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# TENSOR APPROACH TO MULTIDIMENSIONAL WEBS

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*Abstract.* An anholonomic $(n+1)$-web of dimension $r$ is considered as an $(n+1)$-tuple of $r$-dimensional distributions in general position. We investigate a family of $(1,1)$-tensor fields (projectors and nilpotents associated with a web in a natural way) which will be used for characterization of all linear connections on a manifold preserving the given web.

*Keywords*: manifold, connection, web

*MSC 1991*: 53A60, 53B05

## 0. INTRODUCTION

A $d$-web on a manifold $M$ is usually introduced as an ordered family of $d$ differentiable foliations of the same dimension which satisfy additional conditions (the tangent distributions are in general position in $TM$). The theory of $(n+1)$-webs of codimension $r$ on a smooth $nr$-dimensional manifold $M$ was summarized by V. V. Goldberg [G]. The reached results were obtained by applying the theory of systems of differential forms and Cartan methods. A more general and in a way dual case was investigated by I. G. Shandra. His paper [Sh] is devoted to non-holonomic $(n+1)$-webs of dimension $r$ on $M_{nr}$ (the web distributions are non-holonomic in general), and to connections preserving web distributions. A web is substituted by a family of 1-forms (affinors) satisfying a set of conditions. This approach was previously used in [Ng] and [Va] where invariant tensor fields associated with a 3-web were investigated.

Our aim is to use a family of tensor fields $H_\alpha^\beta$ forming a $\{H_\alpha^\beta\}$-structure, instead of web foliations or tanget distributions, to characterize $r$-dimensional (or

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$r$-codimensional) webs on manifolds and web-preserving connections. In many considerations (e.g. the existence of canonical connections) the role of the fields $H_\alpha^\beta$ is essential but the integrability conditions are not used. So we will introduce the definition of a web in a more general setting. Distinguished web-preserving connections, the canonical $\gamma$-connections, can play a similar role by characterization of the most important classes of $(n+1)$-webs as the so called Chern connection by classification of 3-webs [Ki, Ch, Ak, G].

The existence of a $\{H_\alpha^\beta\}$-structure on a manifold $M$ is equivalent with the existence of a $Gl_r$-structure on $M$. A $\{H_\alpha^\beta\}$-structure induces both an $r$-dimensional and an $r$-codimensional $(n+1)$-webs.

Note that in general, it is possible to consider $d$-webs of dimension $r$ on an $m$-dimensional manifold where $m$ is not a multiple of $r$, or even webs consisting of foliations of different dimensions. By technical reasons, it is hardly possible to expect a nice tensor theory in the general case although special examples are known since G. Bol, and many papers of V. V. Goldberg and others are devoted to this subject. Note that if the number of foliations is not "sufficiently high" the local situation is trivial (a given web is equivalent to a web formed by parallel plane surfaces). On the other hand if a web consists of "too many" foliations it can be investigated through its sub-webs. In the case of an $r$-dimensional web on an $nr$-dimensional manifold, it is convenient to assume $d = n+1$.

We will suppose that manifolds, bundles, vector and tensor fields under consideration are smooth (of the class $C^\infty$). $M$ will denote a manifold, $TM$ its tangent bundle, $\mathfrak{X}(M)$ denotes the set of all vector fields on $M$.

## 1. Tensor fields associated with $(n+1)$-webs of dimension $r$

**Definition 1.1.** An anholonomic $(n+1)$-web of dimension $r$ (or of codimension $r$, respectively) on a $C^\infty$-$nr$-manifold $M$ is a family $\mathcal{W} = (D_0, D_1, \dots, D_n)$ of distributions of dimension (codimension) $r$ which are in general position.[1]

Web distributions $D_\alpha$, $\alpha = 0, \dots, n$ are $r$-dimensional subbundles $D_\alpha \to M$ of the tangent bundle $TM \to M$. If all subbundles $D_0, \dots, D_n$ are integrable (that is, if $X, Y \in D_\alpha$ then $[X,Y] \in D_\alpha$) we say that $\mathcal{W}$ is *holonomic*.

As morphisms, we take diffeomorphisms $f\colon M \to M'$ which preserve web distributions, $Tf(D_\alpha) = D'_\alpha$.

[1] *In general position* means that at any point, the intersection $D_\alpha \cap D_\beta$ is trivial for $\beta \neq \alpha$.

Any ordered[2] anholonomic $(n+1)$-web $\mathcal{W} = (D_0, \dots, D_n)$ of dimension $r$ is in a correspondence with a family of $(1,1)$-tensor fields $\{H_\alpha^\beta;\ \alpha, \beta \in 1, \dots, n\}$ which will be described in the following.

Any $n$-tuple of web distributions forms an almost product structure on $M_{nr}$. Let us fix an almost product structure

$$[D_1, \dots, D_n]. \tag{1.1}$$

Denote by $P_\alpha$ the corresponding projectors where $\alpha = 1, \dots, n$. Then $TM = \sum D_\alpha$, $P_\alpha : TM \to D_\alpha$, $P_\alpha X = X_\alpha$ for any vector field $X \in \mathfrak{X}(M)$. These projectors satisfy $\operatorname{im} P_\alpha = D_\alpha$,

$$P_\alpha{}^2 = P_\alpha, \quad P_\alpha P_\beta = 0, \quad \sum P_\alpha = \mathrm{id} \qquad (\alpha \neq \beta,\ \alpha, \beta = 1, \dots, n). \tag{1.2}$$

Let us choose a fixed basis

$$X_0^1, \dots X_0^r \tag{1.3}$$

of the distribution $D_0$, and let us decompose base vector fields with respect to the almost product structure (1.1):

$$P_\alpha(X_0^i) = (X_0^i)_\alpha = X_\alpha^i \in D_\alpha \qquad (\alpha = 1, \dots, n,\ \ i = 1, \dots r) \tag{1.4}$$

where we write $X_\alpha^i$ instead of $(X_0^i)_\alpha$ for the sake of simplicity. A correspondence $X_\alpha^i \mapsto X_\beta^i$, $\alpha \neq \beta$, $i = 1, \dots, r$ can be extended by linearity into a bundle isomorphism

$$B_\alpha^\beta \ :\ D_\alpha \to D_\beta.$$

Evidently, the definition of the above mappings is independent of the choice of a basis in $D_0$. With respect to composition, these bundle isomorphisms satisfy the equalities

$$B_\beta^\gamma \circ B_\alpha^\beta = B_\alpha^\gamma, \quad B_\beta^\alpha \circ B_\alpha^\beta = \mathrm{id}_{D_\alpha}, \quad B_\kappa^\gamma \circ B_\alpha^\beta = 0 \quad \text{for } \kappa \neq \beta,$$
$$P_\beta \circ B_\alpha^\beta = B_\alpha^\beta, \quad P_\kappa \circ B_\alpha^\beta = 0 \quad \text{for } \kappa \neq \beta \neq \alpha.$$

Remark 1.1. In particular, if $n = 2$ the isomorphisms $B_1^2$, $B_2^1$ can be extended by linearity to an involutory isomorphism $B$ of the whole tangent space at any point,

$$B : TM \to TM, \quad \forall X \in \mathfrak{X}(M) \quad BX = B_1^2 P_1 X + B_2^1 P_2 X, \quad B^2 X = X.$$

This is not the case for $n > 3$.

[2] By *ordered* we mean here "with a fixed order of web distributions".

Now let us introduce (1,1)-tensor fields $H_\alpha^\beta : TM \to D_\beta$ by

$$H_\alpha^\beta = B_\alpha^\beta \circ P_\alpha \qquad (\beta \neq \alpha, \quad \alpha, \beta \in \{1, \dots, n\}).$$

It can be verified that the following equalities are satisfied for $\alpha, \beta, \gamma, \kappa \in \{1, \dots, n\}$:

(1.5) $$H_\beta^\alpha \circ H_\alpha^\beta = P_\alpha, \quad \beta \neq \alpha,$$

(1.6) $$H_\beta^\gamma \circ H_\alpha^\beta = H_\alpha^\gamma, \quad \gamma \neq \beta \neq \alpha \neq \gamma,$$

(1.7) $$H_\kappa^\gamma \circ H_\alpha^\beta = 0, \qquad \gamma \neq \kappa \neq \beta \neq \alpha,$$

(1.8) $$(H_\alpha^\beta)^2 = 0, \qquad \beta \neq \alpha,$$

(1.9) $$H_\alpha^\beta \circ P_\alpha = H_\alpha^\beta, \quad \beta \neq \alpha,$$

(1.10) $$H_\beta^\gamma \circ P_\alpha = 0, \qquad \gamma \neq \beta \neq \alpha,$$

(1.11) $$H_\alpha^\beta \mid D_\alpha = B_\alpha^\beta, \quad \operatorname{im} H_\alpha^\beta = D_\beta, \quad \beta \neq \alpha.$$

The kernel of the endomorphism $H_\alpha^\beta$ is $\ker H_\alpha^\beta = \sum_\gamma D_\gamma$, $\gamma$ runs over all indexis $\{1, \dots, \hat{\alpha}, \dots, n\}$ where the symbol $\hat{\alpha}$ means that $\alpha$ is omitted. Let us use the notation

$$H_\alpha^\alpha = P_\alpha, \qquad \alpha = 1, \dots, n.$$

Then the above conditions (1.5)–(1.10), (1.2) can be rewritten in a shorter form[3]

(1.12) $$\sum H_\alpha^\alpha = \mathrm{id}, \qquad H_\kappa^\gamma \circ H_\alpha^\beta = \delta_\kappa^\beta H_\alpha^\gamma \qquad (\alpha, \beta, \gamma, \kappa \in \{1, \dots, n\})$$

where $\delta_\kappa^\beta$ is the Kronecker symbol.

**Definition 1.2.** The family of (1,1)-tensor fields satisfying (1.12) will be called a *$\{H_\alpha^\beta\}_{\alpha,\beta=1}^n$-structure of dimension $r$ on $M_{nr}$.*

Tensor fields $H_\alpha^\beta$, $\beta \neq \alpha$ are nilpotent by (1.8). Each of them determines an almost tangent structure on $M_{nr}$ and satisfies

$$D_\beta = \operatorname{im} H_\alpha^\beta \subseteq \ker H_\alpha^\beta = \sum\nolimits_\gamma D_\gamma, \quad \gamma \in \{1, \dots, \hat{\alpha} \dots, n\}.$$

Let us define $P_0$ by the formula

(1.13) $$P_0 = \tfrac{1}{n} \sum_{\alpha,\beta} H_\alpha^\beta.$$

[3] A family of 1-forms $\{H_\alpha^\beta\}_{\alpha,\beta=1,\dots,n}$ on $M$ satisfying (1.12) is called *an isotranslated $n\pi$-structure* in [Sh].

Then $P_0$ is a projector onto $D_0$. In fact,

$$\tag{1.14} \begin{aligned} P_0^2 &= \frac{1}{n^2} \sum_{\alpha,\beta,\kappa,\gamma} H_\kappa^\gamma H_\alpha^\beta = \frac{1}{n^2} \sum_{\alpha,\beta,\kappa,\gamma} \delta_\kappa^\beta H_\alpha^\gamma \\ &= \frac{1}{n^2} \sum_{\beta=1}^{n} \Big( \sum_{\alpha,\gamma} H_\beta^\gamma H_\alpha^\beta \Big) = \frac{1}{n} \sum_{\alpha,\gamma} H_\alpha^\gamma = P_0, \end{aligned}$$

and $\operatorname{im} P_0 = D_0$ since $\operatorname{im} P_0 | D_\gamma = D_0$ for any $\gamma \in \{1, \dots n\}$. In fact, using notation introduced in (1.3), (1.4) we verify that

$$\begin{aligned} P_0(X_\gamma^i) &= \frac{1}{n} \sum_{\alpha,\beta} H_\alpha^\beta X_\gamma^i = \frac{1}{n} \sum_{\alpha,\beta} H_\alpha^\beta P_\gamma X_0^i = \frac{1}{n} \sum_{\beta} H_\gamma^\beta X_0^i \\ &= \frac{1}{n} \Big( H_\gamma^\gamma X_0^i + \sum_{\beta \neq \gamma} H_\gamma^\beta X_0^i \Big) = \frac{1}{n} \Big( X_\gamma^i + \sum_{\beta \neq \gamma} X_\beta^i \Big) = \frac{1}{n} X_0^i \in D_0. \end{aligned}$$

Therefore $P_0 \colon TM \to D_0$ and $P_0 \,|\, D_0 = \mathrm{id}$.

## 2. The anholonomic $(n+1)$-web corresponding to a $\{H_\alpha^\beta\}$-structure of dimenison $r$

On the other hand, a family of $(1,1)$-tensor fields (1.12) defines an anholonomic $(n+1)$-web of dimension $r$ (or of codimension $r$, respectively). In fact, let $\{H_\alpha^\beta\}$ be a system of $(1,1)$-tensor fields satisfying (1.12) on $M$. Then $\{H_\alpha^\alpha\}_{\alpha=1}^n$ is a system of mutually orthogonal projectors:

$$(H_\alpha^\alpha)^2 = H_\alpha^\alpha, \qquad H_\alpha^\alpha H_\beta^\beta = 0 \qquad (\beta \neq \alpha).$$

Let us verify that the system yields an almost product structure

$$[D_1 = \operatorname{im} H_1^1, \dots, D_n = \operatorname{im} H_n^n].$$

Assume $X \in D_\alpha \cap D_\beta$ ($\beta \neq \alpha$, $\alpha, \beta = 1, \dots, n$). Then $X = H_\beta^\beta X = H_\beta^\beta (H_\alpha^\alpha) X = 0$. So couples of different distributions have trivial intersections. Moreover, $TM = \oplus \operatorname{im} H_\alpha^\alpha$. Further, $\{H_\alpha^\beta\}$, $\beta \neq \alpha$ is a family of almost tangent structures $H_\alpha^\beta \colon TM \to D_\beta$ on $M$,

$$(H_\alpha^\beta)^2 = 0, \qquad H_\beta^\gamma H_\alpha^\beta = H_\alpha^\gamma, \qquad H_\kappa^\gamma H_\alpha^\beta = 0 \quad (\kappa \neq \beta)$$

and the restriction $H_\alpha^\beta \,|\, D_\alpha \colon D_\alpha \to D_\beta$ is a bundle isomorphism. In fact, let $X_\alpha \in D_\alpha$. Then $H_\alpha^\beta X = H_\beta^\beta H_\alpha^\beta X \in D_\beta$. Suppose $H_\alpha^\beta X = 0$ for some $X \in D_\alpha$. Then

$X = H^\alpha_\alpha X = H^\alpha_\beta H^\beta_\alpha X = H^\alpha_\beta 0 = 0$ which proves that $\ker(H^\beta_\alpha \,|\, D_\alpha)$ is trivial. Denote these bundle isomorphisms by $B^\beta_\alpha = H^\beta_\alpha \,|\, D_\alpha$, and the rank $r = \operatorname{rk} H^\beta_\alpha = \operatorname{rk} B^\beta_\alpha$. It follows $\dim M = nr$ where $r$ is the common dimension of all $D_\alpha$, $\alpha = 1, \dots, n$. Now let us introduce

$$H^0_0 = \tfrac{1}{n} \sum_{\alpha,\beta} H^\beta_\alpha. \tag{2.1}$$

Then $H^0_0$ is a projector which can be verified by evaluation similar to those in (1.14). This projector determines an $r$-dimensional distribution, $D_0 = \operatorname{im} H^0_0$. It can be verified that $\operatorname{rk} H^0_0 = \dim D_0 = r$. In fact, let us start from any basis $(X^i_\gamma)$ of $D_\gamma$; $H^0_0 X^i_\gamma = \frac{1}{n} \sum\limits_{\beta=1}^{n} X^i_\beta$. If $X \in D_\gamma$, $X = \sum_i A_i X^i_\gamma$ we obtain equivalences

$$H^0_0 X = 0 \iff \sum_i \sum_\beta A_i X^i_\beta = 0 \iff A_i = 0$$

which prove that $H^0_0 X = 0$ for $X \in D_\gamma$ if and only if $X = 0$. Thus $H^0_0 \,|\, D_\gamma$ are isomorphisms for $\gamma = 1, \dots, n$. Using decomposition of any basis $X^i_0$ of $D_0$ with respect to the almost product structure $[D_1, \dots, D_n]$ we obtain isomorphisms $B^\alpha_0$ given by $B^\alpha_0 : X^i_0 \mapsto H^\alpha_\alpha X^i_0$, and $B^0_\alpha = (B^\alpha_0)^{-1}$.

**Proposition 2.1.** *Let $\{H^\beta_\alpha\}$ be a system of $(1,1)$-tensor fields satisfying (1.12) on an $nr$-dimensional manifold $M$. In the above notation, let $D_\alpha = \operatorname{im} H^\alpha_\alpha$, $\alpha = 0, 1, \dots, n$. Then $(D_0, D_1, \dots, D_n)$ is an anholonomic $(n+1)$-web of dimenison $r$ on $M$.*

Proof. It was verified above that $\dim D_\alpha = \operatorname{rk} H^\alpha_\alpha = r$ for $\alpha = 0, 1, \dots, n$ and that $D_\alpha \cap D_\beta = 0$ for $\alpha \neq \beta$, $\alpha, \beta = 1, \dots, n$. Now let $X \in D_0 \cap D_\alpha$, $\alpha \in \{1, \dots, n\}$. Then $X = H^\alpha_\alpha X = H^\alpha_\alpha H^0_0 H^\alpha_\alpha X = H^\alpha_\alpha \big(\frac{1}{n} \sum\limits_{\beta,\gamma} H^\gamma_\beta (H^\alpha_\alpha X)\big) = \frac{1}{n} \sum\limits_{\beta,\gamma} \delta^\gamma_\alpha \delta^\alpha_\beta H^\gamma_\beta = \frac{1}{n} H^\alpha_\alpha X$, that is, $X = \frac{1}{n} X$ which proves $D_0 \cap D_\alpha = 0$. So the distributions $D_0, \dots, D_n$ of dimension $r$ are in general position. □

Remark 2.1. Similarly, we can prove that a $\{H^\beta_\alpha\}$-structure on $M$ gives rise also to an anholonomic $(n+1)$-web of codimension $r$ formed by distributions in general position $\tilde{D}_\alpha = \ker H^\alpha_\alpha$, $\alpha = 0, 1, \dots, n$,

$$\tilde{D}_0 = \ker H^0_0 = \ker \Big(I + \sum_{\beta \neq \alpha} H^\alpha_\beta\Big), \quad \tilde{P}_0 = I - H^0_0 = \tfrac{1}{n}\Big((n-1)\, I - \sum_{\beta \neq \alpha} H^\alpha_\beta\Big),$$

$$\tilde{D}_\alpha = \ker H^\alpha_\alpha = \sum_\gamma (1 - \delta^\gamma_\alpha)\, D_\gamma, \qquad \tilde{P}_\alpha = \sum_\gamma (1 - \delta^\gamma_\alpha)\, P_\gamma, \quad \alpha = 1, \dots, n$$

where $\tilde{P}_\alpha$ denote the corresponding projectors. We can say that given an $(n+1)$-web $(D_\alpha)$ of dimension $r$ (or $(\tilde{D}_\alpha)$ of codimesion $r$) the normal bundles form a web $(TM/D_\alpha)$ of codimension $r$ (respectively a web $(TM/\tilde{D}_\alpha)$ of dimension $r$).

## 3. The principal bundle of web-adapted frames

Let $\mathcal{W}$ denote an anholonomic $(n+1)$-web of dimension $r$.

**Definition 3.1.** A frame $(X_1^i|\dots|X_n^i)$ is called *adapted* with respect to an almost product structure $[D_1,\dots,D_n]$ if $X_\alpha^i \in D_\alpha$ for $i=1,\dots,r$, $\alpha\in\{1,\dots,n\}$.

**Definition 3.2.** A frame will be called $\mathcal{W}$-*adapted*, or adapted with respect to an anholonomic web $\mathcal{W}=(D_0,D_1,\dots,D_n)$ if it is adapted to (1.1) and is "normed" in such a way that tensor fields

$$X_0^i = \sum_{\alpha=1}^{n} X_\alpha^i \qquad (i=1\dots r) \tag{3.1}$$

form a basis of $D_0$.

The family $WM$ of all $\mathcal{W}$-adapted frames constitutes a $G$-structure on $M$. Its structure group

$$\underbrace{GL(r,\mathbb{R})\overset{\times}{=}\dots\overset{\times}{=}GL(r,\mathbb{R})}_{n\text{-times}}:\begin{pmatrix} A & 0 & \dots & 0\\ 0 & A & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & A\end{pmatrix},\quad A\in GL(r,\mathbb{R})$$

(the diagonal product of $GL(r,\mathbb{R})$ $n$-times) is a subgroup of $GL(nr,\mathbb{R})$ isomorphic with $GL(r,\mathbb{R})$.

**Definition 3.3.** A web $\mathcal{W}$ will be called *regular* if the corresponding $G$-structure $WM$ of web-adapted frames is integrable (=locally flat).

**Definition 3.4.** A frame is *adapted with respect to an* $\{H_\alpha^\beta\}_{\alpha,\beta=1}^n$*-structure* if

$$H_\alpha^\beta X_\alpha^i = X_\beta^i,\qquad i=1,\dots,r\qquad (\beta\neq\alpha,\quad \beta,\alpha\in\{1,\dots,n\}). \tag{3.2}$$

It can be easily seen that a frame is $\mathcal{W}$-adapted iff it is adapted to the corresponding $\{H_\alpha^\beta\}$-structure. So all $\{H_\alpha^\beta\}$-adapted frames form a $GL(r,\mathbb{R})$-structure on $M$.

With respect to an $\{H_\alpha^\beta\}$-adapted frame, the components of the tensor $H_\alpha^\beta$ are $(H_\alpha^\beta)_{\kappa i}^{\gamma j} = \delta_\alpha^\kappa \delta_\gamma^\beta \delta_i^j$ and the matrix representation of the endomorphism $(H_\alpha^\beta)_x$: $T_x M \to (D_\beta)_x$, $x \in M$ is[4]

$$\mathbf{H}_\alpha^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where the $(r \times r)$-identity matrix $I_r$ stands at the position $(\alpha, \beta)$.

## 4. Connections

Let $M$ be a manifold endowed with an anholonomic web $\mathcal{W}$, let $P_\alpha$ denote the corresponding projectors and $H_\alpha^\beta$ the adjoined $(1,1)$-tensor fields.

**Definition 4.1.** We say that a linear connection $\nabla$ on $M$ is *$\mathcal{W}$-preserving* if all projectors are covariantly constant,

$$\nabla P_\alpha = 0, \qquad \alpha = 0, 1, \dots, n.$$

All $\mathcal{W}$-preserving linear connections will be described in Theorem 4.2.

Remark 4.1. A distribution $D$ on $M$ is called *parallel* with respect to a connection $\nabla$ if the following condition is satisfied:

$$\forall\, X, Y \in \mathfrak{X}(M) \quad (Y \in D \Longrightarrow \nabla_X Y \in D).$$

If $D$ is both integrable and parallel to a connection $\nabla$ then $\nabla$ can be reduced to the integral submanifolds of $D$.

It can be easily verified that a connection $\nabla$ is web-preserving if and only if all web-distributions $D_\alpha$, $\alpha = 0, 1, \dots, n$ are parallel with respect to $\nabla$. The web-preserving connections are exactly the linear connections on $M$ reducible to the subbundle $WM$ of adapted frames.

In a similar way we introduce the following definition.

**Definition 4.2.** A connection $\nabla$ *preserves* an $\{H_\alpha^\beta\}$-structure if

$$\nabla H_\alpha^\beta = 0 \qquad \text{for all pairs} \quad \alpha, \beta \in \{1, \dots n\}.$$

The above condition can be written as

$$\text{(4.1)} \qquad \forall\, \alpha, \beta \quad \forall\, X, Y \in \mathfrak{X}(M) \qquad 0 = \nabla H_\alpha^\beta(X; Y) = \nabla_X H_\alpha^\beta Y - H_\alpha^\beta \nabla_X Y.$$

[4] The notation corresponds to the right action $H_\alpha^\beta(u) = u \cdot \mathbf{H}_\alpha^\beta$, $u \in T_x M$.

**Proposition 4.1.** *A linear connection $\nabla$ on $M$ is $\{H_\alpha^\beta\}$-preserving if and only if $\nabla$ is $\mathcal{W}$-preserving.*

Proof. Let $Y \in D_\alpha$, $\alpha \in \{1,\dots,n\}$. Then $H_\alpha^\alpha \nabla_X Y = \nabla_X H_\alpha^\alpha Y = \nabla_X Y$. Now let $Y \in D_0$. Then $Y = \frac{1}{n}\sum_{\alpha,\beta} H_\alpha^\beta Y$, and $H_0^0 \nabla_X Y = \frac{1}{n}\sum_{\alpha,\beta} H_\alpha^\beta \sum_\gamma H_\kappa^\gamma \nabla_X H_\gamma^\kappa Y = \frac{1}{n}\sum_{\gamma,\beta} \nabla_X (H_\kappa^\beta H_\gamma^\kappa Y) = \nabla_X H_0^0 Y = \nabla_X Y$.

On the other hand, let $\nabla P_\alpha = 0$ for $\alpha = 0,1,\dots,n$. Let us choose an adapted frame $(X_1^i|\dots|X_n^i)$, $X_0^i = \sum_\gamma X_\gamma^i$. Then $\nabla_X X_0^i = \sum_\gamma \nabla_X X_\gamma^i$ where $\nabla_X X_0^i \in D_0$ and $\nabla_X X_\gamma^i \in D_\gamma$ by the assumptions. That is, $(\nabla_X X_1^i|\dots|\nabla_X X_n^i)$ is also adapted and we obtain $B_\gamma^\beta \nabla_X X_\gamma^i = \nabla_X X_\beta^i = \nabla_X B_\gamma^\beta X_\gamma^i$. Consequently, $\nabla B_\gamma^\beta P_\gamma = \nabla H_\gamma^\beta = 0$, $\gamma = 1,\dots n$. □

**Proposition 4.2.** *A linear connection preserves an $\{H_\alpha^\beta\}$-structure if and only if the following formula holds:*

$$\forall \beta \in \{1,\dots,n\} \ \ \forall X, Y \in \mathfrak{X}(M) \quad \nabla_X Y = \sum_\alpha H_\beta^\alpha \nabla_X H_\alpha^\beta Y. \tag{4.2}$$

Proof. Let $\nabla$ preserve the structure. Then $H_\alpha^\beta \nabla_X Y = \nabla_X H_\alpha^\beta Y$, which follows by (4.1). We evaluate

$$\nabla_X Y = \sum_\alpha H_\alpha^\alpha \nabla_X Y = \sum_\alpha H_\beta^\alpha H_\alpha^\beta \nabla_X Y = \sum_\alpha H_\beta^\alpha \nabla_X H_\alpha^\beta Y, \quad \beta \in \{1,\dots n\}.$$

On the other hand, let the condition (4.2) be satisfied for all $\beta$. Then we can write for arbitrary indices $\beta$, $\gamma$, $\kappa$

$$H_\kappa^\gamma \nabla_X Y = \sum_\alpha H_\kappa^\gamma H_\beta^\alpha \nabla_X H_\alpha^\beta Y = H_\beta^\gamma \nabla_X H_\kappa^\beta Y. \tag{4.3}$$

However,

$$H_\beta^\gamma \nabla_X H_\kappa^\beta Y = \sum_\alpha H_\beta^\alpha \nabla_X (\delta_\alpha^\gamma H_\kappa^\beta Y) = \sum_\alpha H_\beta^\alpha \nabla_X H_\alpha^\beta H_\kappa^\gamma Y = \nabla_X H_\kappa^\gamma Y. \tag{4.4}$$

Taking into account (4.3), (4.4) we obtain $\nabla H_\kappa^\gamma = 0$. □

An arbitrary linear connection $\Gamma$ on a web-manifold yields a web-preserving connection as follows [Sh].

**Proposition 4.3.** *Let $\Gamma$ be a linear connection on a manifold $M_{nr}$ endowed with an $\{H^\beta_\alpha\}$-structure of dimension $r$. Then for any $\kappa \in \{1,\dots,n\}$ the following formula defines an $\{H^\beta_\alpha\}$-preserving connection $\nabla = \nabla(\Gamma;\kappa)$:*

$$\nabla_X Y = \sum_{\alpha=1}^{n} H^\alpha_\kappa \Gamma_X(H^\kappa_\alpha Y).$$

Proof. By standard evaluation, it can be checked that $\nabla$ is a connection. Moreover, it satisfies the condition (4.2). □

The so called Chern canonical connection [Ch, Ki] on a three-web manifold admits the following generalization to our case. Denote by $\gamma$ a mapping satisfying

$$\gamma\colon \{1,\dots,n\} \to \{1,\dots,n\}, \qquad \gamma(\alpha) \in \{1,\dots,\hat{\alpha},\dots,n\}. \tag{4.5}$$

There exist $(n-1)^n$ such mappings. Now let $M$ be a manifold endowed with an anholonomic $(n+1)$-web of dimension $r$. For any function $\gamma$ described above, we can construct a connection $\overset{\gamma}{\nabla}$ which parallelizes all distributions $D_0,\dots,D_n$ and is unique in the following sense [Sh].

**Theorem 4.1.** *Let $M$ be a manifold endowed with an anholonomic $(n+1)$-web of dimension $r$, $\mathcal{W} = (D_0,\dots,D_n)$, let $\{H^\beta_\alpha\}^n_{\alpha,\beta=1}$ be the corresponding structure, and let $\gamma\colon \{1,\dots,n\} \to \{1,\dots,n\}$, $\alpha \mapsto \gamma(\alpha)$ be a function such that $\gamma(\alpha) \neq \alpha$. Then there exists a unique connection $\nabla = \overset{\gamma}{\nabla}$ which is $H^\beta_\alpha$-preserving and its torsion tensor $\mathcal{T}$ satisfies*

$$H^{\gamma(\alpha)}_{\gamma(\alpha)} \mathcal{T}(H^\alpha_\alpha X, H^{\gamma(\alpha)}_{\gamma(\alpha)} Y) = 0 \qquad (\alpha = 1,\dots n). \tag{4.6.}$$

*This connection is given by the formula*

$$\nabla_X Y = \sum_{\alpha,\beta} H^\beta_{\gamma(\alpha)}[H^\alpha_\alpha X, H^{\gamma(\alpha)}_\beta Y]. \tag{4.7}$$

The proof of the theorem was partially and very briefly sketched in [Sh] (with some indices missing on page 65). Since the theorem is important for the theory let us present the proof with all details here.

Proof. First let us prove that if such a connection exists it is necessarily given by the formula (4.7). So let $\nabla$ satisfy the above conditions. Then $\nabla_X Y =$

$\sum_\kappa H^\kappa_\gamma \nabla_X H^\gamma_\kappa Y$ where $\gamma$ is a fixed index ($\gamma \in \{1, \dots, n\}$). By the assumption (4.6) we obtain

$$\begin{aligned} 0 &= H^\gamma_\gamma\Big(\sum_\kappa H^\kappa_\gamma \nabla_{H^\alpha_\alpha X} H^\gamma_\kappa H^\gamma_\gamma Y - \sum_\kappa H^\kappa_\gamma \nabla_{H^\gamma_\gamma Y} H^\gamma_\kappa H^\alpha_\alpha X - [H^\alpha_\alpha X, H^\gamma_\gamma Y]\Big) \\ &= H^\gamma_\gamma \nabla_{H^\alpha_\alpha X} H^\gamma_\gamma Y - H^\gamma_\gamma \nabla_{H^\gamma_\gamma Y} H^\gamma_\gamma H^\alpha_\alpha X - H^\gamma_\gamma [H^\alpha_\alpha X, H^\gamma_\gamma Y]. \end{aligned}$$

Since $H^\gamma_\gamma \nabla_{H^\alpha_\alpha X} H^\gamma_\gamma Y = (H^\gamma_\gamma)^2 \nabla_{H^\alpha_\alpha X} Y = H^\gamma_\gamma \nabla_{H^\alpha_\alpha X} Y$ we obtain

$$H^\gamma_\gamma \nabla_{H^\alpha_\alpha X} Y = H^\gamma_\gamma [H^\alpha_\alpha X, H^\gamma_\gamma Y]. \tag{4.8}$$

Now

$$\begin{aligned} \nabla_X Y &= \nabla_{\sum_\alpha H^\alpha_\alpha X}\Big(\sum_\beta H^\beta_\beta Y\Big) = \sum_{\alpha,\beta} H^\beta_\beta \nabla_{H^\alpha_\alpha X} Y \\ &= \sum_{\alpha,\beta} H^\beta_\gamma H^\gamma_\gamma H^\gamma_\beta \nabla_{H^\alpha_\alpha X} Y = \sum_{\alpha,\beta} H^\beta_\gamma H^\gamma_\gamma \nabla_{H^\alpha_\alpha X} H^\gamma_\beta Y. \end{aligned} \tag{4.9}$$

Substituting $H^{\gamma(\alpha)}_\beta Y$ instead of $Y$ to the formula (4.9) and $\gamma(\alpha)$ instead od $\gamma$ in the above formula for the connection (4.8) we obtain

$$\nabla_X Y = \sum_{\alpha,\beta} H^\beta_{\gamma(\alpha)} H^{\gamma(\alpha)}_{\gamma(\alpha)} [H^\alpha_\alpha X, H^{\gamma(\alpha)}_{\gamma(\alpha)} H^{\gamma(\alpha)}_\beta Y] = \sum_{\alpha,\beta} H^\beta_{\gamma(\alpha)} [H^\alpha_\alpha X, H^{\gamma(\alpha)}_\beta Y].$$

Now let us verify that the formula (4.7) defines a linear connection on $M$. Linearity is obvious. An evaluation shows that

$$\begin{aligned} \nabla_{fX} Y &= f\nabla_X Y - \sum_{\alpha,\beta} H^\beta_{\gamma(\alpha)} (H^{\gamma(\alpha)}_\beta Y f) \cdot (H^\alpha_\alpha X) \\ &= f\nabla_X Y - \sum (H^{\gamma(\alpha)}_\beta Y f) H^\beta_{\gamma(\alpha)} H^\alpha_\alpha X = f\nabla_X Y \end{aligned}$$

and

$$\begin{aligned} \nabla_X fY &= f\nabla_X Y + \sum_{\alpha,\beta} (H^\alpha_\alpha X f)(H^\beta_{\gamma(\alpha)} H^{\gamma(\alpha)}_\beta Y) \\ &= f\nabla_X Y + (Xf)Y. \end{aligned}$$

It remains to prove (4.6). Let us verify (4.2). Let $\kappa \in \{1, \dots, n\}$. Then

$$\begin{aligned} \sum_\mu H^\mu_\kappa \nabla_X H^\kappa_\mu Y &= \sum_{\alpha,\beta,\mu} H^\mu_\kappa H^\beta_{\gamma(\alpha)} [H^\alpha_\alpha X, H^{\gamma(\alpha)}_\beta H^\kappa_\mu Y] \\ &= \sum_{\alpha,\mu} H^\mu_{\gamma(\alpha)} [H^\alpha_\alpha X, H^{\gamma(\alpha)}_\mu Y] = \nabla_X Y. \end{aligned}$$

Now

$$\begin{aligned}H^{\gamma(\kappa)}_{\gamma(\kappa)}\mathcal{T}(H^{\kappa}_{\kappa}X, H^{\gamma(\kappa)}_{\gamma(\kappa)}Y) = {} & \sum_{\alpha,\beta} H^{\gamma(\kappa)}_{\gamma(\kappa)} H^{\beta}_{\gamma(\alpha)}[\delta^{\alpha}_{\kappa} H^{\kappa}_{\kappa} X, \delta^{\gamma(\kappa)}_{\beta} H^{\gamma(\alpha)}_{\gamma(\kappa)} Y] \\ & - \sum_{\alpha,\beta} H^{\gamma(\kappa)}_{\gamma(\kappa)} H^{\kappa}_{\kappa} H^{\beta}_{\gamma(\alpha)}[\delta^{\gamma(\kappa)}_{\alpha} H^{\gamma(\kappa)}_{\gamma(\kappa)} Y, H^{\gamma(\alpha)}_{\beta} X] \\ & - H^{\gamma(\kappa)}_{\gamma(\kappa)}[H^{\kappa}_{\kappa}X, H^{\gamma(\kappa)}_{\gamma(\kappa)}Y] = 0.\end{aligned}$$

□

The linear connection $\overset{\gamma}{\nabla}$ introduced by the formula (4.7) will be called *the canonical $\gamma$-connection for $\mathcal{W}$.*

Let us evaluate components of the $\gamma$-connection with respect to a web-adapted frame $(X^i_\alpha)$, $\alpha = 1, \dots, n$, $i = 1, \dots, r$. Let $\varrho$, $\mu$ be fixed, $X^j_\mu \in D_\mu$, $X^k_\varrho \in D_\varrho$. Let us denote $\nabla_{X^j_\mu} X^k_\varrho = \sum_{i,\kappa} \Gamma^{jk;\kappa}_{\mu\varrho;i} X^i_\kappa$ and $[X^j_\mu, X^k_\varrho] = \sum_{i,k} c^{jk;\kappa}_{\mu\varrho;i}(x) X^i_\kappa$. According to the formula (4.7) we obtain

$$\Gamma^{jk;\kappa}_{\mu\varrho;i} = \delta^k_\varrho c^{jk;\gamma(\mu)}_{\mu\gamma(\mu);i}.$$

Many ivestigations in web geometry are devoted to the problem of local equivalence of webs. The canonical $\gamma$-connection on a web manifold can play an important role in the classification of webs.

**Theorem 4.2.** [Sh] *The following conditions are equivalent:*

1. *The $(n+1)$-web $\mathcal{W}$ is regular.*
2. *There is an atlas on $M$ such that the corresponding holonomic frames $(\frac{\partial}{\partial x^i_\alpha})$ are web-adapted.*
3. *The $G$-structure of all $\{H^\beta_\alpha\}$-adapted frames is locally flat.*
4. *For any canonical linear $\gamma$-connection, the torsion and curvature tensors are equal to zero, $\overset{\gamma}{\mathcal{T}} = \overset{\gamma}{\mathcal{R}} = 0$.*

Remark 4.1. According to (3) any regular web is holonomic, the coordinate vector fields $\{\frac{\partial}{\partial x^i_\alpha}, i = 1, \dots, r\}$ form a basis of the distribution $D_\alpha$, $\alpha = 1, \dots, n$. It is well known that an $(n+1)$-web is regular if and only if it is locally diffeomorphic to a web formed by $n+1$ foliations of parallel $r$-dimensional affine subspaces (in general position) in $\mathbb{R}^{nr}$.

Let $\nabla$, $\tilde{\nabla}$ be a couple of $\mathcal{W}$-preserving connections. Then the difference tensor $S = \nabla - \tilde{\nabla}$ satisfies

$$S(X, H^\beta_\alpha Y) = H^\beta_\alpha S(X, Y), \tag{4.10}$$

which follows by the evaluation

$$0 = \tilde{\nabla} H^{\beta}_{\alpha}(X;Y) = \nabla H^{\beta}_{\alpha}(X;Y) + S(X, H^{\beta}_{\alpha}Y) - H^{\beta}_{\alpha}S(X,Y). \tag{4.11}$$

For any fixed $X \in \mathfrak{X}(M)$, let us introduce a vector 1-form on $M$ by $\Phi X = S(X,-)\colon Y \mapsto S(X,Y)$. Then $\Phi\colon X \mapsto \Phi X$ yields a homomorphism $X_x \mapsto (\Phi X)_x$, $T_xM \to \mathrm{End}(T_xM)$ at any point $x \in M$. According to (4.10), $\Phi X$ commutes with all mappings $H^{\beta}_{\alpha}$. For any $\kappa \in \{1,\dots,n\}$ and $X \in T_xM$ the restriction $\Phi_\kappa X_x = \Phi X_x|(D_\kappa)_x \in \mathrm{End}(D_\kappa)_x$ is an endomorphism of $(D_\kappa)_x$. Moreover, $S(X,Y) = \sum\limits_{\alpha} H^{\alpha}_{\kappa}(\Phi_\kappa X)(H^{\kappa}_{\alpha}Y)$. In fact, $\Phi\colon TM \to \mathrm{End}(TM)$ is a vector bundle morphism and similarly, $\Phi_\kappa\colon TM \to \mathrm{End}(D_\kappa)$ is a bundle morphism of a vector bundle $TM \to M$ into a vector bundle $\mathrm{End}(D_\kappa) \to M$. Obviously, it is sufficient to define the values of $\Phi X$ on an arbitrary distribution $D_\kappa$.

If one linear web-preserving connection is given, the above considerations enable us to describe the $nr^2$-dimensional bundle of all web-preserving connections as follows.

**Theorem 4.3.** *Let $\nabla$ be a web-preserving linear connection on $M$. Let us choose $\kappa \in \{1,\dots,n\}$. Any web-preserving linear connection is of the form $\tilde{\nabla} = \nabla + S$ where $S$ is a $(1,2)$-tensor field on $M$ given by the formula*

$$S(X,Y) = \sum_{\alpha=1}^{n} H^{\alpha}_{\kappa}(\Phi_\kappa X)(H^{\kappa}_{\alpha}Y), \qquad X,Y \in \mathfrak{X}(M) \tag{4.12}$$

*where $\Phi_\kappa\colon TM \to \mathrm{End}(D_\kappa)$ is a differentiable vector bundle morphism.*

Proof. Let $\nabla$, $\tilde{\nabla}$ be $\mathcal{W}$-preserving connections, $S = \tilde{\nabla} - \nabla$. Then $\Phi_\kappa$ introduced by $\Phi_\kappa X = \Phi X|D_\kappa$, $\kappa \in \{1,\dots,n\}$ satisfies the conditions required by the theorem. On the other hand, let $\nabla$ be $\mathcal{W}$-preserving and let $\Phi_\kappa\colon TM \to \mathrm{End}(D_\kappa)$ be a bundle morphism. Let us introduce $S$ by the formula (4.12). An evaluation shows that $S$ satisfies (4.10): $H^{\mu}_{\beta}S(X,Y) = \sum\limits_{\alpha} H^{\mu}_{\beta}H^{\alpha}_{\kappa}(\Phi_\kappa X)(H^{\kappa}_{\alpha}Y) = H^{\mu}_{\kappa}(\Phi_\kappa X)(H^{\kappa}_{\beta}Y) = \sum\limits_{\alpha} H^{\alpha}_{\kappa}(\Phi_\kappa X)H^{\kappa}_{\alpha}(H^{\mu}_{\beta}Y) = S(X, H^{\mu}_{\beta}Y)$. So (4.11) holds, and $\nabla + S$ is $\mathcal{W}$-preserving. □

In particular, let $n = 2$. The isomorphisms $B_1^2$, $B_2^1$ can be extended by linearity to an involutory tangent bundle isomorphism

$$B\colon TM \to TM, \quad \forall X \in \mathfrak{X}(M) \quad BX = B_1^2 P_1 X + B_2^1 P_2 X, \quad B^2 X = X.$$

A 3-web can be given as a couple $\{P_1, B\}$ of $(1,1)$ tensor fields satisfying

$$P_1{}^2 = P_1, \quad P_1 B + B P_1 = B, \quad B^2 = \mathrm{id}.$$

Here $P_1 = H_1^1$ is a projector onto $D_1$, $P_2 = H_2^2 = \mathrm{id} - P_1$ is a projector onto $D_2$.

A 3-web is holonomic if and only if $[P_1, P_1] = 0$ and $B[B,B](P_1X, P_1Y) = [B,B](P_1X, P_1Y)$ for $X$, $Y \in \mathfrak{X}(M)$ (here [ ] denotes the Nijenhuis bracket).

There exists a unique function $\gamma\colon \{1,2\} \to \{1,2\} \to$ with $\gamma(\alpha) \neq \alpha$ given by

$$\gamma(1) = 2, \qquad \gamma(2) = 1.$$

That is, for an anholonomic 3-web (with a fixed order of web distributions) the above construction yields a unique canonical $\gamma$-connection

$$\begin{aligned}\nabla_X Y &= H_2^1[H_1^1 X, H_1^2 Y] + H_1^2[H_2^2, H_2^1 Y] + H_1^1[H_2^2 X, H_1^1 Y] + H_2^2[H_1^1 X, H_2^2 Y] \\ &= BP_2[P_1X, BP_1Y] + BP_1[P_2X, BP_2Y] + P_1[P_2X, P_1Y] + P_2[P_1, P_2Y]\end{aligned}$$

which coincides with the connection introduced by S. S. Chern [Ch] and reconstructed by P. T. Nagy in [Ng].

A 3-web is called *parallelizable* if it is equivalent (locally diffeomorphic) to a regular (*parallel*) 3-web formed by three systems of parallel affine $r$-planes in an affine space $\mathbb{R}^{2r}$ which are in general position.

Parallelizable webs are equivalently characterized either by vanishing of both the torsion and the curvature tensor of the Chern connection, $\mathcal{T} = \mathcal{R} = 0$ [Ak], [A&S], or by the closing of the Thomsen figure, [Ch], [Ac], or by the condition that all coordinatizing loops are abelian groups [Ac], [A&S].

A (holonomic) 3-web is called *isoclinicly geodesic* if $\mathcal{T} = 0$ [A&S] (in [Ak], *paratactical* was used). It was proved in [Va2] that

$$\begin{aligned}&\mathcal{T}(P_1X, P_1Y) = B[P_1, B](P_1X, P_1Y), \quad \mathcal{T}(P_2X, P_2Y) = -B[P_1, B](P_2X, P_2Y), \\ &\mathcal{T}(P_1X, P_2Y) = B[P_1, B](P_1X, P_2Y) = 0.\end{aligned}$$

Especially, $\mathcal{T} = 0$ if and only if $[P_1, B] = 0$. It can be also verified that $\mathcal{T} = 0$ iff $[H_\alpha^\beta, H_\gamma^\kappa] = 0$ for $\alpha$, $\beta$, $\gamma$, $\kappa \in \{1,2\}$.

A (holonomic) 3-web is called a *Bol web* if the curvature tensor $\mathcal{R}$ is antisymmetric in one couple of arguments, that is, one of the following conditions is satisfied:

$$\mathcal{R}(X,Y)Z = -\mathcal{R}(X,Z)Y, \quad \text{or} \quad = -\mathcal{R}(Y,X)Z, \quad \text{or} \quad = -\mathcal{R}(Z,Y)X.$$

## 6. Examples

Example 6.1. More generally, a (holonomic) $(n+1)$-web of dimension $r$ (of codimension $r$) in $\mathbb{R}^{nr}$ is usually called *parallelizable* if it is equivalent with a web formed by $n+1$ foliations (in general position) of parallel $r$-planes (respectively of parallel $(n-1)r$-planes). With respect to a web-adapted coordinate frame, the corresponding tensor fields have matrix representations

$$H_\alpha^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{(\alpha,\beta)} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $I_{(\alpha,\beta)}$ denotes a unit matrix in the position $(\alpha,\beta)$.

According to Theorem 4.2. parallelizable $r$-dimensional $(n+1)$-webs are in fact regular webs in the sense of Definition 3.3. and can be characterized by $\overset{\gamma}{T} = \overset{\gamma}{R} = 0$.

All coordinate $n$-quasigroups of a parallelizable $r$-codimensional $(n+1)$-web are abelian $n$-groups [G].

Example 6.2. A commutative Lie group $G = (\mathbb{S}^1, \cdot)$ of complex units gives rise to an integrable parallelizable 3-web on the torus $\mathbf{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as follows. Let us consider Lie subgroups

$$G_1 = G \times \{1\}, \qquad G_2 = \{1\} \times G, \qquad G_0 = \{(g,g); g \in G\}.$$

Then the factor spaces $\mathcal{F}_i = (G \times G)/G_i$, $i = 0,1,2$ define a 3-web of dimension one on $G \times G$ with equivalence classes being the leaves (formed by meridians, parallels, and the third system of closed curves). Obviously, local coordinates can be chosen on $\mathbf{T}^2$ so that the *coordinate frame* is web-adapted, and $H_\alpha^\beta$ are given by

$$H_1^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H_1^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_2^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Both the curvature and the torsion tensors of the Chern connection are zero, the web is parallelizable.

More generally, if $\mathcal{G} = (G, \cdot)$ is an $r$-dimensional Lie group with a unit $e$, a 3-web of dimension $r$ can be introduced on the analytic manifold $G \times G$ in a similar way as a triple $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ where $\mathcal{F}_i = (G \times G)/G_i$. The resulting web is the so called *group* 3-web since all coordinatizing loops are associative, the curvature tensor of the Chern connection vanish, $\mathcal{R} = 0$. A group web is parallelizable ($\mathcal{T} = 0$) if and only if the Lie group $G$ is commutative [Ak].

Example 6.3. In $\mathbb{R}^4$, let us introduce web foliations by

$$\begin{aligned}
\mathcal{F}_1 &: \quad x_3 = \text{const}, \qquad x_4 = \text{const}, \\
\mathcal{F}_2 &: \quad x_1 = \text{const}, \qquad x_2 = \text{const}, \\
\mathcal{F}_0 &: \quad \varphi_1 = \frac{x_1 + x_3}{x_2 + x_4} = \text{const}, \quad \varphi_2 = \frac{x_1 - x_3}{x_2 - x_4} = \text{const}.
\end{aligned}$$

The tangent distributions are $D_1 = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $D_2 = (\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})$, and $D_0$ is spanned by any couple $(v_1, v_2)$ of independent vectors satisfying $\mathrm{d}\varphi_1(v_i) = \mathrm{d}\varphi_2(v_i) = 0$, $i = 1, 2$. An evaluation shows that

$$\begin{aligned}
\mathrm{d}\varphi_1 &= \frac{1}{x_2 + x_4} h_1 - \frac{x_1 + x_3}{(x_2 + x_4)^2} h_2 + \frac{1}{x_2 + x_4} h_3 - \frac{x_1 + x_3}{(x_2 + x_4)^2} h_4, \\
\mathrm{d}\varphi_2 &= \frac{1}{x_2 - x_4} h_1 - \frac{x_1 - x_3}{(x_2 - x_4)^2} h_2 - \frac{1}{x_2 - x_4} h_3 + \frac{x_1 - x_3}{(x_2 - x_4)^2} h_4
\end{aligned}$$

and we can choose

$$\begin{aligned}
v_1 &= (x_1 + x_3)\frac{\partial}{\partial x_1} + (x_2 + x_4)\frac{\partial}{\partial x_2} + (x_1 + x_3)\frac{\partial}{\partial x_3} + (x_2 + x_4)\frac{\partial}{\partial x_4}, \\
v_2 &= (x_1 - x_3)\frac{\partial}{\partial x_1} + (x_2 - x_4)\frac{\partial}{\partial x_2} + (-x_1 + x_3)\frac{\partial}{\partial x_3} + (-x_2 + x_4)\frac{\partial}{\partial x_4}.
\end{aligned}$$

It can be easily seen that the tangent vectors

$$\begin{aligned}
e_1 &= (x_1 + x_3)\frac{\partial}{\partial x_1} + (x_2 + x_4)\frac{\partial}{\partial x_2}, \quad e_3 = (x_1 + x_3)\frac{\partial}{\partial x_3} + (x_2 + x_4)\frac{\partial}{\partial x_4}, \\
e_2 &= (x_1 - x_3)\frac{\partial}{\partial x_1} + (x_2 - x_4)\frac{\partial}{\partial x_2}, \quad e_4 = (-x_1 + x_3)\frac{\partial}{\partial x_3} + (-x_2 + x_4)\frac{\partial}{\partial x_4}
\end{aligned}$$

form a web-adapted frame, $v_1 = e_1 + e_3$, $v_2 = e_2 + e_4$, $B_1^2(e_1) = e_3$, $B_1^2(e_2) = e_4$. With respect to this adapted frame $\langle e_1, e_2, e_3, e_4 \rangle$ we have

$$P_1 = H_1^1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_2 = H_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

$$H_1^2 = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad H_2^1 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

An evaluation yields

$$[e_1, e_2] = e_1 - e_2, \qquad [e_1, e_3] = -e_1 + e_3, \qquad [e_1, e_4] = e_2 - e_3,$$
$$[e_2, e_3] = e_1 - e_4, \qquad [e_2, e_4] = -e_2 + e_4, \qquad [e_3, e_4] = e_3 - e_4,$$

$$\nabla_{e_i} e_i = e_i, \qquad i = 1, 2, 3, 4,$$
$$-\nabla_{e_1} e_2 = \nabla_{e_3} e_1 = -\nabla_{e_3} e_2 = e_1, \quad \nabla_{e_1} e_3 = -\nabla_{e_1} e_4 = -\nabla_{e_3} e_4 = e_3,$$
$$-\nabla_{e_2} e_1 = \nabla_{e_4} e_2 = -\nabla_{e_4} e_1 = e_2, \quad \nabla_{e_2} e_4 = -\nabla_{e_2} e_3 = -\nabla_{e_4} e_3 = e_4.$$

The non-zero components of the connection in the adapted frame $\langle e_i \rangle$ are

$$\Gamma^1_{11} = \Gamma^1_{31} = \Gamma^2_{22} = \Gamma^2_{42} = \Gamma^3_{13} = \Gamma^3_{33} = \Gamma^4_{24} = \Gamma^4_{44} = 1,$$
$$\Gamma^1_{12} = \Gamma^2_{21} = \Gamma^3_{14} = \Gamma^4_{23} = \Gamma^3_{34} = \Gamma^4_{43} = \Gamma^1_{32} = \Gamma^2_{41} = -1.$$

The torsion tensor $\mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ does not vanish identically:

$$\mathcal{T}(e_1, e_2) = B[P_1, B](e_1, e_2) = -2e_1 + 2e_2,$$
$$\mathcal{T}(e_3, e_4) = B[P_1, B](e_3, e_4) = -2e_3 + 2e_4,$$
$$\mathcal{T}(e_1, e_3) = \mathcal{T}(e_1, e_4) = \mathcal{T}(e_2, e_3) = \mathcal{T}(e_2, e_4) = 0.$$

The curvature tensor

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

does not vanish identically, e.g.

$$\mathcal{R}(e_2, e_3)e_4 = -\nabla_{e_2} e_3 - \nabla_{e_3} e_4 - \nabla_{e_1} e_4 + \nabla_{e_4} e_4 = 2e_3 + 2e_4 = -\mathcal{R}(e_3, e_2)e_4,$$
$$\mathcal{R}(e_4, e_1)e_3 = -2e_3 - 2e_4 = -\mathcal{R}(e_1, e_4)e_3,$$

and satisfies $\mathcal{R}(Y, X)Z = -\mathcal{R}(X, Y)Z$.

We conclude that the web is neither parallelizable nor paratactical nor a group web, but it belongs to the family of Bol webs.

Remark 6.1. With respect to the coordinate frame $\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle$ the corresponding matrix representations are

$$B_1^2 = Q = \begin{pmatrix} -\dfrac{x_1x_2 - x_3x_4}{x_1x_4 - x_2x_3}, & \dfrac{x_4^2 - x_2^2}{x_1x_4 - x_2x_3} \\ \dfrac{x_1^2 - x_3^2}{x_1x_4 - x_2x_3}, & \dfrac{x_1x_2 - x_3x_4}{x_1x_4 - x_2x_3} \end{pmatrix}, \quad B_2^1 = Q^{-1}, \quad B = \begin{pmatrix} 0 & Q \\ Q^{-1} & 0 \end{pmatrix},$$

$$H_1^1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad H_2^2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad H_1^2 = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}, \quad H_2^1 = \begin{pmatrix} 0 & 0 \\ Q^{-1} & 0 \end{pmatrix},$$

and the evaluations would be more complicated.

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