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ON STRONG REGULARITY OF RELATIONS

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Summary. There exists a natural extension of the notion of preorder from binary relations onto relations whose arities are arbitrary ordinals. In the article we find a condition under which extended preorders coincide with preorders if viewed categorically.

Keywords: Relations of type α, reflexivity, diagonality, strong regularity, homomorphism.

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In [6] the usual n-ary relations (n a positive integer) have been generalized by introducing relations of type α (α an ordinal). In [8] topologies associated with relations of type α are investigated and a property of regularity is defined for ternary relations. In the present note, for relations of type α we introduce and study a property of strong regularity which is useful for dealing with associated topologies.

Following the convention introduced by J. von Neumann, we identify ordinals with the set of their predecessors—see e.g. [1]. We use only some fundamental concepts concerning the category theory—they can be found in [4].

Given a set $G \neq \emptyset$ and an ordinal $\alpha > 0$, by a *relation* of type α on G we understand any subset $R \subseteq G^\alpha$. In other words, a relation of type α on a set G is a set of sequences of type α consisting of elements of G. The pair (G, R) is then called a *relational system* of type α. If (G, R), (H, S) are two relational systems of type α, then a *homomorphism* of (G, R) into (H, S) is any map $f\colon G \to H$ with $(x_i \mid i < \alpha) \in R \implies \big(f(x_i) \mid i < \alpha\big) \in S$.

A relation R of type α on a set G (a relational system (G, R)) is called [7]:

reflexive if R is contains all constant sequences of type α consisting of elements of G,

diagonal, if whenever $x_{ij} \in G$ for all $i, j < \alpha$, then $(x_{ii} \mid i < \alpha) \in R$, provided $(x_{ij} \mid j < \alpha) \in R$ for all $i < \alpha$ and $(x_{ij} \mid i < \alpha) \in R$ for all $j < \alpha$.

Let us note that for binary relations diagonality is equivalent to transitivity. Hence reflexive and diagonal relations of type α can be viewed as extensions of preorders. In [7] it is shown that for any ordinal $\alpha > 0$ the reflexive and diagonal relational systems of type α together with homomorphisms for a cartesian closed topological category (in the sense of [3]).

Definition. Let R be a relation of type α on a set G. The relation R (the relational system (G, R)) is called *strongly regular* if the following condition is satisfied:

if $(x_i \mid i < a) \in G^\alpha$ has the property that for each ordinal i_0, $0 < i_0 < \alpha$, there exists $(y_j \mid j < \alpha) \in R$ and an ordinal j_0, $0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $y_j \in \{x_i;\ i < i_0\}$ for each $j < j_0$, then $(x_i \mid i < \alpha) \in R$.

Unary and binary relations are always strongly regular. A ternary relation R on a set G is strongly regular iff each of the following six conditions is sufficient for $(x, y, z) \in R$:

1^0 $(x,y,t) \in R,\ (x,z,u) \in R$	4^0 $(x,x,y) \in R,\ (x,z,u) \in R$
2^0 $(x,y,t) \in R,\ (y,z,u) \in R$	5^0 $(x,x,y) \in R,\ (y,z,u) \in R$
3^0 $(x,y,t) \in R,\ (y,x,z) \in R$	6^0 $(x,y,t) \in R,\ (y,y,z) \in R.$

In [8] a regularity is defined for ternary relations by requiring only the conditions 1^0 and 2^0 to se sufficient for $(x, y, z) \in R$.

Remark. By a topology (in Čech's sense [2]) on a set G we understand any map $u\colon \exp G \to \exp G$ with $u\emptyset = \emptyset$, $X \subseteq G \implies X \subseteq uX$, $X \subseteq Y \subseteq G \implies uX \subseteq uY$. The pair (G, u) is then called a topological space. With any relation R of type α on a set G a topology u_R on G is associated in [8] as follows:

$X \subseteq G \implies u_R X = X \cup \{x \in G;$ there exists $(x_i \mid i < \alpha) \in R$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in X$ for all $i < i_0\}$.

From the results attained in [8] it immediately follows that for any pair of reflexive and strongly regular relations R, S of type α on a given set we have $R \neq S \implies u_R \neq u_S$. This fact then yields the existence of an embedding of the category of reflexive and strongly regular relational systems of type α (with homomorphisms as morphisms) into the category of topological spaces (with continuous maps [2] as morphisms).

For any ordinal $\alpha > 0$ denote by $\mathscr{R}_\alpha$ the category of reflexive, diagonal and strongly regular relational systems of type α with homomorphisms as morphisms. Obviously, $\mathscr{R}_2$ is the well-known (topological and cartesian closed) category of preordered sets. The following result shows that, in substance, by having defined $\mathscr{R}_\alpha$ for all $\alpha > 2$ we have received no other categories.

Theorem. *$\mathscr{R}_\alpha$ is isomorphic to $\mathscr{R}_2$ for each ordinal $\alpha > 2$.*

Proof. Let $\alpha > 2$ be an ordinal. For each object $(G, \varrho) \in \mathscr{R}_2$ put $F(G, \varrho) = (G, R_\varrho)$ where $R_\varrho \subseteq G^\alpha$ is defined by

$$(x_i \mid i < \alpha) \in R_\varrho \iff x_0 \,\varrho\, x_{i_0}$$

for each ordinal i_0, $0 < i_0 < \alpha$. Next, for each morphism f in $\mathscr{R}_2$ put $Ff = f$. Clearly, the reflexivity of ϱ implies the reflexivity of R_ϱ and the transitivity of ϱ implies the diagonality of R_ϱ.

Let $(x_i \mid i < \alpha) \in G^\alpha$ be a sequence with the property that for each ordinal i_0, $0 < i_0 < \alpha$, there exists $(y_j \mid j < \alpha) \in R_\varrho$ and an ordinal j_0, $0 < j_0 < \alpha$, such that

$$x_{i_0} = y_{j_0} \quad \text{and} \quad y_j \in \{x_i;\ i < i_0\}$$

for each $j < j_0$. Then for any ordinal i_0, $0 < i_0 < \alpha$, there is an ordinal $i_1 < i_0$ such that $x_{i_1} \,\varrho\, x_{i_0}$. Further, if $i_1 > 0$, there is $i_2 < i_1$ such that $x_{i_2} \,\varrho\, x_{i_1}$. Repeating this argument, after a finite number n of steps we get $x_{i_n} \,\varrho\, x_{i_{n-1}}$ where $i_n = 0$. Thus, we have

$$x_0 \,\varrho\, x_{i_{n-1}} \,\varrho\, x_{i_{n-2}} \,\varrho \cdots \varrho\, x_{i_1} \,\varrho\, x_{i_0}$$

and therefore, because of the transitivity of ϱ, $x_0 \,\varrho\, x_{i_0}$. Hence $(x_i \mid i < \alpha) \in R_\varrho$ which implies that R_ϱ is strongly regular.

We have shown that $F(G, \varrho) \in \mathscr{R}_\alpha$ whenever $(G, \varrho) \in \mathscr{R}_2$. Clearly, for any pair of objects (G, ϱ), $(H, \varrho) \in \mathscr{R}_2$, f is a homomorphism of (G, ϱ) into (H, σ) iff f is a homomorphism of (G, R_ϱ) into (H, R_σ). Therefore F is a full embedding of $\mathscr{R}_2$ into $\mathscr{R}_\alpha$. We aim at showing that F is surjective on objects.

To this end, let $(G, R) \in \mathscr{R}_\alpha$ be an object. Define $\varrho \subseteq G^2$ as follows:

$$x \,\varrho\, y \iff \text{there is } (x_i \mid i < \alpha) \in R \text{ such that } x = x_0 \text{ and } y = x_{i_0} \text{ for all ordinals } i_0,\ 0 < i_0 < \alpha.$$

The reflexivity of ϱ follows immediately from the reflexivity of R. Let $x, y, z \in G$, $x \,\varrho\, y$, $y \,\varrho\, z$. Put $t_{00} = x$, $t_{0j} = y$ whenever $0 < j < \alpha$, $t_{i0} = y$ whenever $0 < i < \alpha$, $t_{ij} = z$ whenever $i > 0$ and $j > 0$. Then

$$(t_{ij} \mid j < \alpha) \in R \text{ for all } i < \alpha \text{ and } (t_{ij} \mid i < \alpha) \in R \text{ for all } j < \alpha.$$

Consequently, because of the diagonality of R, $(t_{ii} \mid i < \alpha) \in R$. Since $t_{i_0 i_0} = z$ for each ordinal i_0, $0 < i_0 < \alpha$, we have $x \,\varrho\, z$. Hence ϱ is transitive, which yields $(G, \varrho) \in \mathscr{R}_2$.

We will show that $F(G,\varrho) = (G,R)$, i.e. $R_\varrho = R$. To this end, let $(x_i \mid i < \alpha) \in R_\varrho$. Then $x_0 \,\varrho\, x_{i_0}$ for each ordinal i_0, $0 < i_0 < \alpha$. Thus, for each ordinal i_0, $0 < i_0 < \alpha$, there is $(y_j \mid j < \alpha) \in R$ such that $x_0 = y_0$ and $x_{i_0} = y_{j_0}$ for all ordinals j_0, $0 < j_0 < \alpha$, in particular for $j_0 = 1$. Consequently, because of the strong regularity of R, we get $(x_i \mid i < \alpha) \in R$. Therefore $R_\varrho \subseteq R$.

Conversely, let $(x_i \mid i < \alpha) \in R$. For any ordinal i_0, $0 < i_0 < \alpha$, let $s(i_0) \in G^\alpha$ denote the sequence given by $s(i_0) = (y_j \mid j < \alpha)$ where $y_0 = x_0$ and $y_{j_0} = x_{i_0}$ for each ordinal j_0, $0 < j_0 < \alpha$. Clearly, $s(1) \in R$ results immediately from the strong regularity of R. Let i_0 be an ordinal with $1 < i_0 < \alpha$ and assume $s(i_0') \in R$ whenever $0 < i_0' < i_0$. For each $i < i_0$ and each $j \geqslant i_0$ put $u_{ij} = x_i$, for each $j < i_0$ and each $i \geqslant i_0$ put $u_{ij} = x_j$, for each i, j with $i_0 \leqslant i = j < \alpha$ put $u_{ij} = x_{i_0}$, and, finally, for all the other $i, j < \alpha$ put $u_{ij} = x_0$. We get a matrix (u_{ij}), $i, j < \alpha$, of the following form:

$$\begin{array}{ccccccccc}
x_0 & x_0 & x_0 & \cdots & x_0 & x_0 & x_0 & \cdots \\
x_0 & x_0 & x_0 & \cdots & x_1 & x_1 & x_1 & \cdots \\
x_0 & x_0 & x_0 & \cdots & x_2 & x_2 & x_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\
x_0 & x_1 & x_2 & \cdots & x_{i_0} & x_0 & x_0 & \cdots \\
x_0 & x_1 & x_2 & \cdots & x_0 & x_{i_0} & x_0 & \cdots \\
x_0 & x_1 & x_2 & \cdots & x_0 & x_0 & x_{i_0} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots
\end{array}$$

It is easy to see that the condition $s(i_0') \in R$ whenever $0 < i_0' < i_0$, the reflexivity of R and the strong regularity of R imply that all rows and all columns of the matrix are elements of R. Hence the diagonal is an element of R, too. But then $s(i_0) \in R$ in virtue of the strong regularity of R. Thus, according to the principle of transfinite induction, we have proved that $s(i_0) \in R$ for all ordinals i_0, $0 < i_0 < \alpha$. Consequently, $x_0 \,\varrho\, x_{i_0}$ for each ordinal i_0, $0 < i_0 < \alpha$. This yields $(x_i \mid i < \alpha) \in R_\varrho$, so we have $R \subseteq R_\varrho$. Therefore $R_\varrho = R$ and the proof is complete. □

Example. Let $(G,\varrho) \in \mathscr{R}_2$ where $G = \{0,1\}$ and $\varrho = \leqslant$. Then for $\alpha = 3$ the relation $R_\varrho \subseteq G^\alpha$ defined in the proof of Theorem clearly fulfils $R_\varrho = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,1,1)\}$.

References

[1] *Y. Bar-Hillel, A.A. Fraenkel, A. Lévy*: Foundations of Set Theory. North Holland, Amsterdam, 1973.

[2] *E. Čech*: Topological papers of Eduard Čech, Ch. 8. Academia, Prague, 1968, pp. 436–472.

[3] *H. Herrlich*: Cartesian closed topological categories. Math. Coll. Univ. Cape Town *9* (1974), 1–16.

[4] *S. Mac Lane*: Categories for the Working Mathematician. Springer-Verlag, Heidelberg-New York, 1971.
[5] *V. Novák*: On a power of relational structures. Czech. Math. J. *35* (1985), 167–172.
[6] *J. Šlapal*: Relations of type α. Zeitschr. f. math. Logik und Grundl. d. Math. *34* (1988), 563–573.
[7] *J. Šlapal*: Cartesian closedness in categories of relational systems. Arch. Math. (Basel) *52* (1989), 603–606.
[8] *J. Šlapal*: Relations and topologies. Czech. Math. J. *43* (1993), 141–150.

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