\frac{\partial F g}{\partial y_2} - 5\frac{\partial g}{\partial y_1}, \quad b_5^{13} = 4\frac{\partial F g}{\partial y_3}, \quad b_5^{23} = 5\frac{\partial g}{\partial y_3}.$$

Then the requirements (9) with $k = 0, \ldots, 4$ read

$$a^{23} = a^{13} = a^{12} + a^{03} = \frac{\partial g}{\partial y_3} a^{12} - 2a^{02}$$
$$= 2a^{01} - 3\frac{\partial g}{\partial y_3} a^{02} + 3\frac{\partial g}{\partial y_2} a^{03} + \frac{\partial F g}{\partial y_3} a^{03} = 0$$

and determine the functions

$$a^{23} = a^{13} = 0, \quad a^{12} = -a^{03} = u, \quad a^{02} = uP, \quad a^{01} = uQ$$

and thus the restricted form

$$\mathrm{d}\tilde{\xi} = u(\mathrm{d}y_0 \wedge (Q\,\mathrm{d}y_1 + P\,\mathrm{d}y_2 - \mathrm{d}y_3) + \mathrm{d}y_1 \wedge \mathrm{d}y_3)$$

up to an unknown factor $u = u(y_0, \dots, y_3)$. Here we have denoted

$$P = \frac{1}{2}\frac{\partial g}{\partial y_3}, \quad Q = \frac{1}{2}\left(\frac{2}{3}\left(\frac{\partial g}{\partial y_3}\right)^2 + 3\frac{\partial g}{\partial y_3} - \frac{\partial Fg}{\partial y_3}\right).$$

One can see that the nonvanishing $u \neq 0$ is necessary and sufficient for regularity ($\mathcal{B}$ is expressed by $\mathrm{d}\tilde{\xi} \wedge \mathrm{d}\tilde{\xi} \neq 0$) and that the next requirement (9) with $k = 5$ yields the necessary compatibility condition

$$\frac{\partial F^2 g}{\partial y_3} + \frac{9}{2}\frac{\partial g}{\partial y_3}\frac{\partial Fg}{\partial y_3} - 4\frac{\partial F}{\partial y_2} + \frac{15}{2}\frac{\partial g}{\partial y_3}\left(\frac{1}{2}\left(\frac{\partial g}{\partial y_3}\right)^2 + \frac{\partial g}{\partial y_2}\right) + 5\frac{\partial g}{\partial y_1} = 0$$

for this nonvanishing. If this condition is identically satisfied, all requirements (9) with $k > 5$ may be omitted. At last, the unknown factor u should satisfy the closedness condition $\mathrm{d}^2\tilde{\xi} = 0$ (cf. (iv) Section 5) which may be easily converted into the system

$$\frac{\partial u}{\partial y_0} = u\left(\frac{\partial(Q-P)}{\partial y_3} + \frac{\partial Q}{\partial y_2} - \frac{\partial P}{\partial y_1}\right), \quad \frac{\partial u}{\partial y_1} = u\frac{\partial Q}{\partial y_3}, \quad \frac{\partial u}{\partial y_2} = u\frac{\partial P}{\partial y_3}, \quad \frac{\partial u}{\partial y_3} = 0$$

(quite analogous to (15)) with the conclusion that the nonvanishing solution $u \neq 0$ exists if and only if the form

$$\left(\frac{\partial(Q-P)}{\partial y_3} + \frac{\partial Q}{\partial y_2} - \frac{\partial P}{\partial y_1}\right)\mathrm{d}y_0 + \frac{\partial Q}{\partial y_3}\mathrm{d}y_1 + \frac{\partial P}{\partial y_3}\mathrm{d}y_2$$

with $x = \text{const.}$ kept fixed is a total differential (of $\ln|u|$).

26. Several variable functions. We introduce the space of variables x, y_s^i ($i = 1, \dots, m$; $s = 0, 1, \dots$) equipped with the contact forms $\vartheta_s^i \equiv \mathrm{d}y_s^i - y_{s+1}^i\,\mathrm{d}x$, total derivative $D = \partial/\partial x + \sum y_{s+1}^i$, and the variational integral

(43) $$\int f(x, y_0^1, \dots, y_0^m, \dots, y_n^1, \dots, y_n^m)\,\mathrm{d}x \to \text{extremum}, \quad y_s^i \equiv d^s y^i/\mathrm{d}x^s.$$

First assume the regularity $\det(\partial^2 f/\partial y_n^i \partial y_n^j) \neq 0$. Then, if $e^i \equiv \sum(-D)^s \partial f/\partial y_s^i$ are the $\mathcal{EL}$ operators, the $\mathcal{EL}$ system $e^i \equiv 0$ can be uniquely represented by an equivalent system of equations of the kind

(44) $$y_{2n}^i \equiv g^i(x, y_0^1, \dots, y_0^m, \dots, y_{2n-1}^1, y_{2n-1}^m)$$

with the highest derivatives of order $2n$ separated on the left. In the inverse problem, we search for (43) if (44) is given in advance.

Let E be the subspace defined by $D^k e^i \equiv 0$ ($i = 1, \ldots, m$; $k = 0, 1, \ldots$) of the total space. It may be equivalently defined by the equations $y^i_{2n+k} \equiv D^k g^i$ and it follows that the functions x, y^i_s ($i = 1, \ldots, m$; $s = 0, \ldots, 2n-1$) may serve for coordinates on E. Quite analogously as above, our aim is to determine the $\mathcal{PC}$ form

$$\xi = f\,\mathrm{d}x + \sum a^i_s \vartheta^i_s \quad (a^i_{n-1} \equiv \partial f/\partial y^i_n,\ a^i_{s-1} \equiv \partial f/\partial y^i_s - Da^i_s)$$

satisfying the congruences

$$\xi \cong f\,\mathrm{d}x \text{ (mod all } \vartheta^i_s), \quad D\rfloor \mathrm{d}\xi \cong 0 \text{ (mod all } \vartheta^i_0).$$

It may be regarded as a differential form on E. Owing to the property $F\rfloor \mathrm{d}\xi = 0$ where

$$F = \partial/\partial x + \sum y^i_0 \partial/\partial y^i_0 + \ldots + \sum y^i_{2n-1}\partial/\partial y^i_{2n-2} + \sum g^i \partial/\partial y^i_{2n-1}$$

is the restriction of D on E, it is sufficient to determine the form $\mathrm{d}\tilde{\xi}$, the restriction of the form $\mathrm{d}\xi$ on a fixed hyperplane $x = $ const. The form $\mathrm{d}\tilde{\xi}$ should provide a symplectical structure on the hyperplane and should satisfy the conditions (9) where

$$\mu = \mathrm{d}y^1_0 \wedge \ldots \wedge \mathrm{d}y^m_0 \wedge \ldots \wedge \mathrm{d}y^1_{n-1} \wedge \ldots \wedge \mathrm{d}y^m_{n-1}, \quad G = F - \partial/\partial x.$$

The proof of all these assertions may be omitted.

27. On non-regular cases. Assuming $m = n = 2$, let us first mention the variational integral

$$\int (Ay^1_2 + By^2_2)\,\mathrm{d}x \to \text{ extremum}$$

where A, B are dependent on x, y^1_0, y^2_0, y^1_1, y^2_1. Omitting all lower order terms, the $\mathcal{EL}$ operators are

$$e^1 = \ldots + Cy^2_3,\ e^2 = \ldots - Cy^1_3 \quad (C = \partial A/\partial y^2_1 - \partial B/\partial y^1_1),$$

and assuming $C \neq 0$, the $\mathcal{EL}$ system can be uniquely expressed by certain formulae of the kind $y^i_3 \equiv g^i$ ($i = 1, 2$) with the highest derivatives separated on the left. So, introducing the vector field

$$F = \partial/\partial x + \sum y^i_1 \partial/\partial y^i_0 + \sum y^i_2 \partial/\partial y^i_1 + \sum g^i \partial/\partial y^i_2$$

on the subspace E defined by $y^i_{3+k} \equiv D^k g^i$, the $\mathcal{PC}$ form ξ (defined quite analogously as in the previous section) may be regarded as a differential form on E satisfying $F \rfloor \mathrm{d}\xi = 0$. If E is given in advance and we search for the variational integral, it is sufficient to determine the restriction $\mathrm{d}\tilde{\xi}$ of $\mathrm{d}\xi$ on a hyperplane $x =$ const. The form $\mathrm{d}\tilde{\xi}$ should be a symplectical one and should satisfy (9) with $\mu = \mathrm{d}y^1_0 \wedge \mathrm{d}y^2_0 \wedge \mathrm{d}y^1_1 \wedge \mathrm{d}y^2_1$ and $G = F - \partial/\partial x$. So regardless of the non-regularity, the results are of the same nature as in Section 26.

On the contrary, the non-regular variational integral

$$\int f(x, y^1_0, y^2_0, \dots, y^1_2, y^2_2)\,\mathrm{d}x \to \text{ extremum } (f_{11} \neq 0, f_{11}f_{22} = (f_{12})^2),$$

where we temporarily denote $f_{ij} \equiv p^2 f/\partial y^i_2 \partial^j_2$, leads to analogous results as in Sections 16–20. In particular, the $\mathcal{EL}$ system involves the equation $(e =)\ e^1 - ce^2 = 0$ $(c = f_{12}/f_{11})$ of at most the third order and if $\partial e/\partial y^1_3 \neq 0$, then $(\bar{e} =)$ $f_{11}De - e^1 \partial e/\partial y^1_3 = 0$ is another third order equation of the (prolonged) $\mathcal{EL}$ system. One can observe that the $\mathcal{PC}$ form cannot be identified with its restriction on the relevant subspace E.

It follows that the common definition of regular problems is not the best possible one. Some deeper interrelations between the $\mathcal{PC}$ forms and $\mathcal{EL}$ systems should be taken into account.

Formal calculus of variations

28. Prelude. The common approach to the constrained variational problems through the admissible variations resembles a little the vicious circle: in order to obtain the extremals, certain boundary conditions are chosen in advance but the derived $\mathcal{EL}$ system does not depend on them and on the contrary, the possible shape of boundary conditions is to a large extent determined by the width of the family of the resulting extremals. It seems that without some assumption of regularity, the general constrained variational problems are lying beyond the scope of this method. For this reason, we shall propose another approch. Temporarily, it may be regarded as a mere formal variant of the usual Lagrange multipliers rule but the multipliers will be soon eliminated.

We begin at a very general level. Let $\mathbf{F}$ be a function on a manifold $\mathbf{M}$. (In reality, an infinite-dimensional space of certain curves will be substituted for $\mathbf{M}$ but we prefer a certain ambiguity here since only very general properties of $\mathbf{M}$ and other relevant objects to follow will be needed.) Let $\mathbf{G}\colon \mathbf{M} \to \mathbf{V}$ be a mapping into a vector space $\mathbf{V}$, $\mathbf{P}_0 \in \mathbf{M}$ a fixed point with $\mathbf{G}\mathbf{P}_0 = 0 \in \mathbf{V}$, $\mathrm{d}\mathbf{F}$ and $\mathrm{d}\mathbf{G}$ differentials at $\mathbf{P}_0$. Let

$$\mathbf{P}(\tau) \in \mathbf{M}\ (-\varepsilon < \tau < \varepsilon, \varepsilon > 0), \quad \mathbf{P}(0) = \mathbf{P}_0 \tag{45}$$

be a one-parameter family of points, $\mathbf{Z} = d\mathbf{P}(\tau)/\,\mathrm{d}\tau\,|_{\tau=0}$ its tangent vector at $\mathbf{P}_0$.

We introduce the following properties I-IV of $\mathbf{P}_0$:

I: $\mathrm{d}\mathbf{F}\,\mathbf{P}(\tau)/\,\mathrm{d}\tau\,\big|_{\tau=0} = 0$ for every family (45) with $\mathbf{G}\mathbf{P}(\tau) \equiv 0$,

II: $\mathrm{d}\mathbf{F}\,\mathbf{P}(\tau)/\,\mathrm{d}\tau\,\big|_{\tau=0} = 0$ for every family (45) with $\mathrm{d}\mathbf{G}\,\mathbf{P}(\tau)/\,\mathrm{d}\tau\,\big|_{\tau=0} = 0$,

III: $\mathrm{d}\mathbf{F}(\mathbf{Z}) = 0$ for every $\mathbf{Z}$ with $\mathrm{d}\mathbf{G}(\mathbf{Z}) = 0$,

IV: there exists a linear function $\mathbf{L}$ on $\mathbf{V}$ satisfying $\mathrm{d}\mathbf{F} = \mathbf{L} \circ \mathrm{d}\mathbf{G}$.

Property I recalls the usual concept of a *critical point* $\mathbf{P}_0$ *of* $\mathbf{F}$ at the level subset $\mathbf{G}^{-1}(0) \subset \mathbf{M}$ of the mapping $\mathbf{G}$. Owing to the common definition of differentials, II and III are equivalent. The other neighbouring properties are very near one to the other and clearly equivalent if the data $\mathbf{F}$, $\mathbf{G}$, $\mathbf{M}$, $\mathbf{V}$ are favourable enough. The property IV resembles the concept of a *standard* $\mathbf{G}$*-critical point of* $\mathbf{F}$ [Ch]. It may (and will) be used for a convenient substitute of the original concept of a critical point from now on.

We continue with a particular realization of the previous abstract scheme. Let $\mathbf{N}$ be a manifold, Φ the module of all differential 1-forms on $\mathbf{N}$, $\lambda \in \Phi$ a fixed 1-form, $\Omega \subset \Phi$ a subset, $\mathbf{V} = \bigoplus \mathbf{R}_\omega$ the direct product of real lines $\mathbf{R}_\omega \equiv \mathbf{R}$ indexed by the family of forms $\omega \in \Omega$ and equipped with the direct product topology. Let $\mathbf{M}$ be the space of all curves $\mathbf{P}\colon p = p(t) \in \mathbf{N}$, $0 \leqslant t \leqslant 1$, embedded into the manifold $\mathbf{N}$. We introduce the function $\mathbf{F}\colon \mathbf{M} \to \mathbf{R}$ and the mapping $\mathbf{G}\colon \mathbf{M} \to \mathbf{V}$ as follows:

$$\mathbf{F}\mathbf{P} = \int_0^1 p^*\lambda, \quad \mathbf{G}\mathbf{P} = \left\{\int_0^1 p^*\omega\right\}_{\omega\in\Omega}.$$

Then a moving point (45) is realized as a one-parameter family of embedded curves $\mathbf{P}(\tau)\colon p = p(t,\tau) \in \mathbf{N}$, $0 \leqslant t \leqslant 1$, $-\varepsilon < \tau < \varepsilon$, and the relevant tangent vector $\mathbf{Z}$ may be visualized as a family of vector fields

$$Z_{p(t,0)} = dp(t,\tau)/\,\mathrm{d}\tau\,\big|_{\tau=0}$$

along $\mathbf{P}_0\colon p = p(t,0)$. For technical reasons, it is appropriate to extend the family into a global vector field (noted Z) on $\mathbf{N}$. Occasionally abbreviating $p(t,0) = p_0$, the differentials are as follows:

$$\mathrm{d}\mathbf{F}(\mathbf{Z}) = \int_0^1 p_0^*\mathcal{L}_Z\lambda = \int_0^1 p_0^* Z\rfloor\,\mathrm{d}\,\lambda + \lambda(Z_{p(1,0)}) - \lambda(Z_{p(0,0)}),$$

$$\mathrm{d}\mathbf{G}(\mathbf{Z}) = \left\{\int_0^1 p_0^*\mathcal{L}_Z\omega\right\} = \left\{\int_0^1 p_0^* Z\rfloor\,\mathrm{d}\omega + \omega(Z_{p(1,0)}) - \omega(Z_{p(0,0)})\right\}.$$

We shall suppose $Z_{p(1,0)} = Z_{p(0,0)} = 0$ (being not interested in the boundary conditions and refering to [Ch] for the general case). Since every linear function $\mathbf{L}$ on $\mathbf{V}$ can be explicitly expressed by a finite sum

$$\mathbf{L}(\{v_\omega\}_{\omega\in\Omega}) = \sum \ell_\omega v_\omega \quad (\ell_\omega, v_\omega \in \mathbf{R})$$

for appropriate constants l_ω, we have

$$\mathbf{L} \circ \mathrm{d}\mathbf{G}(\mathbf{Z}) = \sum \ell_\omega \int_0^1 p_0^* Z \rfloor \,\mathrm{d}\omega = \int_0^1 p_0^* Z \rfloor \,\mathrm{d}\bar{\omega},$$

where $\bar{\omega} = \sum \ell_\omega \omega \in \Omega$. It follows that the requirement $\mathbf{dF} = \mathbf{L} \circ \mathbf{dG}$ (cf. IV) can be expressed by

$$\int_0^1 p_0^* Z \rfloor \,\mathrm{d}(\lambda - \bar{\omega}) \equiv 0 \quad (\text{appropriate } \bar{\omega}, \text{ arbitrary } Z),$$

that is,

$$(46) \qquad p_0^* Z \rfloor d(\lambda - \bar{\omega}) \equiv 0 \quad (\text{appropriate } \bar{\omega} \in \Omega, \text{ arbitrary } Z).$$

It is to be noted that the level set $\mathbf{G}^{-1}(0)$ coincides with the set of all embedded curves $\mathbf{P}\colon p = p(t) \in \mathbf{N}$, $0 \leqslant t \leqslant 1$, which satisfy Pfaff's system $\omega \equiv 0$ ($\omega \in \Omega$, easy verification). Altogether taken, the theory of constrained variational integral (3) is identified with the study of standard $\mathbf{G}$-critical points of $\mathbf{F}$. The curves $\mathbf{P}_0$: $p = p_0(t) \in \mathbf{N}$, $0 \leqslant t \leqslant 1$, satisfying the constraints $p_0^*\omega \equiv 0$ ($\omega \in \Omega$) and the requirement (46) will be called *extremals* (see also [Gr, I d 14]).

As yet the form $\bar{\omega}$ appearing in (46) may in principle depend on the choice of the extremal $\mathbf{P}_0$ under consideration. But such a form $\bar{\omega}$ is *not* unique and our next aim will be to find a *universal* $\bar{\omega}$ such that (46) is valid for *all* extremals. In a certain sense, this $\bar{\omega}$ is already unique (cf. Section 38) and can be abstractly characterized by a certain congruence (55). The construction of this universal $\bar{\omega}$ will be easy and explicit but some rather unusual preliminaries are needed.

29. Ordinary differential equations. An arbitrary system of such equations can be converted into an equivalent first order system if the higher order derivatives are taken for new unknowns. So without any loss of generality, we may deal only with systems of the kind

$$\frac{\mathrm{d}y^{c+k}}{\mathrm{d}x} \equiv g^k\left(x, y^1, \ldots, y^m, \frac{\mathrm{d}y^1}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}y^c}{\mathrm{d}x}\right); \; k = 0, \ldots, m - c.$$

The infinite prolongation then arises if we introduce the infinite family of variables

$$y_0^i \equiv y^i \quad (i = 1, \ldots, m), \; y_s^j \equiv \frac{\mathrm{d}^s y^j}{\mathrm{d}x^s} \quad (j = 1, \ldots, c; s = 0, 1, \ldots),$$

and it consists of the infinite system of differential equations

$$\frac{\mathrm{d}y_s^j \,\mathrm{d}y_s^j}{\mathrm{d}x} \equiv y_{s+1}^j \; (j = 1, \ldots, c; s = 0, 1, \ldots), \quad \frac{\mathrm{d}y_0^{c+k}}{\mathrm{d}x} \equiv g^k \; (k = 0, \ldots, m - c).$$

But in order to eliminate the effect of the particular choice of coordinate systems, this equations will be represented as an equivalent Pfaffian system, namely by

$$\omega_s^j \equiv 0 \ (j = 1, \ldots, c; \ s = 0, 1, \ldots), \quad \omega_0^{c+k} \equiv 0 \ (k = 0, \ldots, m - c),$$

where $\omega_s^j \equiv \mathrm{d}y_s^j - y_{s+1}^j \,\mathrm{d}x$, $\omega_0^{c+k} \equiv \mathrm{d}y_0{}^{c+k} - g^k \,\mathrm{d}x$, and even by the module Ω generated by all the forms mentioned. Thus the original differential equations are equivalent to the infinite Pfaffian system $\omega \equiv 0$ ($\omega \in \Omega$), where $\Omega \subset \Phi$ is a given submodule of the module Φ of all differential 1-forms. It is interesting that the submodules Ω arising in this manner can be abstractly characterized.

First, $\Omega \subset \Phi$ is of codimension 1 since Φ is clearly generated by Ω and the differential $\mathrm{d}x \notin \Omega$. It follows that the module $\Omega^\perp$ of all vector fields Z satisfying $\omega(Z) \equiv 0$ ($Z \in \Omega$) is 1-dimensional: every $Z \in \Omega^\perp$ is a multiple of the vector field

$$D = \partial/\partial x + \sum g^k \partial/\partial y_0^{c+k} + \sum y_{s+1}^j \partial/\partial y_s^j.$$

One can also observe that $\mathcal{L}_D \Omega \subset \Omega$, explicitly

$$\mathcal{L}_D \omega_s^j = \omega_{s+1}^j, \ \mathcal{L}_D \omega_0^{c+k} \cong 0 \ (\text{mod all } \omega_0^{c+k}, \omega_1^j).$$

The just mentioned properties of Ω prove to be typical and quite sufficient.

30. Definition. Let $\mathbf{N} = \mathbf{R}^\infty$ be the infinite product of real lines, i.e., the space of real infinite sequences $\{t^1, t^2, \ldots\}$. Denote by $\mathcal{F}$ the module of all functions $f = f(t^1, \ldots, t^{m(f)})$, let Φ be the module of all differential 1-forms $\varphi = \sum f^i dg^i$ ($f^i, g^i \in \mathcal{F}$, finite sum), let $\hat{\Phi}$ be the dual module of all vector fields $Z = \sum z^i \partial/\partial t^i$ ($z^i \in \mathcal{F}$, infinite sum). A submodule $\Omega \subset \Phi$ of codimension 1 is called a *diffiety* if there exist $\omega^1, \ldots, \omega^c \in \Omega$ such that the family of all forms

$$\mathcal{L}_D^k \omega^j \quad (k = 0, 1, \ldots; \ j = 1, \ldots, c; \ 0 \neq D \in \Omega^\perp)$$

may be used for generators of Ω. Concerning the vector field D, there does exist $\mathrm{d}x \in \Phi$, $\mathrm{d}x \notin \Omega$. Then every $\varphi \in \Phi$ can be uniquely expressed as $\varphi = g\,\mathrm{d}x + \omega$ (for appropriate $g \in \mathcal{F}$, $\omega \in \Omega$ depending on φ) and the identity $\varphi(D) \equiv g$ defines a vector field $D \in \Omega^\perp$ which may be used in the above definition.

It is to be noted that infinite prolongations of underdetermined systems of differential equations are diffieties (cf. the previous section). The converse is also true (but not needed here and therefore we omit the elementary but lengthy proof) so that the diffieties provide an abstract substitute for the infinite prolongations mentioned.

31. Filtrations. For every diffiety Ω, there exist many filtrations

$$\Omega^* : 0 = \bigcap \Omega^\ell \subset \ldots \subset \Omega^\ell \subset \Omega^{\ell+1} \subset \ldots \subset \Omega = \bigcup \Omega^\ell$$

by finitely generated submodules $\Omega^\ell \subset \Omega$ satisfying

$$\Omega^{\ell+1} \supset \mathcal{L}_D\Omega^\ell \text{ (all } \ell), \quad \Omega^{\ell+1} = \Omega^\ell + \mathcal{L}_D\Omega^\ell \text{ } (l \text{ large}). \tag{49}$$

(In practice Ω^ℓ may be generated by all forms $\mathcal{L}_D^k\omega^j$ ($k \leqslant \ell$) where $\omega^1, \dots, \omega^c$ are the forms in the definition of a diffiety.) But our aim is to find the *normal filtrations* satisfying moreover

$$\mathcal{L}_D\Omega^{-1} \subset \Omega^{-1}, \ D\colon \Omega^\ell/\Omega^{\ell-1} \to \Omega^{\ell+1}/\Omega^\ell \text{ is injective if } \ell \geqslant 0. \tag{50}$$

Here the mapping denoted by D is the natural *module homomorphism* induced by $\mathcal{L}_D$ for a nonvanishing vector field $D \in \Omega^\perp$. In more detail, we put

$$D[\omega] = [\mathcal{L}_D\omega] \in \Omega^{\ell+1}/\Omega^\ell \text{ for every } \omega \in \Omega^\ell, \ [o]n\Omega^\ell/\Omega^{\ell-1}$$

where the square brackets denote the relevant classes (the first inclusion (49) is employed here).

The properties (50) can be achieved by a simple successive change of a finite number of terms of an arbitrary filtration Ω^* satisfying (49). In fact, clearly D: $\Omega^\ell/\Omega^{\ell-1} \to \Omega^{\ell+1}/\Omega^\ell$ is a surjection and hence a bijection for all ℓ large enough, say, for $\ell \geqslant L$. One can then put $\bar{\Omega}^\ell \equiv \Omega^\ell$ ($\ell \geqslant L$) and inductively with $\ell = L-1, L-2, \dots$

$$\bar{\Omega}^\ell = \text{ the kernel of the composition } \bar{\Omega}^{\ell+1} \to \bar{\Omega}^{\ell+2} \to \bar{\Omega}^{\ell+2}/\bar{\Omega}^{\ell+1}, \tag{51}$$

(factorization after $\mathcal{L}_D$). The proper inclusions $\dots \supset \bar{\Omega}^k \supset \bar{\Omega}^{k-1} \supset \dots$ necessarily terminate by the equalities $\bar{\Omega}^K = \bar{\Omega}^{K-1} = \dots$. Then a mere change of indices yields the desired result.

32. Theorem. *The "residual term" $\mathcal{R}(\Omega) = \Omega^{-1}$ of a normal filtration is identical with the maximal element of any of the following families of modules Ψ_1 – Ψ_3:*

(i) *the family of all finite-dimensional and flat submodules $\Psi_1 \subset \Omega$,*

(ii) *the family of all finite-dimensional submodules $\Psi_2 \subset \Omega$ satisfying $\mathcal{L}_D\Psi_2 \subset \Psi_2$ $(0 \neq D \in \Omega^\perp)$,*

(iii) *the family of all finite-dimensional submodules $\Psi_3 \subset \Phi$ satisfying $\mathcal{L}_{fD}\Psi_3 \subset \Psi_3$ $(0 \neq D \in \Omega^\perp)$ for all $f \in \mathcal{F}$. Especially, the module Ω^{-1} does not depend on the choice of the normal filtration Ω^* of Ω.*

Proof. Let $\mathcal{R}(\Omega)$ be the set of all $\varphi \in \Phi$ such that the family of all forms $\mathcal{L}_{fD}^k\varphi$ ($k = 0, 1, \dots; f \in \mathcal{F}$) lies in a finite-dimensional submodule of Φ. Clearly $\mathcal{R}(\Omega) \subset \Phi$ is a submodule and $\mathcal{L}_{fD}\mathcal{R}(\Omega) \subset \mathcal{R}(\Omega)$. Then the formula

$$\mathcal{L}_{fD}\varphi = f\mathcal{L}_D\varphi + \varphi(D)\,\mathrm{d}f$$

implies that necessarily $\varphi(D) = 0$ for every $\varphi \in \mathcal{R}(\Omega)$, hence $\mathcal{R}(\Omega) \subset \Omega$. Clearly $\Omega^{-1} \subset \mathcal{R}(\Omega)$ for any normal filtration Ω^*. On the other hand, if $\omega \in \Omega^{\ell+1}$ but $\omega \notin \Omega^\ell$ where $\ell \geqslant -1$, then the class $[\omega] \in \Omega^{\ell+1}/\Omega^\ell$ satisfies

$$(52) \qquad 0 \neq D^{\ell+k}[\omega] = [\mathcal{L}_D^k \omega] \in \Omega^{\ell+k}/\Omega^{\ell+k-1}$$

for any $k = 0, 1, \ldots$ and thus $\omega \notin \mathcal{R}(\Omega)$. It follows that $\mathcal{R}(\Omega) \subset \Omega^{-1}$ and (iii) is verified.

The equivalence (ii) $\iff$ (iii) follows from $\mathcal{L}_{fD}\omega = f\mathcal{L}_D\omega$ $(\omega \in \Omega)$.

It is clear that a finite-dimensional submodule $\Psi \subset \Omega$, $\Psi \not\subset \Omega^{-1}$ cannot be flat (use (52) with $\omega \in \Psi$, $\omega \notin \Omega^{-1}$). On the other hand, if $\Psi \subset \Omega$ is a submodule and $\mathcal{L}_Y\Psi \subset \Psi$ for a certain vector field Y, then $\mathcal{L}_Y \operatorname{Adj}\Psi \subset \operatorname{Adj}\Psi$. (Proof. $\operatorname{Adj}\Psi$ is generated by Ψ and all $Z \rfloor \mathrm{d}\Psi$ $(Z \in \Psi^\perp)$ whence $\mathcal{L}_Y\Psi \subset \Psi \subset \operatorname{Adj}\Psi$. Moreover,

$$\mathcal{L}_Y(Z \rfloor \mathrm{d}\Psi) = [Y, Z] \rfloor \mathrm{d}\Psi + Z \rfloor \mathrm{d}\mathcal{L}_Y\Psi$$

where $Z \rfloor \mathrm{d}\mathcal{L}_Y\Psi \subset Z \rfloor \mathrm{d}\Psi \subset \operatorname{Adj}\Psi$ and $[Y, Z] \rfloor \mathrm{d}\Psi \subset \operatorname{Adj}\Psi$ (since $[Y, Z] \in \Psi^\perp$ as follows from $0 = \mathcal{L}_Y(Z \rfloor \Psi) = [Y, Z] \rfloor \Psi + Z \rfloor \mathcal{L}_Y\Psi = [Y, Z] \rfloor \Psi$). This concludes the proof of the inclusion.) Choosing $\Psi = \Omega^{-1}$ and $Y = D$, we obtain $\mathcal{L}_D \operatorname{Adj}\Omega^{-1} \subset \Omega^{-1}$ hence $\operatorname{Adj}\Omega^{-1} \subset \Omega^{-1}$ (cf. (ii)) and thus $\operatorname{Adj}\Omega^{-1} = \Omega^{-1}$. This concludes the proof of (i). □

33. The crucial construction. We begin with the choice of a special basis of a diffiety Ω suitably adapted to a given normal filtration Ω^*. First of all, by virtue the injectivity of D (cf. (50)), there exist forms

$$\omega_{(r)}^j \in \Omega^r,\ \omega_{(r)}^j \notin \Omega^{r-1} \quad (r = 0, 1, \ldots;\ j = 1, \ldots, j_r)$$

such that the classes $D^k[\omega_{(r)}^j] = [\mathcal{L}_D^k\omega_{(r)}^j] \in \Omega^\ell/\Omega^{\ell-1}$ (where $k = 0, \ldots, \ell - r$ $j = 1, \ldots, j_r$) may be used for a basis of the module $\Omega^\ell/\Omega^{\ell-1}$ for any $\ell \geqslant 0$. Choosing moreover a basis $\mathrm{d}x^1, \ldots, \mathrm{d}x^a$ $(a \geqslant 0)$ of Ω^{-1}, all forms

$$\mathrm{d}x^i\ (i = 1, \ldots, a), \quad \mathcal{L}_D^k\omega_{(r)}^j\ (k + r \leqslant \ell;\ j = 1, \ldots, j_r)$$

constitute a basis of Ω^ℓ (for any $\ell \geqslant -1$). However, it is useful to introduce also the alternative notation

$$\omega_{k+r}^{j_1+\ldots+j_{r-1}+j} = \mathcal{L}_D^k\omega_{(r)}^j \in \Omega^{k+r} \quad (j = 1, \ldots, j_r).$$

This is a mere formal measure ensuring the simple rule $\mathcal{L}_D\omega_s^i \equiv \omega_{s+1}^i$. Recall that the forms $\mathrm{d}x^1, \ldots, \mathrm{d}x^a$ together with all ω_s^i $(0 \leqslant s \leqslant \ell,\ 1 \leqslant i \leqslant j_1 + \ldots + j_\ell)$ may be used for a basis of Ω^ℓ $(\ell \geqslant -1)$. Especially the *initial forms*

$$\omega_{(r)}^j \equiv \omega_r^{i+j} \quad (r = 0, 1, \ldots; i = j_1 + \ldots + j_{r-1}; j = 1, \ldots, j_r)$$

(with $k = 0$ in the above formulae) should be pointed out. Since $j_\ell = 0$ for all ℓ large enough (as follows from the second requirement (49)), there is only a finite number $\mu = \mu(\Omega) = j_0 + j_1 + \dots$ (a finite sum) of them. (The constant $\mu(\Omega)$ does not actually depend on the choice of the filtration but it will not be needed in future.)

Returning now to the topic of Section 28, we choose $\mathbf{N} = \mathbf{R}^\infty$ and a diffiety for the relevant subset $\Omega \subset \Phi$. Let an embedded curve

$$\mathbf{P}_0 : p = p_0(t) = \{t^1(t), t^2(t), \dots\} \in \mathbf{R}^\infty \quad (0 \leqslant t \leqslant 1)$$

be "nearly" an extremal in the sense that $p_0^*\omega \equiv 0$ ($\omega \in \Omega$) but (46) is satisfied only "modulo Ω^{-1}". In correct terms: denoting

$$d(\lambda - \bar\omega) \cong \sum a_s^j \,\mathrm{d}x \wedge \omega_s^j \quad (\text{modulo } \Omega^{-1} \text{ and } \Omega \wedge \Omega) \tag{53}$$

(where $\mathrm{d}x \notin \Omega$ and the basis ω_s^j of Ω/Ω^{-1} are fixed) we suppose only

$$p_0^* Z \rfloor \sum a_s^j \,\mathrm{d}x \wedge \omega_s^j \equiv 0 \quad (Z \text{ arbitrary}), \tag{54}$$

that is, $p_0^* a_s^j \equiv 0$ for all j and s.

Since $da_s^j \cong Da_s^j \cdot \mathrm{d}x$ (mod Ω), the last identity implies $0 = p_0^* da_s^j = p_0^* Da_s^j \cdot p_0^* \,\mathrm{d}x$ hence $p_0^* Da_s^j \equiv 0$ and thus $p_0^* D^k a_s^j \equiv 0$ for all k. Consequently, if $a_S^J \omega_S^J$ is a particular summand in (54) with $\omega_S^J = \mathcal{L}_D \omega_{S-1}^J$ *not* an initial form, then

$$p_0^* Z \rfloor d(a_S^J \omega_{S-1}^J) = p_0^* Z \rfloor (Da_S^J \cdot \mathrm{d}x \wedge \omega_{S-1}^J + a_S^J \,\mathrm{d}x \wedge \omega_S^J) = 0$$

and it follows that the original form $\bar\omega$ in (53) can be replaced by $\bar\omega - a_S^J \omega_{S-1}^J$ without destroying (54) but then the original summand $a_S^J \omega_S^J$ turns into the lower order term $-Da_S^J \omega_{S-1}^J$. Repeatedly applying this reduction, the procedure terminates when only the initial summands survive in (53), that is, when we obtain

$$\mathrm{d}(\lambda - \bar\omega) \cong \sum e_{(r)}^j \cdot \omega_{(r)}^j \wedge \mathrm{d}x \quad (\text{mod } \mathcal{R}(\Omega) \text{ and } \Omega \wedge \Omega) \tag{55}$$

in a slightly changed notation. Clearly $p_0^* e_{(r)}^j = 0$ (and thus $p_0^* D^k e_{(r)}^j \equiv 0$) is true for our curve $\mathbf{P}_0$ as before. But the point is that the form $\bar\omega$ satisfying the congruence (55) is unique modulo Ω^{-1} (trivial, use the basis ω_s^j together with the rule $\mathrm{d}\omega_s^j \cong \mathrm{d}x \wedge \omega_{s+1}^j$). So the resulting form $\bar\omega$ does not depend on $\mathbf{P}_0$ or, equivalently, $p^* e_s^j \equiv 0$ for all curves $\mathbf{P} : p = p(t) \in \mathbf{R}^\infty$ which are "nearly" extremals.

Finally, let $\bar\omega = \sum b_s^j \omega_s^j + \sum b^i \,\mathrm{d}x^i$ be a form satisfying (55). Recall that $b_s^j \in \mathcal{F}$ are uniquely determined but $b^i \in \mathcal{F}$ may be (as yet) arbitrary. If $\mathbf{P}_0$ is a "nearly" extremal and $\mathrm{d}\,\lambda \cong \sum c^i \,\mathrm{d}x^i \wedge \mathrm{d}x$ (modulo all ω_s^j) then the original requirement (46) reads

$$p_0^* Z \rfloor d(\lambda - \bar\omega) = p_0^* \sum (c^i + Db^i) Zx^i \cdot p_0^* \,\mathrm{d}x = 0. \tag{56}$$

However, the identities $p_0^*(c^i + Db^i) = p_0^* c^i + \frac{\mathrm{d}}{\mathrm{d}t} p_0^* b^i \equiv 0$ can be always satisfied by a proper choice of b^i. We conclude that $\mathbf{P}_0$ is in reality a true extremal.

34. Summary and definition. *To every $\lambda \in \Phi$ there exists a unique modulo $\mathcal{R}(\Omega)$ form $\bar{\omega} \in \Omega$ satisfying* (55). *Then a curve $\mathbf{P}_0 \colon p = p_0(t) \in \mathbf{R}^\infty$ $(0 \leqslant t \leqslant 1)$ is an extremal if and only if $p_0^* \omega \equiv 0$ $(\omega \in \Omega)$ and $p_0^* e^j_{(r)} \equiv 0$ (hence $p_0^* D^k e^j_{(r)} \equiv 0$) are satisfied.* The form $\xi = \lambda - \bar{\omega}$ may be regarded as a true generalization of the concept of *$\mathcal{PC}$ forms for the constrained variational integral* (3), $e^j_{(r)}$ (or even all $D^k e^j_{(r)}$) are *$\mathcal{EL}$ operators*, $D^k e^j_{(r)} \equiv 0$ is the (*infinitely prolonged*) *$\mathcal{EL}$ system*. The *inverse problem* consists in the reverse determination of the form $\lambda \in \Phi$ (or better, of the $\mathcal{PC}$ form ξ) if the diffiety Ω and the subspace $\mathbf{E} \subset \mathbf{R}^\infty$ consisting of all points which satisfy the $\mathcal{EL}$ system $D^k e^j_{(r)} \equiv 0$ are given in advance.

35. Example. Denoting by $x, y, y_0, z_0, y_1, z_1, \ldots$ the coordinates in $\mathbf{R}^\infty$, let $\Omega \subset \Phi$ be the submodule generated by all forms $\eta = \mathrm{d}y - u\,\mathrm{d}x$, $\eta_s \equiv \mathrm{d}y_s - y_{s+1}\,\mathrm{d}x$, $\zeta_s \equiv \mathrm{d}z_s - z_{s+1}\,\mathrm{d}x$ $(s = 0, 1, \ldots)$ where $u = u(x, y, y_0, z_0) \in \mathcal{F}$. Clearly $\Omega^\perp$ consists of all multiples of $D = \partial/\partial x + u\partial/\partial y + \sum y_{s+1}\partial/\partial y_s + \sum z_{s+1}\partial/\partial z_s$. One can easily see that $\mathcal{L}_D \eta_s \equiv \eta_{s+1}$, $\mathcal{L}_D \zeta_s \equiv \zeta_{s+1}$ so that Ω is a diffiety (namely the infinite prolongation of the underdetermined differential equation $\mathrm{d}y \,/\, \mathrm{d}x = u(x, y, y_0, z_0)$ with unknown functions y, y_0, z_0). We shall suppose $\partial u/\partial z_0 \neq 0$, for certainty. Then the modules $\Omega^\ell \equiv 0$ $(\ell < 0)$, $\Omega^0 = \{\eta\}$ of all multiples of η, and $\Omega^\ell = \{\eta, \eta_0, \zeta_0, \ldots, \eta_{\ell-1}, \zeta_{\ell-1}\}$ $(\ell > 0)$ provide a normal filtration of Ω. The forms $\omega^1_{(0)} = \eta \in \Omega^0$, $\omega^1_{(1)} = \eta_0 \in \Omega^1$ are the initial ones (hence $j_0 = j_1 = 1$, $j_r \equiv 0$ if $r > 1$) and the family $\omega^1_k \equiv \mathcal{L}^k_D \eta \in \Omega^k$, $\omega^2_k \equiv \mathcal{L}^k_D \eta_0 \in \Omega^{k+1}$ $(k = 0, 1, \ldots)$ may serve for a suitable basis of Ω.

We shall deal with the variational problem (3) where $\lambda = f(x, y, y_0, z_0)\,\mathrm{d}x$. One can then easily find the $\mathcal{PC}$ form $\xi = f\,\mathrm{d}x + (\partial f/\partial z_0)/(\partial u/\partial z_0) \cdot \eta$ and the $\mathcal{EL}$ operators

$$e^1_0 = \frac{\partial f}{\partial y_0} - \frac{\partial f}{\partial z_0}\frac{\partial u}{\partial y_0} Big/ \frac{\partial u}{\partial z_0}, \quad e^2_1 = \frac{\partial f}{\partial y} - \frac{\partial u}{\partial y}\Big/\frac{\partial u}{\partial z_0} - D\left(\frac{\partial f}{\partial z_0}\Big/\frac{\partial u}{\partial z_0}\right)$$

of the zeroth and first order, respectively. After this preparatory result, we pass to the inverse problem proper.

Given data for the inverse problem consist of the prescribed constraint equation $\mathrm{d}y \,/\, \mathrm{d}x = u$ (i.e., of the diffiety Ω) together with the $\mathcal{EL}$ system generated by a zeroth order equation $z_0 = v(x, y, y_0)$ and a special first order equation $\mathrm{d}y_0 \,/\, \mathrm{d}t = w(x, y, y_0)$ (since the equations $z_0 = v$, $\mathrm{d}z_0 \,/\, \mathrm{d}x = Dv$ permit to eliminate the arguments z^0, $\mathrm{d}z^0 \,/\, \mathrm{d}z$). The subspace $\mathbf{E} \subset \mathbf{R}^\infty$ corresponding to the infinite prolongation of the $\mathcal{EL}$ system is defined by the equations $D^k(z_0 - v) = z_k - D^k v = 0$, $D^k(y_1 - w) = y_{k+1} - D^k w = 0$ and it follows that x, y, y_0 may be used for coordinates on $\mathbf{E}$. Alternatively, x, h^1, h^2 (where $h^i \equiv h^i(x, y, y_0)$ are independent first integrals of extremals) may be used as well. Recall the property $\partial h^i/\partial x + \bar{u}\partial h^i/\partial y + w\partial h^i/\partial y_0 \equiv 0$ ($\bar{u} = u(x, y, y_0, v)$) of these first integrals. It is equivalent to the useful identity

$$(57) \qquad Dh^i \equiv (u - \bar{u})\partial h^i/\partial y + (y_1 - w)\partial h^i/\partial y_0.$$

In order to determine the variational integral, we shall search for the relevant $\mathcal{PC}$ form ξ or better, for its differential $\mathrm{d}\xi$.

According to the above results, $\mathrm{d}\xi$ can be expressed in terms of the variables x, y, y_0, z_0 or, alternatively, in terms of x, h^1, h^2, $e = z_0 - v$. Following the general suggestions of the subsequent Section 37, we may assume

$$\mathrm{d}\xi = H\,\mathrm{d}h^1 \wedge \mathrm{d}h^2 + dK \wedge de + \mathrm{d}A \wedge \mathrm{d}x + \sum \mathrm{d}B^i \wedge \mathrm{d}h^i + \mathrm{d}C \wedge de$$

where $H = H(h^1, h^2)$, $K = K(x, h^1, h^2)$, and the functions A, B^1, B^2, C of variables x, h^1, h^2, e vanish at the hyperplane $e = 0$ and satisfy $\partial A/\partial e = \partial K/\partial x$ at the hyperplane $e = 0$. Then, if we deal with a $\mathcal{PC}$ form (i.e., if the congruence (55) with $\lambda - \bar{\omega} = \xi$, $\omega^1_{(0)} = \eta$, $\omega^1_{(1)} = \eta_0$ and appropriate $e^1_{(0)}, e^1_{(0)} \in \mathcal{F}$ is satisfied), the inverse problem is "almost resolved" in the sense that the $\mathcal{EL}$ system corresponding to $\mathrm{d}\xi$ always vanish on the prescribed subspace $\mathbf{E}$ (see Section 37 below) so that only some additional "nondegeneracy" is needed to ensure the exact coincidence.

Let us turn to explicit calculations. The condition $\partial A/\partial e = \partial K/\partial x$ (ate $= 0$) is equivalent to $A = e(\partial K/\partial x + L)$ where L is an unknown function vanishing at the hyperplane $e = 0$. Inserting this A into $\mathrm{d}\xi$, we conclude

$$(58) \qquad \begin{aligned} \mathrm{d}\xi &= H\,\mathrm{d}h^1 \wedge \mathrm{d}h^2 + \left(dK - \frac{\partial K}{\partial x}\,\mathrm{d}x\right) \wedge \mathrm{d}e \\ &+ \left(ed\frac{\partial K}{\partial x} + d(eL)\right) \wedge \mathrm{d}x + \sum \mathrm{d}B^i \wedge \mathrm{d}h^i + \mathrm{d}C \wedge \mathrm{d}e\,. \end{aligned}$$

The remaining condition (55) is clearly equivalent to the congruence

$$D\rfloor\,\mathrm{d}\xi \cong 0 \quad (\mathrm{mod}\ \mathrm{d}h^1 - Dh^1\,\mathrm{d}x, \mathrm{d}h^2 - Dh^2\,\mathrm{d}x).$$

hence to the congruence $D\rfloor\,\mathrm{d}\xi \cong 0$ (mod $\mathrm{d}x, \mathrm{d}h^1, \mathrm{d}h^2$), since always $D\rfloor\,\mathrm{d}\xi \in \Omega$. In terms of the coordinates x, h^1, h^2, e, we obtain the sole condition

$$(59) \qquad \left(\frac{\partial K}{\partial h^1} - \frac{\partial B^1}{\partial e}\right) Dh^1 + \left(\frac{\partial K}{\partial h^2} - \frac{\partial B^2}{\partial e}\right) Dh^2 + \frac{\partial(eL)}{\partial e} + DC = 0,$$

the coefficient of de in the form $D\rfloor\,\mathrm{d}\xi$. Here only the summand $DC = \ldots + \partial C/\partial e \cdot De = \ldots + \partial C/\partial e.z_1$ may depend on the variable z_1 (as follows from (57)) so that necessarily $\partial C/\partial e = 0$ whence $C = 0$ (since $C = 0$ at the hyperplane $e = 0$). Inserting $C = 0$ and (57) into (59) one can obtain two requirements

$$
\begin{gathered}
\left(\frac{\partial K}{\partial h^1} - \frac{\partial B^1}{\partial e}\right)\frac{\partial h^1}{\partial y_0} + \left(\frac{\partial K}{\partial h^2} - \frac{\partial B^2}{\partial e}\right)\frac{\partial h^2}{\partial y_0} = 0, \\
\left(\frac{\partial K}{\partial h^1} - \frac{\partial B}{\partial e}\right)\frac{\partial h^1}{\partial y} + \left(\frac{\partial K}{\partial h^2} - \frac{\partial B^2}{\partial e}\right)\frac{\partial h^2}{\partial y_0} = \frac{1}{u-\bar{u}}\frac{\partial (eL)}{\partial e}
\end{gathered} \tag{60}
$$

for the unknown functions K, B^1, B^2, L (by looking for the variable y_1). They constitute a very favourable system with a lot of solutions. In any case, the form

$$\xi = \int H\,\mathrm{d}h^1 \cdot \mathrm{d}h^2 + K\mathrm{d}e + A\,\mathrm{d}x + \sum B^i h^i + \mathrm{d}V \;\; (A = e(\frac{\partial K}{\partial x} + L))$$

arising by integration is a $\mathcal{PC}$ form to the variational integral (3) with $\lambda = f\,\mathrm{d}x$,

$$f = \int H\,\mathrm{d}h^1 \cdot Dh^2 + KDe + e\left(\frac{\partial K}{\partial x} + L\right) + \sum B^i Dh^i + DV \tag{61}$$

and the relevant $\mathcal{EL}$ system vanishes on the given subspace $\mathbf{E} \subset \mathbf{R}^\infty$. Altogether taken, two troubles still remain to be discussed.

First, the function (61) may depend on the higher order variables y_1, z_1. Using (57), the explicit formula

$$f = \ldots + \left\{\int H\,\mathrm{d}h^1\frac{\partial h^2}{\partial y_0} + \sum B^i \tfrac{\partial h^i}{\partial y_0} + \tfrac{\partial V}{\partial y_0}\right\} y_1 + \left\{K + \frac{\partial V}{\partial e}\right\}\left(z_1 - y_1\frac{pv}{\partial y_0}\right)$$

gives two necessary and sufficient conditions $\{\ldots\} = 0$ if we insist on the functions of the special kind $f = f(x, y, y_0, z_0)$. It seems to be not impossible to analyze these conditions in more detail (e.g., the latter gives $V = U - eK$ where U and K depend only on x, h^1, h^2) but we are not daring enough to do it here.

Second, let us look for the $\mathcal{EL}$ system corresponding to the solution (61) of the inverse problem. According to the general theory, it is generated by the coefficients e_1^0, e_1^1 of the form $D\rfloor\,\mathrm{d}\xi = \sum e^j_{(r)}\omega^j_{(r)} = e_1^0\eta + e_1^1\eta_0$. But it is simpler to use the equivalent development $D\rfloor\,\mathrm{d}\xi \cong P\,\mathrm{d}h^1 + Q\,\mathrm{d}h^2$ (mod $\mathrm{d}x$). One can then find

$$
\begin{aligned}
P &= \left(-H + \frac{\partial B^1}{\partial h^2} - \frac{\partial B^2}{\partial h^1}\right)Dh^2 + \left(\frac{\partial B^1}{\partial e} - \frac{\partial K}{\partial h^1}\right)De - \frac{\partial A}{\partial h^1} + \frac{\partial B^1}{\partial x}, \\
Q &= \left(H - \frac{\partial B^1}{\partial h^2} + \frac{\partial B^2}{\partial h^1}\right)Dh^1 + \left(\frac{\partial B^2}{\partial e} - \frac{\partial K}{\partial h^2}\right)De - \frac{\partial A}{\partial h^2} + \frac{\partial B^2}{\partial x}.
\end{aligned}
$$

Here $\partial A/\partial h^i \equiv e(\partial K/\partial x + L)$, Dh^i can be expressed by (57), and $B^i \equiv eC^i$ (with appropriate $C^i \equiv C^i(x, h^1, h^2, e)$) vanish on $\mathbf{E}$ by supposition, so that both functions

P and Q do vanish on $\mathbf{E}$. But in general the system $P = Q = 0$ (clearly equivalent to the $\mathcal{EL}$ system $e_1^0 = e_1^1 = 0$) need not be equivalent to the prescribed system $y_1 - w = z_0 - v$ $(= e) = 0$ owing to the presence of the terms De. A necessary condition for the equivalence is

$$\frac{\partial B^1}{\partial e} - \frac{\partial K}{\partial h^1} = \frac{\partial B^2}{\partial e} - \frac{\partial K}{\partial h^2} = 0.$$

If this condition is satisfied then the system $P = Q = 0$ is an algebraic consequence of the prescribed system $y_1 - w = z_0 - v = 0$ so that the converse (and thus the equivalence) takes place if the solution is nondegenerate enough (a certain Jacobian should not vanish). It is interesting to note that it implies both requirements (60) (and also $\partial(eL)/\partial e = 0$ hence $L = 0$) and permits to determine the solution of the inverse problem in terms of quadratures. We will not pass to obvious details here.

So altogether taken, in addition to all solutions of the inverse problem, even certain "weakened solutions" can be obtained.

36. Example. Let $x, z, y_0, y_1, \ldots$ be coordinates in $\mathbf{R}^\infty$. We shall deal with the submodule $\Omega \subset \Phi$ generated by all forms $\eta = \mathrm{d}y - u\,\mathrm{d}x$, $\eta_s \equiv \mathrm{d}y_s - y_{s+1}\,\mathrm{d}x$ $(s = 0, 1, \ldots)$ where $u = u(x, y, y_0, y_1)$. Clearly $\Omega^\perp$ consists of all multiples of the vector field $D = \partial/\partial x + u\partial/\partial y + \sum y_{s+1}\partial/\partial y_s$. One can easily see that Ω is a diffiety (the infinite prolongation of the *Monge equation* $\mathrm{d}y\,/\,\mathrm{d}x = u(x, y, y_0, \mathrm{d}y_0\,/\,\mathrm{d}x)$) and the submodules $\Omega^\ell \equiv 0$ $(\ell < 0)$, $\Omega^\ell \equiv \{\eta - \partial u/\partial y_1 \cdot \eta_0, \eta_0, \ldots, \eta_{\ell-1}\}$ $(\ell \geqslant 0)$ provide a normal filtration of Ω. There is only one initial form, e.g., the form $\omega^1_{(0)} = \eta - \partial u/\partial y_1 \cdot \eta_0$ (thus $j_0 = 1$, $j_r \equiv 0$ if $r > 0$).

We shall deal with the constrained variational integral (3) where $\lambda = f\,\mathrm{d}x$, $f = f(x, y, y_0)\,\mathrm{d}x$. One can then easily find the $\mathcal{PC}$ form $\xi = f\,\mathrm{d}x + a\omega^1_{(0)}$ where

$$a = \left(\frac{\partial f}{\partial y_0} + \frac{\partial f}{\partial y}\frac{\partial u}{\partial y_1}\right) \Big/ \left(\frac{\partial u}{\partial y_0} + \frac{\partial u}{\partial y_1}\frac{\partial u}{\partial y} - D\frac{\partial u}{\partial y_1}\right),$$

and the $\mathcal{EL}$ operator $e_0^1 = \partial f/\partial y - a\partial u/\partial z - Da$ of exactly the third order so that the equation $e_0^1 = 0$ may be equivalently expressed as $y_3 - g = 0$ where $g = g(x, y, y_0, y_1, y_2)$ and the infinitely prolonged $\mathcal{EL}$ system consists of all equations $y_{k+3} = D^k g$. The variables x, y, y_0, y_1, y_2 may be used for coordinates on $\mathbf{E}$.

Passing to the inverse problem, the functions u (that is, the diffiety Ω) and the equation $y_3 - g = 0$ (the subspace $\mathbf{E}$) are given in advance and we search for the relevant $\mathcal{PC}$ form ξ. It may be expressed in terms of the variables x, y, y_0, y_1, y_2 but also alternatively, in terms of the variables $x, h^1, \ldots, h^3$ where $h^1, \ldots, h^3$ are independent first integral for extremals. As follows from the next Section 37, we may assume $\mathrm{d}\xi = \sum dH^i \wedge \mathrm{d}h^i$ and thus

$$\xi = \sum H^i\,\mathrm{d}h^i + \mathrm{d}V \ \ (H^i \equiv H^i(h^1, \ldots, h^4), V = V(x, h^1, \ldots, h^4)).$$

Then (55) means that $D\rfloor\,\mathrm{d}\xi \cong 0 \pmod{\omega^1_{(0)}}$. This congruence can be expressed as $\sum(DH^i\,\mathrm{d}h^i - Dh^i\,\mathrm{d}h^i) =$

$$(y_3 - g)\sum H^{ij}\left(\frac{\partial h^i}{\partial y_2}\,\mathrm{d}h^i - \frac{\partial h^i}{\partial y_2}\,\mathrm{d}h^j\right) \cong 0 \quad \left(H^{ij} \equiv \frac{\partial H^j}{\partial h^i} - \frac{\partial H^i}{\partial h^j}\right)$$

if one employs the identities $DH^i \equiv \sum \partial H^i/\partial h^j \cdot Dh^j$, $Dh^j \equiv (y_3 - g)\partial h^j/\partial y_2$. Even more explicitly, the last congruence is equivalent to the system

$$\sum\left(\frac{\partial h^j}{\partial y_2}\left(\frac{\partial h^i}{\partial y_0} + \frac{\partial u}{\partial y_1}\frac{\partial h^i}{\partial z}\right) - \frac{\partial h^i}{\partial y_2}\left(\frac{\partial h^j}{\partial y_0} + \frac{\partial u}{\partial y_1}\frac{\partial h^i}{\partial z}\right)\right)H^{ij} \equiv 0,$$

$$\sum\left(\frac{\partial h^j}{\partial y_2}\frac{\partial h^i}{\partial y_1} - \frac{\partial h^j}{\partial y_2}\frac{\partial h^i}{\partial y_1}\right)H^{ij} \equiv 0,$$

as we obtain by separating the variable y_3. Here the coefficients may depend on x but the sought functions H^{ij} must be independent of it. So we find ourselves in a quite analogous situation as in Sections 11, 12 and the following analysis leads to a similar compatibility problem as in the Douglas case of Section 14. (Instead of first integrals, one may also use the original coordinates x, y, y_0, y_1, y_2. Then the resulting theory closely simulates Sections 3–5, 7.) We omit all details.

37. Concluding directions to the inverse problem. Given are a diffiety $\Omega \subset \Phi$ and a subspace $\mathbf{E} \subset \mathbf{R}^\infty$. We suppose that the generating vector field $D \in \Omega^\perp$ is *tangent* to $\mathbf{E}$ (if some $e \in \mathcal{F}$ vanishes on $\mathbf{E}$ then De also vanishes on $\mathbf{E}$) so that there is a vector field F on $\mathbf{E}$, the restriction of D. We choose a fixed normal filtration Ω^* and the relevant initial forms $\omega^j_{(r)}$ as a mere technical tool.

First assume $\mathbf{E}$ to be of a finite dimension $a+1$ (that is, the $\mathcal{EL}$ system to be a determined one) which is the common classical case. Concerning the coordinates, let $h^1, \ldots, h^a \in \mathcal{F}$ be independent first integrals of extremals (i.e., $\mathrm{d}h^1 \wedge \ldots \wedge \mathrm{d}h^a \neq 0$ and all Dh^i vanish on $\mathbf{E}$ or, equivalently, $Fh^i \equiv 0$) and let $x \in \mathcal{F}$ be a function transverse to extremals (i.e., satisfying $Dx \neq 0$, and we shall even suppose $Dx = 1$ for technical reasons). Then $x, h^1, \ldots, h^a$ may be used for coordinates on $\mathbf{E}$. We shall moreover suppose that there are $e^1, e^2, \ldots \in \mathcal{F}$ vanishing on $\mathbf{E}$ such that the complete family $x, h^1, \ldots, h^a, e^1, e^2, \ldots$ may be used for coordinates on the total space $\mathbf{R}^\infty$. (The choice of the coordinates is a matter of art for every problem under consideration. In principle they may be quite arbitrary but in reality a suitable choice essentially simplifies the calculations.) Given Ω and $\mathbf{E}$, we search for the $\mathcal{PC}$ form ξ modulo the residuum $\mathcal{R}(\Omega)$. Since $F\rfloor\,\mathrm{d}\xi \cong 0$ (cf. (55) and realize that $e^j_{(r)} \equiv 0$ on $\mathbf{E}$), the restriction of $\mathrm{d}\xi$ on $\mathbf{E}$ can be expressed in terms of $h^1, \ldots, h^a$ whence

$$\xi \cong \sum H^i\,\mathrm{d}h^i + \mathrm{d}V \quad (H^i \equiv H^i(h^1, \ldots, h^a),\ V = V(x, h^1, \ldots, h^a))$$

is valid modulo $\mathcal{R}(\Omega)$ for the *restriction of* ξ *to the subspace* $\mathbf{E}$. It follows that *at the points of* $\mathbf{E}$ we have

$$\xi \cong \sum H^i \,\mathrm{d}h^i + \sum K^i \,\mathrm{d}e^i + \mathrm{d}V \quad (K^i \equiv K^i(x, h^1, \dots, h^a))$$

and at quite general points, that is, in the *whole space* necessarily

$$\xi \cong \sum H^i \,\mathrm{d}h^i + \sum K^i \,\mathrm{d}e^i + \varphi + \mathrm{d}V$$

where $\varphi \in \Phi$ is a certain form vanishing at the points of $\mathbf{E}$, that is, $\varphi = A\,\mathrm{d}x + \sum B^i\,\mathrm{d}h^i + \sum C^i\,\mathrm{d}e^i$ with the coefficients A, B^i, C^i vanishing on $\mathbf{E}$. Consequently,

$$\text{(62)} \quad \mathrm{d}\xi \cong \sum dH^i \wedge \mathrm{d}h^i + \sum dK^i \wedge \mathrm{d}e^i + \mathrm{d}A \wedge \mathrm{d}x + \sum \mathrm{d}B^i \wedge \mathrm{d}h^i + \sum \mathrm{d}C^i \wedge \mathrm{d}e^i$$

with coefficients of the above mentioned special kind. For the form (62), the original requirement $F \rfloor \mathrm{d}\xi \cong 0$ simplifies to the boundary condition

$$\text{(63)} \qquad \frac{\partial K^i}{\partial x} \equiv \frac{\partial A}{\partial e^i} \quad \text{at the subspace } \mathbf{E},$$

as one can see by direct calculation. Let us turn to the condition (55) which ensures that ξ is indeed a $\mathcal{PC}$ form. It may be also expressed by the congruence $D \rfloor \mathrm{d}\xi \cong 0$ (mod $\mathcal{R}(\Omega)$ and all $\omega^j_{(r)}$). But using (62), we explicitly obtain

$$\begin{aligned} D \rfloor \mathrm{d}\xi \cong & \sum \left(\frac{\partial H^i}{\partial h^j} - \frac{\partial H^j}{\partial h^i} \right) Dh^j \alpha^j \\ & + \sum \frac{\partial K^i}{px} \beta^i + \sum \frac{\partial K^i}{\partial h^j} (Dh^j \beta^i - De^i \alpha^j) \\ & - \sum \frac{\partial A}{\partial h^i} \alpha^i - \sum \frac{\partial A}{\partial e^i} \beta^i \\ & + \sum \frac{\partial B^i}{\partial x} \alpha^i + \sum \left(\frac{\partial B^i}{\partial h^j} - \frac{\partial B^j}{\partial h^i} \right) Dh^j \alpha^i + \sum \frac{\partial B^i}{\partial e^j} (De^j \alpha^i - Dh^i \beta^j) \\ & + \sum \frac{pC^i}{\partial x} \beta^i + \sum \frac{\partial C^i}{\partial h^j} (Dh^j \beta^i - Dh^i \alpha^j) + \sum \left(\frac{\partial C^i}{\partial e^j} - \frac{\partial C^j}{\partial e^i} \right) De^j \beta^i \end{aligned}$$

(mod $\mathcal{R}(\Omega)$) where $\alpha^i \equiv \mathrm{d}h^i - Dh^i\,\mathrm{d}x$, $\beta^i \equiv \mathrm{d}e^i - De^i\,\mathrm{d}x \in \Omega$. The forms a^i, β^i can be expressed as linear combinations of the forms ω^j_s. Then the condition (55) means that in the complete sum only the initial summands $\omega^j_{(r)}$ survive. This provides a large system of linear partial differential equations for the unknown functions H^i, K^i, A, B^i, C^i. If this system is satisfied then we have a formula of the kind

$$D \rfloor \mathrm{d}\xi \cong \sum e^j_{(r)} \omega^j_{(r)} \quad (\mathrm{mod}\ \mathcal{R}(\Omega)),$$

and thus the form ξ under consideration is a $\mathcal{PC}$ form to a variational integral. Moreover, the coefficients $e^j_{(r)}$ (the $\mathcal{EL}$ operators) are linearly dependent on the variables

$$Dh^j,\ De^i,\ \frac{\partial K^i}{\partial k} - \frac{\partial A}{\partial e^i},\ \frac{\partial A}{\partial h^i},\ \frac{\partial B^i}{\partial x},\ \frac{\partial C^i}{\partial x}$$

that all vanish on $\mathbf{E}$. So, altogether taken, we have a "weakened" solution of the inverse problem: the given system of equations determining the subspace $\mathbf{E}$ implies all relations $e^j_{(r)} \equiv 0$ (and thus the prolongation $D^k e^j_{(r)} \equiv 0$) but the converse need not be true. In order to ensure the exact coincidence, some additional "nondegeneracy" conditions of algebraic nature are needed: the lowest order equations of the prolonged system $D^k e^j_{(r)} \equiv 0$ (in current cases already the equations $e^j_{(r)} \equiv 0$) should be equivalent to the lowest order equations which determine the subspace $\mathbf{E}$. We abstain from more details since they cannot bring any new ideas.

If $\mathbf{E}$ is of infinite dimension (that is, we deal with an underdetermined system of differential equations) then the residuum $\mathcal{R}(\Omega/_{\mathbf{E}})$ of the restriction $\Omega/_{\mathbf{E}}$ of the diffiety Ω on $\mathbf{E}$ is generated by certain total differentials which assume the role of the previous first integrals $h^1, \ldots, h^a$. At the place of the previous variable x, a series of variables $x^1, x^2, \ldots$ should be introduced to obtain a system of coordinates $h^1, \ldots, h^a, x^1, x^2, \ldots$ on $\mathbf{E}$. With this change, the previous directions (especially the formula (62), (63)) may be accepted.

APPENDIX

38. Uniqueness of $\mathcal{PC}$ forms. Let us consider two normal filtrations $\Omega^*, \bar{\Omega}^*$ of the same diffiety Ω with the corresponding families of initial forms $\omega^j_{(r)}$ ($j = 1, \ldots, j_r$), $\bar{\omega}^j_{(r)}$ ($j = 1, \ldots, \bar{j}_r$). To a given constrained variational integral (3), we obtain the relevant $\mathcal{PC}$ forms $\xi, \bar{\xi}$ together with the families of $\mathcal{EL}$ operators e^j_r ($j = 1, \ldots, j_r$), $\bar{e}^j_r$ ($j = 1, \ldots, \bar{j}_r$), respectively. It is self-evident that the resulting $\mathcal{EL}$ systems $D^k e^j_r \equiv 0$, $D^k \bar{e}^j_r \equiv 0$ appearing after prolongation are equivalent, that is, the relevant subspaces $\mathbf{E} = \bar{\mathbf{E}} \subset \mathbf{R}^\infty$ are identical (since they can be defined in terms of the intrinsical concept (46) of an extremal).

However, the $\mathcal{PC}$ forms $\xi, \bar{\xi}$ need not be equal. Assume for certainty $\bar{\xi} \cong \xi + \sum c^j_s \omega^j_s$ (modulo $\mathcal{R}(\Omega)$). Then

$$\sum \bar{e}^j_{(r)} \bar{\omega}^j_{(r)} \wedge \mathrm{d}x \cong \left(\sum e^j_r \omega^j_{(r)} - Dc^j_s \omega^j_s - \sum c^j_s \omega^j_{s+1} \right) \wedge \mathrm{d}x$$

(modulo $\mathcal{R}(\Omega)$ and $\Omega \wedge \Omega$) after taking the exterior differential (cf. (55)). Here the forms $\bar{\omega}^j_{(r)}$ on the left can be expressed by linear combinations of the forms ω^j_s and it follows that the higher order nonvanishing coefficients c^j_s (with maximal s) of the term $\omega^j_{s+1} \wedge \mathrm{d}x$ on the right can be expressed by linear combinations of functions $\bar{e}^j_r$.

In particular, these coefficients are all vanishing on $\mathbf{E}$. Since then $Dc_s^j = 0$ for these coefficients on the subspace $\mathbf{E}$, too, we may proceed with lower order coefficients c_s^j in an analogous manner.

As a final result, all coefficients c_s^j prove to be vanishing on the subspace $\mathbf{E}$ and we conclude that $\xi \cong \bar{\xi}$ (modulo $\mathcal{R}(\Omega)$) at every point of $\mathbf{E}$. Consequently, *the restrictions of the $\mathcal{PC}$ forms $\xi, \bar{\xi}$ and thus of their differentials* $\mathrm{d}x$, $\mathrm{d}\,\bar{\xi}$ *on the subspace* $\mathbf{E} \subset \mathbf{R}^\infty$ *are identical* modulo $\mathcal{R}(\Omega)$. So it does not matter which $\mathcal{PC}$ form is used for investigations of properties of extremals since they all lie on $\mathbf{E}$.

39. Vanishing of $\mathcal{PC}$ system. Assume $\mathbf{E} = \mathbf{R}^\infty$, hence $e_r^j \equiv 0$ is identically vanishing. In virtue of (55) we have $\mathrm{d}\xi \cong \sum a_{rs}^{ij} \omega_r^i \wedge \omega_s^j$ (modulo $\mathcal{R}(\Omega)$) where the sum may be taken only over $r < s$ and $r = s$ with $i < j$. Then applying the identity $\mathrm{d}^2\,\xi \cong 0$ (mod $\mathcal{R}(\Omega)$) and using the formulae $\mathrm{d}\omega_s^j \cong \mathrm{d}x \wedge \omega_{s+1}^j$ (mod $\mathcal{R}(\Omega)$ and $\Omega \wedge \Omega$), one can easily find that necessarily $a_{rs}^{ij} \equiv 0$ are identically vanishing. It follows that $\mathrm{d}\xi \cong 0$ (modulo $\mathcal{R}(\Omega)$), hence $\xi = \sum a^i\,\mathrm{d}x^i + \mathrm{d}V$ for appropriate a^i, $V \in \mathcal{F}$, $\mathrm{d}x^i \in \mathcal{R}(\Omega)$. Choosing a function $x \in \mathcal{F}$ transverse to D, i.e., satisfying $Dx = 1$, we may write $\lambda \cong (\sum a^i Dx^i + DV)\,\mathrm{d}x$ (mod Ω) for the constrained variational integral and in particular, $\lambda \cong DV \cdot \mathrm{d}x$ is a *generalized divergence* if $\mathcal{R}(\Omega) = 0$.

40. E. Noether's theory. Let a vector field S be a *divergence symmetry* of the variational problem (3) in the sense that we suppose $\mathcal{L}_S\Omega \subset \Omega$, $\mathcal{L}_S\lambda \cong dg$ (mod Ω) for appropriate $g \in \mathcal{F}$. Let $\mathbf{P}_0 \colon p = p_0(t) \in \mathbf{R}^\infty$ $(0 \leqslant t \leqslant 1)$ be an extremal, hence $p_0^* Z \rfloor \mathrm{d}\xi = 0$ for all vector fields Z where ξ is a $\mathcal{PC}$ form (cf. (46)). It follows that

$$\begin{aligned} dp_0^* S\rfloor\xi &= p_0^*\,\mathrm{d}S\rfloor\xi = p_0^*(S\rfloor\mathrm{d}\xi + \mathrm{d}S\rfloor\xi) \\ &= p_0^*\mathcal{L}_S\xi = p_0^*\mathcal{L}_S\lambda = p_0^*\,\mathrm{d}g = \mathrm{d}p_0^* g \end{aligned}$$

and hence $S\rfloor\xi - g =$ const. *on every extremal.*

41. Variational formula. Let a two-dimensional surface lying in the space $\mathbf{R}^\infty$ be constituted by a one-parameter family of curves $\mathbf{P}(\tau) \colon p = p(t,\tau) \in \mathbf{R}^\infty$ $(a(\tau) \leqslant t \leqslant b(\tau),\ \alpha \leqslant \tau \leqslant \beta)$ that are solutions of Pfaff's system $\omega \equiv 0$ $(\omega \in \Omega)$ for every fixed value of the parameter τ. Assume moreover that $\mathrm{d}x^i \equiv 0$ $(i = 1, \ldots, a)$ on the surface (where $\mathcal{R}(\Omega) = \{\mathrm{d}x^1, \ldots, \mathrm{d}x^a\}$ as above). Then

$$\oint \xi = \iint \mathrm{d}\xi = \sum \iint e_r^j \omega_{(r)}^j \wedge \mathrm{d}x \tag{64}$$

for the $\mathcal{PC}$ form ξ by virtue of Green's formula on (55). The double integral on the right is taken over the surface, the curvilinear integral on the left is taken over its boundary: the curves $\mathbf{P}(\alpha) \colon p = p(t,\alpha)$ $(a(\alpha) \leqslant t \leqslant b(\alpha))$ and $\mathbf{P}(\beta) \colon p = p(t,\beta)$

$(a(\beta) \leqslant t \leqslant b(\beta))$ completed by the "end point curves" $p = (a(\tau), \tau)$ and $p = (b(\tau), \tau)$ $(a \leqslant t \leqslant b)$ together constitute the boundary.

If $\beta = \alpha + \psi\varepsilon$ is "infinitesimally near" to α, the surface degenerates into a thin band and the above formula yields

$$(65) \qquad \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{\mathbf{P}(\alpha+\varepsilon)} \xi\right)_{\varepsilon=0} - \xi(Z_{b(\alpha)}) + \xi(Z_{a(\alpha)}) = \sum \int_{\mathbf{P}(\alpha)} e^j_\tau Z_t \rfloor (\omega^j_{(\tau)} \wedge \mathrm{d}x)$$

where $Z_t = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} p(a(t), \alpha + \varepsilon)\big|_{\varepsilon=0}$ is the "variational vector". For the *transverse variations* satisfying $Z_t x = 0$, the right hand side simplifies to

$$(66) \qquad \sum \int_{\mathbf{P}(\alpha)} e^j_\tau \omega^j_{(\tau)}(Z_t)\, \mathrm{d}x$$

and resembles the common variational term for classical unconstrained variational problems. Like in the case of the above problems, the functions $\omega^j_{(\tau)}(Z_t)$ can be made quite arbitrary by appropriate choice of the vector field Z_t, which implies $e^j_\tau \equiv 0$ on all extremals by elementary classical arguments.

42. Integral invariants. If the curves $\mathbf{P}(\tau)$ of the preceding section are extremals, (64) simplifies to $\oint \xi = 0$. It follows easily that this identity is satisfied for any closed curve lying in the above mentioned surface or more generally, for any closed curve lying in a two-dimensional simply connected surface constituted by a one-parameter family of extremals. In equivalent terms, we conclude that $\oint \xi =$ const. if the closed curve of integration depends on a parameter and moves in the direction of the flow of extremals. (That is, ξ is a *relative integral invariant* for the $\mathcal{EL}$ system.) In still other terms, $\iint \mathrm{d}\xi =$ const. if the two-dimensional integration domain moves in the direction of the flow of extremals. (That is, $\mathrm{d}\xi$ is an *absolute integral invariant.*) Since we are working in the subspace $\mathbf{E}$, we in fact deal with uniquely determined forms ξ, $\mathrm{d}\xi$ here.

43. Geodesics field theory. A $(k+1)$-dimensional subspace $\mathbf{F} \subset \mathbf{E}$ is called a *field of extremals* if $\mathbf{F}$ is constituted by a k-parameter family of extremals $\mathbf{P}(\tau)\colon p = p(t, \tau) \in \mathbf{R}^\infty$ $(a(\tau) \leqslant t \leqslant b(\tau), \tau = (\tau^1, \dots, \tau^k)$ is varying in a subdomain of $\mathbf{R}^k)$ and moreover $\oint \xi = 0$ for every closed curve lying in $\mathbf{F}$ (where ξ is a $\mathcal{PC}$ form). Since the llast condition can be expressed by $\iint \mathrm{d}\xi = 0$ for every two-dimensional integration domain in $\mathbf{F}$, we conclude that $\mathrm{d}\xi \equiv 0$ is vanishing on $\mathbf{F}$. By comparison with classical theory, Pfaff's equation $\mathrm{d}\xi = 0$ appears as a far going generalization of the *Hamilton-Jacobi equation* for all constrained variational integrals (3). Analogously ξ may serve as a convenient substitute of the *Hilbert invariant integral* which immediately leads to the *generalized Weierstrass theory.* It seems that the 23rd *Hilbert problem* (suggestively explained but not explicitly formulated by Hilbert, cf. [Ma]) which might have consisted in a reasonable generalization of the geodesics field theory can be resolved for the constrained integrals of the kind (3). However, it is to be noted that the multiple variational integrals cause much more difficulties. Although the concept of a diffiety can be carried over to several independent variables without troubles, the investigation of $\mathcal{PC}$ forms and inverse problems still belongs to the most important and urgent mathematical impositions at all.

References

[An] *Ian. M. Anderson*: Aspects of the inverse problem to the calculus of variations. Archivum Mathematicum (Brno), vol. 24 (1988), 181–202.

[Br] *R.L. Bryant – S.S. Chern – R.B. Gardner – H.L. Goldschmidt – P.A. Griffiths*: Exterior differential systems. Math. Sc. Research Inst. Publ. 18, Springer Verlag 1991.

[Ca] *E. Cartan*: Les systémes différentielles extérieurs et leurs applications. Actualités scient. et. ind. no. 944, 1945.

[Ch] *J. Chrastina*: Formal calculus of variations on fibered manifolds. Folia Fac. Scient. Nat. Univ. Purkynianae Brunensis, Mathematica 2, Brno 1989.

[Da] *G. Darboux*: Leçons sur la théorie génerale des surfaces. Paris 1894, §§604 and 605.

[Do] *J. Douglas*: Solution of the inverse problem of the calculus of variations. Transactions AMS *50* (1941), 71–128.

[Ma] Mathematical developments arising from Hilbert problems. Proc. Symp. in Pure and Appl. Math. AMS, Vol. XXVII, 1976.

[Ol] *Peter J. Olver*: Application of Lie groups to differential equations. Graduate Text in Mathematics 107, Springer-Verlag 1989.

[Sa] *R.M. Santilli*: Foundation of theoretical mechanics I., The inverse problems in Newtonian mechanics. Texts and Monographs in Physics 18, Springer-Verlag 1978.

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