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ON LIOUVILLE THEOREM AND HÖLDER CONTINUITY OF WEAK SOLUTIONS TO SOME QUASILINEAR ELLIPTIC SYSTEMS OF HIGHER ORDER

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Summary. The aim of this paper is to show that the Liouville-type property is a sufficient and necessary condition for the regularity of weak solutions of quasilinear elliptic systems of higher orders.

Keywords: regularity of weak solutions, quasilinear elliptic systems

AMS classification: 35J60, 35D10

INTRODUCTION

In this paper we shall deal with quasilinear elliptic systems. More precisely we shall consider the following problem.

Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$, $n \geqslant 2$. Let us denote $\sigma(n,k) = \binom{n+k-1}{k}$, $\varrho(n,k) = \binom{n+k}{k}$, $n, k \in \mathbb{N}$. We shall study weak solutions $u \in H^{\underline{m}}(\Omega) \cap H^{\underline{m-1},\infty}(\Omega)$ to the system

$$(0.1)\qquad \sum_{j=1}^{N} \sum_{\substack{|\alpha| \leqslant m_i \\ |\beta| = m_j}} (-1)^{|\alpha|} D^\alpha (A_{ij}^{\alpha\beta}(x, \delta(u)) D^\beta u^j) = \sum_{|\alpha| \leqslant m_i} (-1)^{|\alpha|} D^\alpha g_\alpha^i,$$
$$i = 1, \ldots, N, \text{ in } \Omega.$$

By a weak solution of (0.1) we mean a function $u \in H^{\underline{m}}(\Omega)$ ($H^{\underline{m}}(\Omega) = H^{m_1}(\Omega) \times \ldots \times H^{m_N}(\Omega)$, $H^{m_i}(\Omega)$ — Sobolev space, $m_i \geqslant 1$ for $i = 1, \ldots, N$, $u = (u^1, \ldots, u^N)$—see

[5]) such that

$$\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|\leqslant m_i \\ |\beta|=m_j}} \int_{\Omega} A_{ij}^{\alpha\beta}(x,\delta(u)) D^{\beta}u^{j} D^{\alpha}\varphi^{i}\,\mathrm{d}x = \sum_{i=1}^{N} \sum_{|\alpha|\leqslant m_i} \int_{\Omega} g_{\alpha}^{i} D^{\alpha}\varphi^{i}\,\mathrm{d}x, \tag{0.2}$$

$$\varphi \in [\mathcal{D}(\Omega)]^{N}.$$

$$\delta(u) = \{D^{\alpha}u^{i} : |\alpha| \leqslant m_i - 1,\ i = 1,\dots,N\}.$$

We shall assume that

$$A_{ij}^{\alpha\beta} \in C(\overline{\Omega} \times \mathbf{R}^{\kappa}), \quad \kappa = \sum_{i=1}^{N} \varrho(n, m_i - 1), \tag{0.3}$$

there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} A_{ij}^{\alpha\beta}(x,\zeta)\xi_i^{\alpha}\xi_j^{\beta} \geqslant \nu\|\xi\|^2, \tag{0.4}$$

$$(x,\zeta) \in \overline{\Omega} \times \mathbf{R}^{\kappa}, \quad \xi \in \mathbf{R}^{\vartheta}, \quad \vartheta = \sum_{i=1}^{N} \sigma(n, m_i),$$

$$g_{\alpha}^{i} \in L^{p_{\alpha}^{i}}(\Omega), \quad p_{\alpha}^{i} = \frac{p}{m_i - |\alpha| + 1}, \tag{0.5}$$

where $p > n, \quad p \geqslant 2\bigl(\max_i\{m_i\} + 1\bigr).$

For $M > 0$, $G > 0$ let us denote

$$[M] = \{u \in H^{\underline{m}}(\Omega) \cap H^{\underline{m-1},\infty}(\Omega) : u \text{ is a solution to (0.1) and } \|u\|_{H^{\underline{m-1},\infty}(\Omega)} \leqslant M\},$$

$$[G] = \{g_{\alpha}^{i} \in L^{p_{\alpha}^{i}}(\Omega) : \sum_{i=1}^{N} \sum_{|\alpha|\leqslant m_i} \|g_{\alpha}^{i}\|_{L^{p_{\alpha}^{i}}(\Omega)} \leqslant G\},$$

$$A = A(M) = \sup_{\substack{|\zeta|\leqslant M \\ x\in\Omega}} \Bigl\{ \sum_{i,j,\alpha,\beta} |A_{ij}^{\alpha\beta}(x,\zeta)| \Bigr\},$$

$$\delta_2(u) = \{D^{\alpha}u^{i} : |\alpha| = m_i - 1,\ i = 1,\dots,N\},$$

$$\delta_1(u) = \delta(u) \setminus \delta_2(u).$$

Let $\underline{s} = (s_1,\dots,s_N)$, $s_i \in \mathbf{N} \cup \{0\}$, $i = 1, \dots, N$. We shall use the notation $P_{\underline{s}}^{N} = \{(P_1,\dots,P_N) : P_i$ is a polynomial such that $\deg(P_i) \leqslant s_i\}$. Denote $B(x^0, R) = \{x \in \mathbf{R}^n : |x - x^0| < R\}$ and $\tau = \sum_{i=1}^{n} \varrho(n, m_i - 2)$ (we put $\varrho(n,-1) = 0$).

Definition 0.6. We say that the system (0.1) has Liouville's property (L), if for every $x^0 \in \Omega$, $\xi \in \mathbf{R}^r$ every function $v \in H^{\underline{m}}_{\mathrm{loc}}(\mathbf{R}^n)$ with bounded derivatives of order $\underline{m-1}$, solving in $\mathbf{R}^n$ the system

$$(0.7) \qquad \sum_{j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} (-1)^{|\alpha|} D^\alpha \big(A_{ij}^{\alpha\beta}(x^0, \xi, \delta_2(v))\, D^\beta v^j(x)\big) = 0, \quad i = 1, \ldots, N$$

$$\left(\text{i.e. } \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{\mathbf{R}^n} A_{ij}^{\alpha\beta}(x^0, \xi, \delta_2(v))\, D^\beta v^j(x) D^\alpha \varphi^i(x)\, \mathrm{d}x = 0, \ \varphi \in [\mathcal{D}(\mathbf{R}^n)]^N \right)$$

is a polynomial from the set $P^N_{\underline{m-1}}$.

Definition 0.8. We say that the system (0.1) has the property of regularity (R) if for every $x^0 \in \Omega$, $\xi \in \mathbf{R}^r$, $M > 0$ there exist $\eta > 0$, $c > 0$ and $\mu \in (0,1)$ such that every weak solution u (in $\mathbf{R}^n$) of the system (0.7) with $|D^\alpha u^i| \leqslant M$, $i = 1, \ldots, N$, $|\alpha| = m_i - 1$ belongs to the space $C^{\underline{m-1},\mu}\big(\overline{B(0,\eta)}\big)$ and $\|u\|_{C^{\underline{m-1},\mu}(\overline{B(0,\eta)})} \leqslant c$.

It will be proved in this paper that the property (L) implies the interior regularity of solutions to the system (0.1), i.e. if u is a weak solution to (0.1) then $u \in C^{\underline{m-1},\mu}(\overline{\Omega'})$, where $\overline{\Omega'} \subset \Omega$, $\mu \in \big(0, 1 - \frac{n}{p}\big)$.

It will be also shown that (R)$\Rightarrow$ (L).

These results generalize the results of [4]. In [4] the analogous assertions are proved for quasilinear elliptic systems of the second order.

The history of the regularity problem and Liouville's property is described in [2], [4].

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1. Some lemmas

Let us denote

$$U(x^0, R) = R^{-n} \int_{B(x^0,R)} \left(\sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} |D^\alpha u^i(x) - (D^\alpha u^i)_{x^0,R}|^2 \right) \mathrm{d}x,$$

$u \in H^{\underline{m-1}}(B(x^0, R))$, where by $(D^\alpha u^i)_{x^0,R}$ we mean the integral mean value $D^\alpha u^i$ in $B(x^0, R)$.

Lemma 1.1. *Let $A_{ij}^{\alpha\beta}$ be constants with $|A_{ij}^{\alpha\beta}| \leqslant L$, $L > 0$ and let (0.4) be satisfied for $A_{ij}^{\alpha\beta}$. Let $u \in H^{\underline{m}}_{\mathrm{loc}}(B(0,1)) \cap H^{\underline{m-1}}(B(0,1))$ be a solution to the system*

$$(1.2)\qquad \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i\\|\beta|=m_j}} \int_{B(0,1)} A_{ij}^{\alpha\beta} D^\beta u^j D^\alpha \varphi^i \mathrm{d}x = 0, \quad \varphi \in [\mathcal{D}(B(0,1))]^N.$$

Then there exists a constant $\Lambda = \Lambda(n, N, L, \underline{m}, \nu)$ such that for all $0 < \varrho \leqslant 1$

$$(1.3)\qquad U(0,\varrho) \leqslant \Lambda \varrho^2 U(0,1).$$

The proof of this lemma is analogous to that of Lemma 2 in [3]. Using the Lax-Milgram lemma we could prove

Lemma 1.4. *Suppose that $u \in [M]$, $x^0 \in \Omega$. Let (0.3), (0.4), (0.5) be satisfied and let the right-hand sides of the system (0.1) belong to $[G]$. Then there exists $R_0 = R_0(A, M)$, $0 < R_0 \leqslant \operatorname{dist}(x^0, \partial\Omega)$ such that for all $R \in (0, R_0]$ the linear elliptic system*

(1.5)

$$\sum_{j=1}^{N} \sum_{\substack{|\alpha|\leqslant m_i\\|\beta|=m_j}} (-1)^{|\alpha|} D^\alpha \left(A_{ij}^{\alpha\beta}(x, \delta(u)) D^\beta v_R^j\right) = \sum_{|\alpha|\leqslant m_i} (-1)^{|\alpha|} D^\alpha g_\alpha^i, \quad i = 1, \ldots, N,$$

has a unique weak solution in $H_0^{\underline{m}}(B(x^0, R))$.

Since (1.5) is uniquely solvable for $R \leqslant R_0$ we may decompose any solution u of the quasilinear system (0.1) in the following manner:

$$(1.6)\qquad u = v_R + w_R,$$

where $v_R \in H_0^{\underline{m}}(B(x^0, R))$ solves the system (1.5) and

$$(1.7)\qquad \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|\leqslant m_i\\|\beta|=m_j}} \int_{B(x^0,R)} A_{ij}^{\alpha\beta}(x, \delta(u)) D^\beta w_R^j D^\alpha \varphi^i \mathrm{d}x = 0, \quad \varphi \in [\mathcal{D}(B(x^0, R))]^N.$$

Now we shall investigate v_R, w_R.

Lemma 1.8. *Let the assumptions of Lemma 1.4 be satisfied. Let v_R be defined as above with $0 < R \leqslant R_0$, $\Omega' \subset\subset \Omega$. There exists a constant $c_1 =$*

$c_1(n, N, \underline{m}, A, M, \nu, R_0, G)$ *such that the following holds uniformly with respect to* $x^0 \in \Omega'$ *and uniformly with respect to the class* $[M] \cup [G]$:

$$V^R(x^0, R) \leqslant c_1 R^{2-\frac{2n}{p}}, \quad R \in (0, \min\{1, R_0\}]. \tag{1.9}$$

Proof. Let $v_R \in H_0^{\underline{m}}(B(x^0, R))$, $R \in (0, \min\{1, R_0\}]$, be a weak solution to (1.5):

$$\begin{aligned}
&\sum_{i,j=1}^{N} \sum_{\substack{|\alpha| \leqslant m_i \\ |\beta| = m_j}} \int_{B(x^0,R)} A_{ij}^{\alpha\beta}(x, \delta(u)) D^\beta v_R^j D^\alpha \varphi^i \, \mathrm{d}x \\
&= \sum_{i=1}^{N} \sum_{|\alpha| \leqslant m_i} \int_{B(x^0,R)} g_\alpha^i D^\alpha \varphi^i \, \mathrm{d}x, \quad \varphi \in [\mathcal{D}(B(x^0, R))]^N.
\end{aligned} \tag{1.10}$$

Let us denote the left-hand side of (1.10) by $\mathrm{a}(v_R, \varphi)$. Putting $\varphi = v_R$ and using the Hölder inequality, the fact that the norms are equivalent and (0.4) we have

$$\mathrm{a}(v_R, v_R) \geqslant \frac{1}{2}\nu |v_R|^2_{H^{\underline{m}}(B(x^0,R))}, \tag{1.11}$$

where the constant $\frac{1}{2}\nu$ is obtained by the choice of the constant R_0 in Lemma 1.4, and $|\cdot|_{H^{\underline{m}}(B(x^0,R))}$ includes derivatives of order $\underline{m}$ only. The relations (1.10), (1.11), the Hölder inequality and the fact that $p_\alpha^i \geqslant 2$, $(m_i - |\alpha|)(p - n) \geqslant 0$, $i = 1, \ldots, N$ imply

$$\begin{aligned}
\frac{1}{2}\nu |v_R|^2_{H^{\underline{m}}(B(x^0,R))} &\leqslant \sum_{i=1}^{N} \sum_{|\alpha| \leqslant m_i} \int_{B(x^0,R)} g_\alpha^i D^\alpha v_R^i \, \mathrm{d}x \\
&\leqslant c_2 G R^{\frac{n}{2} - \frac{n}{p}} |v_R|_{H^{\underline{m}}(B(x^0,R))}.
\end{aligned}$$

From this inequality we have

$$\begin{aligned}
\|D^\alpha v_R^i\|_{L^2(B(x^0,R))} &\leqslant c_3 \Big\{ \sum_{i=1}^{N} \sum_{|\alpha| \leqslant m_i} \|g_\alpha^i\|_{L^{p_\alpha^i}(\Omega)} \Big\} R^{m_i - |\alpha| + \frac{n}{2} - \frac{n}{p}}, \\
&|\alpha| \leqslant m_i, \quad i = 1, \ldots, N
\end{aligned} \tag{1.12}$$

and

$$|v_R|_{H^{\underline{m}}(B(x^0,R))} \leqslant c_4(A, M, \nu, R_0, G, n, \underline{m}, N) R^{\frac{n}{2} - \frac{n}{p}}. \tag{1.13}$$

Now (1.13) and the inequality

$$V^R(x^0, R) \leqslant R^{-n} c_5 R^2 |v_R|^2_{H^{\underline{m}}(B(x^0,R))}$$

imply (1.9). □

Remark 1.14. In what follows we shall often extract subsequences without changing the notation, if there is no danger of misunderstanding.

We have a fundamental lemma due to E. Giusti [3]:

Lemma 1.15. *Let* $M > 0$, $G > 0$ *and* $u \in [M]$. *Suppose that assumptions* (0.3), (0.4), (0.5) *are satisfied for the system* (0.1). *Let the right-hand sides of* (0.1) *belong to the class* $[G]$ *and let* Λ *be the constant from Lemma* 1.1.

Then for all $\tau \in (0,1)$ *there exist* $\varepsilon_0 = \varepsilon_0(\tau, M)$, $R_0 = R_0(\tau, M)$ *such that for* $x^0 \in \Omega$ *and* $0 < R \leqslant \min\{R_0, \operatorname{dist}(x^0, \partial\Omega)\}$ *we have*

$$W^R(x^0, R) < \varepsilon_0^2 \Rightarrow W^R(x^0, \tau R) \leqslant 2\Lambda\tau^2 W^R(x^0, R). \tag{1.16}$$

Proof. Let us suppose that the lemma is not true for some τ. Then there exist $\{\varepsilon_s\}_{s=1}^{\infty}$, $\varepsilon_s \to 0$, $\{R_s\}_{s=1}^{\infty}$, $R_s \to 0$, $\{x^s\}_{s=1}^{\infty} \subset \Omega$, $x^s \to x^0 \in \overline{\Omega}$ and $\{u_s\}_{s=1}^{\infty} \subset [M]$ such that

$$W^{sR_s}(x^s, R_s) = \varepsilon_s^2$$

and

$$W^{sR_s}(x^s, \tau R_s) > 2\Lambda\tau^2 W^{sR_s}(x^s, R_s) = 2\Lambda\tau^2\varepsilon_s^2. \tag{1.17}$$

For $s = 1, 2, \ldots$ let $q_s \in P^N_{\underline{m-1}}$ be such that

$$\int\limits_{B(x^s,R_s)} D^\alpha q_s^j(x)\,\mathrm{d}x = \int\limits_{B(x^s,R_s)} D^\alpha w^j_{sR_s}(x)\,\mathrm{d}x, \quad j = 1, \ldots, N, \quad |\alpha| \leqslant m_j - 1. \tag{1.18}$$

On the ball $B(0,1)$ define

$$\begin{aligned} h_s^j(y) &= R_s^{1-m_j}\varepsilon_s^{-1}[w^j_{sR_s}(x^s + R_s y) - q_s^j(x^s + R_s y)], \quad j = 1, \ldots, N, \\ h_s(y) &= (h_s^1(y), \ldots, h_s^N(y)), \end{aligned}$$

and put $x = x_s + R_s y$. We have

$$H_s(0,1) = \int\limits_{B(0,1)} \sum_{j=1}^{N} \sum_{|\alpha|=m_j-1} |D^\alpha h_s^j(y) - (D^\alpha h_s^j)_{0,1}|^2 \mathrm{d}y = 1, \quad s = 1, 2, \ldots \tag{1.19}$$

and

$$(1.20)\quad H_s(0,\tau)=$$

$$=\tau^{-n}\int_{B(0,\tau)}\sum_{j=1}^{N}\sum_{|\alpha|=m_j-1}|D^\alpha h_s^j(y)-(D^\alpha h_s^j)_{0,\tau}|^2\mathrm{d}y$$

$$=\varepsilon_s^{-2}\tau^{-n}R_s^{-n}\sum_{j=1}^{N}\sum_{|\alpha|=m_j-1}\int_{B(x^s,\tau R_s)}|D^\alpha w_{sR_s}^j(x)-D^\alpha q_s^j(x)-\varepsilon_s(D^\alpha h_s^j)_{0,\tau}|^2\mathrm{d}x$$

$$\geqslant \varepsilon_s^{-2}W^{sR_s}(x^s,\tau R_s)>2\Lambda\tau^2.$$

Now let $\psi\in[\mathcal{D}(B(0,1))]^N$.

Put $\varphi^i=\varepsilon_s^{-1}R_s^{m_i+1}\psi^i\left(\frac{x-x^s}{R_s}\right)$, $i=1,\ldots,n$, in (1.7), where $w_R=w_{sR_s}$. Using the transformation $x=x^s+R_sy$ and the fact that $D^\beta w_{sR_s}^j(x^s+R_sy)=\varepsilon_sR_s^{-1}D^\beta h_s^j(y)$, $|\beta|=m_j$, we have

$$(1.21)\qquad \sum_{i,j=1}^{N}\sum_{\substack{|\alpha|\leqslant m_i\\|\beta|=m_j}}\int_{B(0,1)}R_s^{m_i-|\alpha|}B_{ijs}^{\alpha\beta}(y)D^\beta h_s^j(y)D^\alpha\psi^i(y)\,\mathrm{d}y=0,$$

where $B_{ijs}^{\alpha\beta}(y)=A_{ij}^{\alpha\beta}\left(x^s+R_sy,\delta\left(u_s(x^s+R_sy)\right)\right)$.

The definition of $h_s^j(y)$ and (1.6) imply

$$(1.22)\qquad D^\alpha u_s^j(x^s+R_sy)=R_s^{m_j-1-|\alpha|}\varepsilon_sD^\alpha h_s^j(y)$$
$$+D^\alpha q_s^j(x^s+R_sy)+D^\alpha v_{sR_s}^j(x^s+R_sy),$$
$$j=1,\ldots,N,\quad \alpha:|\alpha|\leqslant m_j-1.$$

From (1.12) it follows that

$$D^\alpha v_{sR_s}^j(x^s+R_sy)\to 0\quad \text{in } L^2(B(0,1)),\quad \alpha:|\alpha|\leqslant m_j-1,\quad j=1,\ldots,N.$$

and consequently

$$(1.23)\quad D^\alpha v_{sR_s}^j(x^s+R_sy)\to 0\quad \text{a.e. in } B(0,1),\quad \alpha:|\alpha|\leqslant m_j-1,\quad j=1,\ldots,N.$$

Using (1.18), (1.19) we have for $j=1,\ldots,N$

$$(1.24)\qquad \|h_s^j\|_{H^{m_j-1}(B(0,1))}\leqslant c_6,\quad s=1,2,\ldots,$$

and this inequality implies

$$(1.25)\qquad R_s^{m_j-1-|\alpha|}\varepsilon_s D^\alpha h_s^j(y)\to 0 \quad \text{a.e. in } B(0,1),\quad |\alpha|\leqslant m_j-1,\quad j=1,\ldots,N$$

and

$$(1.26)\qquad h_s^j \rightharpoonup h^j \text{ in } H^{m_j-1}(B(0,1)),\quad j=1,\ldots,N,$$
$$\text{i.e. } D^\alpha h_s^j \rightharpoonup D^\alpha h^j \text{ in } L^2(B(0,1)),\quad \alpha\leqslant m_j-1.$$

The polynomials in (1.18) may be written in the form

$$q_s^j(x)=\sum_{|\alpha|\leqslant m_j-1} c_\alpha^{j,s}x^\alpha,\quad x=x^s+R_s y.$$

By induction, using the form of the coefficients $c_\alpha^{j,s}$ and (1.22) we could prove: there exists a constant $K>0$ such that

$$(1.27)\qquad |c_\alpha^{j,s}|\leqslant K,\quad j=1,\ldots,N,\quad \alpha\colon |\alpha|\leqslant m_j-1,\quad s=1,2,\ldots.$$

It follows from (1.27) that for $j=1,\ldots,N$, $|\alpha|\leqslant m_j-1$ there exist subsequences $\{c_\alpha^{j,s}\}_{s=1}^\infty$ such that

$$(1.28)\qquad c_\alpha^{j,s}\to c_\alpha^j,\quad s\to\infty.$$

Put $q^j(x)=\sum_{|\alpha|\leqslant m_j-1} c_\alpha^j x^\alpha$, $j=1,\ldots,N$. It is clear that

$$D^\beta q_s^j \rightrightarrows D^\beta q_j,\quad j=1,\ldots,N,\quad \beta\colon|\beta|\leqslant m_j-1$$

(in Ω).

By the relations $|D^\beta q_s^j(x^s+R_s y)-D^\beta q^j(x^s+R_s y)|\rightrightarrows 0$ and $|D^\beta q^j(x^s+R_s y)-D^\beta q^j(x^0)|\rightrightarrows 0$ in $B(0,1)$ we have

$$(1.29)\qquad D^\beta q_s^j(x^s+R_s y)\rightrightarrows D^\beta q^j(x^0) \text{ in } B(0,1),\quad j=1,\ldots,N,\quad \beta\colon|\beta|\leqslant m_j-1.$$

Using (1.22), (1.23), (1.25) and (1.29) we have

$$D^\beta u_s^j(x^s+R_s y)\to D^\beta q^j(x^0) \text{ a.e. in } B(0,1),\quad \beta\colon|\beta|\leqslant m_j-1,\quad j=1,\ldots,N.$$

This and the fact that $\{u_s\}_{s=1}^{\infty} \subset [M]$ imply that

$$|\delta(q(x^0))|_{\mathbb{R}^n} \leqslant M, \quad q = (q^1, \dots, q^N).$$

(0.3) implies

$$B_{ijs}^{\alpha\beta}(y) \to A_{ij}^{\alpha\beta}(x^0, \delta(q(x^0))) \text{ a.e. in } B(0,1). \tag{1.30}$$

Now let $0 < t < t_1 < 1$, $\chi \in \mathcal{D}(B(0,t_1))$, $0 \leqslant \chi \leqslant 1$ in $B(0,t_1)$ and $\chi = 1$ in $B(0,t)$. Let us put $\psi^i = h_s^i \chi^{2k}$, $k = \max_i\{m_i\}$, $i = 1, \dots, N$, in (1.21). Using the Leibniz formula we have

$$\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|\leqslant m_i \\ |\beta|=m_j}} \sum_{\gamma\leqslant\alpha} \int_{B(0,1)} \binom{\alpha}{\gamma} R_s^{m_i-|\alpha|} B_{ijs}^{\alpha\beta}(y) D^\beta h_s^j(y) D^\gamma h_s^i(y) D^{\alpha-\gamma}(\chi^{2k})\, dy = 0.$$

Using the equality $D^{\alpha-\gamma}(\chi^{2k}) = \chi^k \cdot Z^{\alpha-\gamma}(\chi^{2k})$ where $Z^{\alpha-\gamma}$ contains derivatives of the function χ, we have

(1.31)

$$\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{B(0,1)} A_{ij}^{\alpha\beta}(x^s + R_s y, \delta(u_s(x^s R_s y)))(D^\beta h_s^j(y)\chi^k)(D^\alpha h_s^i(y)\chi^k)\, dy$$

$$= -\sum_{i,j=1}^{N} \sum_{|\beta|=m_j} \Big(\sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} + \sum_{|\alpha|<m_i} \sum_{\gamma\leqslant\alpha} \Big) \int_{B(0,1)} \binom{\alpha}{\gamma} R_s^{m_i-|\alpha|} B_{ijs}^{\alpha\beta}(y)(D^\beta h_s^j(y)\chi^k)$$

$$\times D^\gamma h_s^i(y) Z^{\alpha-\gamma}(\chi^{2k})\, dy.$$

Denoting the left-hand side of (1.31) by (LS) and the right-hand side by (RS) and using (0.4) and the Hölder inequality we have

$$(\mathrm{LS}) \geqslant \nu \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B(0,1)} |D^\alpha h_s^i \cdot \chi^k|^2 dy = \nu \cdot J_s$$

$$|(\mathrm{RS})| \leqslant c_7(t)(J_s)^{\frac{1}{2}} \|h_s\|_{H^{\underline{m-1}}(B(0,1))}.$$

It follows from these inequalities that

$$J_s \leqslant c_8(t)(J_s)^{\frac{1}{2}} \|h_s\|_{H^{\underline{m-1}}(B(0,1))}, \tag{1.32}$$

and using (1.19) we have

$$J_s \leqslant c_9 \|h_s\|^2_{H^{\underline{m-1}}(B(0,1))} \leqslant c_{10} |h_s|^2_{H^{\underline{m-1}}(B(0,1))} \leqslant c_{11} H_s(0,1) = c_{11}(t).$$

The inequality $|h_s|^2_{H^{\underline{m}}(B(0,t))} \leqslant J_s \leqslant c_{11}(t)$ and Poincaré's inequality imply

$$(1.33) \qquad \|h_s\|^2_{H^{\underline{m}}(B(0,t))} \leqslant c_{12}(t), \quad t \in (0,1), \quad s = 1,2,\dots.$$

Using the imbedding theorem we obtain from (1.33) that

$$(1.34) \qquad \begin{cases} h_s \rightharpoonup h & \text{in} \quad H^{\underline{m}}(B(0,t)) \\ D^\alpha h_s^i \rightharpoonup D^\alpha h^i & \text{in} \quad L^2(B(0,t)),\ |\alpha| = m_i,\ i = 1,\dots,N \\ h_s \to h & \text{in} \quad H^{\underline{m-1}}(B(0,t)). \end{cases}$$

Now let us choose $t = t_r = 1 - \frac{1}{r+1}$, $r = 1, 2, \dots$. Thanks to the diagonalization process there exists a subsequence $\{h_s\}_{s=1}^\infty$ such that

$$(1.35) \qquad h_s \rightharpoonup h \text{ in } H^{\underline{m}}(B(0,t_r)), \quad r \in \mathbb{N},$$

$$(1.36) \qquad D^\alpha h_s^i \rightharpoonup D^\alpha h^i \text{ in } L^2(B(0,t_r)), \quad r \in \mathbb{N}, \quad |\alpha| = m_i, \quad i = 1,\dots,N,$$

$$(1.37) \qquad h_s \to h \text{ in } H^{\underline{m-1}}(B(0,t_r)), \quad r \in \mathbb{N}.$$

Let $\psi \in [\mathcal{D}(B(0,1))]^N$. The Dominated Convergence Theorem and (1.30) imply

$$(1.38) \qquad B^{\alpha\beta}_{ijs} \cdot D^\alpha \psi^i \to A^{\alpha\beta}_{ijs}(x^0, \delta(q(x^0)))D^\alpha \psi^i \text{ in } L^2(B(0,1)), \\ i,j = 1,\dots,N, \quad \alpha,\beta\colon |\alpha| = m_i, \quad |\beta| = m_j,$$

$$(1.39) \qquad R_s^{m_i - |\alpha|} B^{\alpha\beta}_{ijs} D^\alpha \psi^i \to 0 \quad \text{in} \quad L^2(B(0,1)), \\ i,j = 1,\dots,N, \quad \alpha,\beta\colon |\alpha| < m_i, \quad |\beta| = m_j.$$

It is clear that for $\psi \in [\mathcal{D}(B(0,1))]^N$ there exists $r \in \mathbb{N}$ such that $\operatorname{supp}\psi \subset B(0,t_r)$. Now using the limiting process and (1.36), (1.38), (1.39) we conclude from (1.21) that

$$(1.40) \qquad \sum_{i,j=1}^N \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{B(0,1)} A^{\alpha\beta}_{ij}(x^0, \delta(q(x^0)))D^\beta h^j(y) D^\alpha \psi^i(y)\, dy = 0.$$

The Hölder inequality, (1.19) and (1.26) imply that $H(0,1) \leqslant 1$. Using the fact that $H_s(0,\tau) \to H(0,\tau)$, $\tau \in (0,1)$ and (1.20) we have

$$(1.41) \qquad H(0,\tau) \geqslant 2\Lambda\tau^2 > 0$$

and

$$(1.42)\qquad H(0,1) \geqslant \tau^n H(0,\tau) > 0.$$

Now (1.41) and Lemma 1.1 (applied to the system (1.40)) imply

$$2\Lambda\tau^2 H(0,1) \leqslant 2\Lambda\tau^2 \leqslant H(0,\tau) \leqslant \Lambda\tau^2 H(0,1),$$

i.e. $H(0,1) = 0$. This assertion contradicts (1.42). □

Remark 1.43. Let $M > 0$, $G > 0$, $u \in [M]$. It follows from the inequality (1.12) that there exist constants $\gamma_1(G)$ and $\overline{R}(M)$ such that for all $x \in \Omega$ and $0 < R \leqslant \min\{\mathrm{dist}(x,\partial\Omega), \overline{R}\}$

$$(1.44)\qquad \Big(\sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} R^{-n} \int_{B(x,R)} |D^\alpha v^i_{xR}(y)|^2 \mathrm{d}y\Big)^{\frac{1}{2}} \leqslant \gamma_1 R^\omega,$$

where

$$\omega = 1 - \frac{n}{p}.$$

Let Λ be the constant from Lemma 1.1 and let $\tau \in (0,1)$ be such that

$$(1.45)\qquad \sqrt{2\Lambda}\tau \leqslant \tau^\omega < \frac{1}{2}.$$

Put $\gamma_2 = \gamma_1(\tau^\omega + \tau^{-\frac{n}{2}})$. It is clear that there exists $k_0 \in \mathbb{N}$ such that $k_0\tau^{\omega(k_0-1)} = \max_{k\in\mathbb{N}} k\tau^{\omega(k-1)} = c_0 \geqslant 1$. Now let $\varepsilon_0 = \varepsilon_0(\tau, M)$, $R_0 = R_0(\tau, M)$ be the constants from Lemma 1.15 and let R_1 be chosen in such a way that

$$(1.46)\qquad 0 < R_1 \leqslant \min\{R_0, \overline{R}\},$$

$$(1.47)\qquad c_0\gamma_2 R_1^\omega < \frac{\varepsilon_0}{2},$$

$$(1.48)\qquad \gamma_1 R_1^\omega < \frac{\varepsilon_0}{6}.$$

Put $\delta = R_1(1 - 2^{-\frac{2}{n}})$.

Lemma 1.49. *Let $\mu \in [0,\omega)$. Then there exists a constant $c > 0$ such that for all $x^0 \in \Omega$, $R_1 \leqslant \mathrm{dist}(x^0,\partial\Omega)$ (R_1 satisfies (1.46), (1.47), (1.48)) and $u \in [M]$ the following assertions hold:*

$$W^{R_1}(x^0, R_1) < \left(\frac{\varepsilon_0}{4}\right)^2 \Rightarrow u \in C^{\underline{m-1},\mu}(\overline{B(x^0,\delta)}),$$

and

$$\|u\|_{C^{m-1,\mu}(\overline{B(x^0,\delta)})} \leqslant c.$$

P r o o f. Let $u \in [M]$ and $x \in B(x^0,\delta)$. Put $R_x = R_1 - |x - x^0| > R_1 - \delta$, $R_x < R_1 \leqslant R_0$. It is clear that $B(x, R_x) \subset B(x^0, R_1)$.

We shall prove that

$$W^{xR_x}(x, R_x) < \varepsilon_0^2. \tag{1.50}$$

Using (1.6), (1.44), (1.48) and the definition of δ we have

$$\begin{aligned}
(W^{xR_x}(x,R_x))^{\frac{1}{2}} &\leqslant \left(R_x^{-n}\sum_{i=1}^{N}\sum_{|\alpha|=m_i-1}\int_{B(x,R_x)} |D^\alpha w^i_{xR_x}(y) - (D^\alpha w^i_{x^0R_1})_{x^0,R_1}|^2 \mathrm{d}y\right)^{\frac{1}{2}} \\
&\leqslant \left(R_x^{-n}\sum_{i=1}^{N}\sum_{|\alpha|=m_i-1}\int_{B(x,R_x)} |D^\alpha w^i_{x^0R_1}(y) - (D^\alpha w^i_{x^0R_1})_{x^0,R_1}|^2 \mathrm{d}y\right)^{\frac{1}{2}} \\
&+ \left(R_x^{-n}\sum_{i=1}^{N}\sum_{|\alpha|=m_i-1}\int_{B(x,R_x)} |D^\alpha v^i_{x^0R_1}(y)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \\
&+ \left(R_x^{-n}\sum_{i=1}^{N}\sum_{|\alpha|=m_i-1}\int_{B(x,R_x)} |D^\alpha v^i_{xR_x}(y)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{R_1}{R_1-\delta}\right)^{\frac{n}{2}}\left[(W^{R_1}(x^0,R_1))^{\frac{1}{2}} + \gamma_1 R_1^\omega\right] + \gamma_1 R_x^\omega < \varepsilon_0.
\end{aligned}$$

It follows from Lemma 1.15 that if $W^{x,R}(x,R) < \varepsilon_0^2$, $0 < R \leqslant R_x$, then

$$W^{x,R}(x,\tau R) \leqslant 2\Lambda\tau^2 W^{x,R}(x,R). \tag{1.51}$$

Using (1.45), (1.44) and (1.51) we have

$$\begin{aligned}
(W^{x,\tau R}(x,\tau R))^{\frac{1}{2}} &\leqslant \left(\tau^{-n}R^{-n}\sum_{i=1}^{N}\sum_{|\alpha|=m_i-1}\int_{B(x,\tau R)} |D^\alpha w^i_{x\tau R}(y) - (D^\alpha w^i_{xR})_{x,\tau R}|^2 \mathrm{d}y\right)^{\frac{1}{2}} \\
&\leqslant \sqrt{2\Lambda}\,\tau(W^{x,R}(x,R))^{\frac{1}{2}} + \gamma_1\tau^{-\frac{n}{2}}R^\omega + \gamma_1(\tau R)^\omega \\
&\leqslant \tau^\omega(W^{x,R}(x,R))^{\frac{1}{2}} + \gamma_2 R^\omega.
\end{aligned}$$

By induction we obtain

$$\begin{aligned}
(W^{x,\tau^k R_x}(x,\tau^k R_x))^{\frac{1}{2}} &\leqslant \tau^{k\omega}(W^{x,R_x}(x,R_x))^{\frac{1}{2}} \\
&\quad + \gamma_2 k\tau^{(k-1)\omega}R_x^\omega, \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{1.52}$$

Using (1.52), (1.47) and (1.50) we have

$$\left(W^{x,\tau^k R_x}(x,\tau^k R_x)\right)^{\frac{1}{2}} \leqslant \tau^{k\mu}\left\{\varepsilon_0 \tau^{k(\omega-\mu)} + \frac{\varepsilon_0}{2}\tau^{-\omega} k\tau^{k(\omega-\mu)}\right\}.$$

Because $\lim\limits_{k\to\infty} \tau^{k(\omega-\mu)} = 0$, $\lim\limits_{k\to\infty} k\tau^{k(\omega-\mu)} = 0$, it is clear that there exists a constant γ_3 such that

$$(1.53) \qquad \left(W^{x,\tau^k R_x}(x,\tau^k R_x)\right)^{\frac{1}{2}} \leqslant \gamma_3 \tau^{k\mu}, \quad k \in \mathbf{N}.$$

Now (1.6), (1.44) and (1.53) imply that

$$\begin{aligned}\left(U(x,\tau^k R_x)\right)^{\frac{1}{2}} &\leqslant \left(W^{x,\tau^k R_x}(x,\tau^k R_x)\right)^{\frac{1}{2}} + \gamma_1(\tau^k R_x)^{\omega} \\ &\leqslant \tau^{k\mu}(\gamma_3 + \gamma_1 \tau^{k(\omega-\mu)} R_1^{\omega}).\end{aligned}$$

It follows from this estimate that there exists a constant γ_4 such that

$$(1.54) \qquad \left(U(x,\tau^k R_x)\right)^{\frac{1}{2}} \leqslant \gamma_4 \tau^{k\mu}, \quad k \in \mathbf{N}.$$

Let $0 < \varrho < R_1 - \delta < R_x$. Then there exists $k \in \mathbf{N}$ such that $\tau^{k+1} R_x \leqslant \varrho < \tau^k R_x$. Using (1.54) we obtain

$$\begin{aligned}\varrho^n U(x,\varrho) &\leqslant \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} \int_{B(x,\varrho)} |D^\alpha u^i(y) - (D^\alpha u^i)_{x,\tau^k R_x}|^2 \mathrm{d}y \\ &\leqslant (\tau^k R_x)^n \cdot U(x,\tau^k R_x)\end{aligned}$$

and

$$\tau^n U(x,\varrho) \leqslant U(x,\tau^k R_x) \leqslant \gamma_4^2 \left(\frac{\varrho}{\tau R_x}\right)^{2\mu}.$$

The latter estimate implies that

$$U(x,\varrho) \leqslant \frac{\gamma_4^2}{\tau^{n+2\mu}(R_1-\delta)^{2\mu}} \cdot \varrho^{2\mu}$$

and

$$\varrho^{-(n+2\mu)} \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} \int_{B(x,\varrho)} |D^\alpha u^i(y) - (D^\alpha u^i)_{x,\varrho}|^2 \mathrm{d}y \leqslant \frac{\gamma_4^2}{\tau^{n+2\mu}(R_1-\delta)^{2\mu}},$$

$$\varrho \in (0, R_1-\delta), \quad x \in B(x^0,\delta).$$

This estimate, the definition of Campanato space and the imbedding theorem imply that the assertion of our lemma is true. □

Remark 1.55. Using (1.6), (1.44) it is a matter of simple calculation to prove that for $x \in \Omega$, $0 < R \leqslant \min\{\operatorname{dist}(x, \partial\Omega), \overline{R}\}$ and $u \in [M]$

$$(W^R(x,R))^{\frac{1}{2}} \leqslant (U(x,R))^{\frac{1}{2}} + \gamma_1 R^{\omega},$$
$$(U(x,R))^{\frac{1}{2}} \leqslant (W^R(x,R))^{\frac{1}{2}} + \gamma_1 R^{\omega}.$$

From these estimates we obtain the identity

$$\lim_{R\to 0+} \inf U(x,R) = \lim_{R\to 0+} \inf W^R(x,R).$$

Lemma 1.49 and Remark 1.55 immediately imply

Lemma 1.56. *Suppose that $u \in [M]$ and the right-hand sides of the system* (0.1) *belong to $[G]$. Let* (0.3), (0.4), (0.5) *be satisfied. Let Ω' be a domain such that $\overline{\Omega'} \subset \Omega$. Let*

$$\lim_{R\to 0+} \inf U(x,R) = 0 \tag{1.57}$$

uniformly with respect to $x \in \overline{\Omega'}$ and $u \in [M]$.

Then $u \in C^{\underline{m-1},\mu}(\overline{\Omega'})$, $\mu \in \left(0, 1 - \frac{n}{p}\right)$ and the a-priori estimate

$$\|u\|_{C^{\underline{m-1},\mu}(\overline{\Omega'})} \leqslant c\big(M, G, A, \nu, \Omega', \operatorname{dist}(\Omega', \partial\Omega)\big) \tag{1.58}$$

holds uniformly with respect to the class $[M] \cup [G]$.

2. Main results

Theorem 2.1. *Let $u \in [M]$ and let the right-hand sides of the system* (0.1) *belong to $[G]$. Let Ω' be a domain such that $\overline{\Omega'} \subset \Omega$. Suppose that* (0.3), (0.4), (0.5) *and the condition* (L) *is satisfied. Then there exists a constant $c = c(M, G, A, \nu, \Omega')$ such that*

$$\|u\|_{C^{\underline{m-1},\mu}(\overline{\Omega'})} \leqslant c, \quad \mu \in \left(0, 1 - \frac{n}{p}\right).$$

Proof. For all $x^0 \in \overline{\Omega'}$ and $R > 0$ we shall define the transformation T_{x^0R}: $y = T_{x^0R}(x) = \frac{x - x^0}{R}$. For $u \in [M]$ we define on $O_{x^0R} = T_{x^0R}(\Omega)$:

$$\begin{cases} u^i_{x^0R}(y) = \dfrac{u^i(x^0 + Ry)}{R^{m_i - 1}} - \displaystyle\sum_{|\gamma| < m_i - 1} \dfrac{D^\gamma u^i(x^0)}{R^{m_i - 1 - |\gamma|}\gamma!} y^\gamma & \text{if } m_i > 1 \\ u^i_{x^0R}(y) = u^i(x^0 + Ry) & \text{if } m_i = 1. \end{cases} \tag{2.2}$$

From (2.2) it follows that

$$(2.3)\qquad D^\alpha u^i_{x^0R}(0)=0,\quad |\alpha|\leqslant m_i-2,\quad m_i>1,\quad i=1,\dots,N,$$

(2.4)
$$D^\alpha u^i(x^0+Ry)=R^{m_i-1-|\alpha|}D^\alpha u^i_{x^0R}(y)+\sum_{\substack{|\gamma|<m_i-1,\\ \alpha\leqslant\gamma}} R^{|\gamma-\alpha|}B_{\gamma,\alpha}\frac{D^\gamma u(x^0)}{\gamma!}y^{\gamma-\alpha},$$
$$|\alpha|\leqslant m_i-2,\ m_i>1,\ i=1,\dots,N,$$

$B_{\gamma,\alpha}$ being constants which are related to the derivative of "y^γ".

$$(2.5)\quad \begin{cases} D^\alpha u^i(x^0+Ry)=D^\alpha u^i_{x^0R}(y) \text{ a.e. in } O_{x^0R}, & |\alpha|=m_i-1,\ i=1,\dots,N,\\ RD^\alpha u^i(x^0+Ry)=D^\alpha u^i_{x^0R}(y) \text{ a.e. in } O_{x^0R}, & |\alpha|=m_i,\ i=1,\dots,N. \end{cases}$$

Let us choose a number $a>0$. Then there exists $R_0>0$ such that for $\forall x^0\in\overline{\Omega'}$ and $0<R\leqslant R_{0}$, $B(0,2a)\subset O_{x^0R}$. From (2.5) and (2.3) it follows that $D^\alpha u^i_{x^0R}$, $|\alpha|\leqslant m_i-1$, $i=1,\dots,N$, are bounded uniformly with respect to $x^0\in\overline{\Omega'}$ and $0<R\leqslant R_0$. Clearly there exists a constant $t>0$ such that for all $x^0\in\overline{\Omega'}$ and $0<R\leqslant R_0$

$$(2.6)\qquad \|u_{x^0R}\|_{H^{\underline{m-1}}(B(0,2a))}\leqslant t.$$

Now let $\varphi\in[\mathcal{D}(O_{x^0R})]^N$. Putting $R^{m_i+1}\cdot\varphi^i(\frac{x-x^0}{R})\in\mathcal{D}(\Omega)$ in (0.2) as a test function and using the transformation $x=x^0+Ry$ we have
(2.7)
$$\sum_{i,j=1}^N\sum_{\substack{|\alpha|\leqslant m_i\\|\beta|=m_j}}\int_{O_{x^0R}} R^{m_i-|\alpha|+1}A^{\alpha\beta}_{ij}(x^0+Ry,\delta(u(x^0+Ry)))D^\beta u^j(x^0+Ry)D^\alpha\varphi^i(y)\,dy$$
$$=\sum_{i=1}^N\sum_{|\alpha|\leqslant m_i}\int_{O_{x^0R}} R^{m_i-|\alpha|+1}g^i_\alpha(x^0+Ry)D^\alpha\varphi^i(y)\,dy.$$

Let $\chi\in\mathcal{D}(B(0,2a))$ be such that $\chi=1$ in $B(0,a)$, $0\leqslant\chi\leqslant1$ in $B(0,2a)$. Using the notation from the introduction, (2.5) and putting $\varphi^i=u^i_{x^0R}\cdot\chi^{2k}$, $i=1,\dots,N$,

$k = \max_i \{m_i\}$ in (2.7) we have

$$\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{B(0,2a)} A_{ij}^{\alpha\beta}\left(x^0 + Ry, \delta_1\left(u(x^0+Ry)\right), \delta_2\left(u_{x^0R}(y)\right)\right) \cdot \left(D^\beta u^j_{x^0R}(y)\chi^k(y)\right)$$

$$\times \left(D^\alpha u^j_{x^0R}(y)\chi^k(y)\right) dy$$

$$= - \sum_{i,j=1}^{N} \sum_{|\beta|=m_j} \left(\sum_{|\alpha|=m_i} \sum_{\gamma<\alpha} + \sum_{|\alpha|\leqslant m_i} \sum_{\gamma\leqslant\alpha} \right)$$

$$\int_{B(0,2a)} \binom{\alpha}{\gamma} R^{m_i-|\alpha|} A_{ij}^{\alpha\beta}\left(x^0 + Ry, \delta_1\left(u(x^0+Ry)\right), \delta_2\left(u_{x^0R}(y)\right)\right)$$

$$\times \left(D^\beta u^j_{x^0R}(y)\chi^k(y)\right) \cdot D^\gamma u^i_{x^0R}(y) Z^{\alpha-\gamma}(\chi^{2k})\, dy$$

$$+ \sum_{i=1}^{N} \sum_{|\alpha|\leqslant m_i} \int_{B(0,2a)} R^{m_i-|\alpha|+1} g^i_\alpha(x^0+Ry) D^\alpha\left(u^i_{x^0R}(y)\chi^{2k}(y)\right) dy,$$

where $Z^{\alpha-\gamma}(\chi^{2k})$ is introduced in (1.31).

Let us denote the left-hand side of this equality by (LS), the first term on the right-hand side by (RS_1) and the second by (RS_2).

Using (0.4) we have

$$\nu \cdot J =: \nu \cdot \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B(0,2a)} |D^\alpha u^i_{x^0R}(y)\chi^k(y)|^2 dy \leqslant \text{(LS)}.$$

Using the fact that $A_{ij}^{\alpha\beta}$ are uniformly bounded and the Hölder inequality we obtain

$$|(\mathrm{RS}_1)| \leqslant c_1 \|u_{x^0R}\|_{H^{\underline{m-1}}(B(0,2a))} J^{\frac{1}{2}}.$$

(0.5), the Leibniz formula, the Hölder inequality and the boundedness of $\chi^k(y)$ imply

$$|(\mathrm{RS}_2)| \leqslant c_2 J^{\frac{1}{2}} + c_3 \|u_{x^0R}\|_{H^{\underline{m-1}}(B(0,2a))}.$$

Using the inequality (LS) $\leqslant |(\mathrm{RS}_1)| + |(\mathrm{RS}_2)|$ and (2.6) we obtain the estimate

$$J \leqslant c_4 J^{\frac{1}{2}} + c_5.$$

This estimate and (2.6) imply that there exists a constant $c_6 = c_6(\nu, a, M, G, A)$ such that for all $x^0 \in \overline{\Omega'}$, $R \in (0, R_0]$, $u \in [M]$

$$\|u_{x^0R}\|_{H^{\underline{m}}(B(0,a))} \leqslant c_6. \tag{2.8}$$

Now we shall prove that (1.57) holds uniformly with respect to $x^0 \in \overline{\Omega'}$ and $u \in [M]$.

Let us suppose the contrary. Then there exist $\{x^k\}_{k=1}^\infty \subset \overline{\Omega'}$, $x^k \to \bar{x} \in \overline{\Omega'}$, $\{R_k\}_{k=1}^\infty \subset \mathbf{R}^+$, $R_k \to 0$, $\{u_k\}_{k=1}^\infty \subset [M]$ and $\varepsilon > 0$ such that

$$U(x^k, R_k) \geqslant \varepsilon. \tag{2.9}$$

The estimate (2.8) implies that there exists a subsequence $\{u_{kx^kR_k}\}_{k=1}^\infty$ such that

$$\begin{aligned} u_{kx^kR_k} &\rightharpoonup P \text{ in } H^{\underline{m}}(B(0,a)),\\ u_{kx^kR_k} &\to P \text{ in } H^{\underline{m-1}}(B(0,a)),\\ D^\alpha u^i_{kx^kR_k} &\to D^\alpha P^i \text{ a.e. in } B(0,a),\ |\alpha| \leqslant m_i - 1,\ i = 1,\dots,N. \end{aligned}$$

Putting $a = r$, $r \in \mathbf{N}$ and using the diagonalization process we obtain for all $r \in \mathbf{N}$

$$u_{kx^kR_k} \rightharpoonup P \text{ in } H^{\underline{m}}(B(0,r)), \tag{2.10}$$

$$u_{kx^kR_k} \to P \text{ in } H^{\underline{m-1}}(B(0,r)), \tag{2.11}$$

$$D^\alpha u^i_{kx^kR_k} \to D^\alpha P^i \text{ a.e. in } B(0,r),\ |\alpha| \leqslant m_i - 1,\ i = 1,\dots,N. \tag{2.12}$$

From (2.5) and (2.12) it follows that there exists a constant $\tau > 0$ such that

$$|D^\alpha P^i| \leqslant \tau, \quad |\alpha| \leqslant m_i - 1, \quad i = 1,\dots,N. \tag{2.13}$$

Now let $\psi \in [\mathcal{D}(\mathbf{R}^n)]^N$. It is clear that there exist r, R_1 such that $\operatorname{supp}\psi \subset B(0,r) \subset O_{x^0R}$ for all $x^0 \in \overline{\Omega'}$ and $0 < R \leqslant R_1$. Putting $\varphi = \psi$ in (2.7) we have

$$\begin{aligned} &\sum_{i,j=1}^N \sum_{\substack{|\alpha|\leqslant m_i\\|\beta|=m_j}} \int_{B(0,r)} R_k^{m_i-|\alpha|} A_{ij}^{\alpha\beta}\big(x^k + R_k y, \delta_1(u_k(x^k + R_k y)), \delta_2(u_{kx^kR_k}(y))\big)\\ &\qquad \times D^\beta u^j_{kx^kR_k}(y) D^\alpha \psi^i(y)\,dy\\ &= \sum_{i=1}^N \sum_{|\alpha|\leqslant m_i} \int_{B(0,r)} R_k^{m_i-|\alpha|+1} g^i_\alpha(x^k + R_k y) D^\alpha \psi^i(y)\,dy. \end{aligned} \tag{2.14}$$

The fact that the functions $A_{ij}^{\alpha\beta}$ and $D^\alpha\psi^i$ are uniformly bounded and the formula (2.10) imply

$$\begin{aligned} &\sum_{i,j=1}^N \sum_{\substack{|\alpha|< m_i\\|\beta|=m_j}} \int_{B(0,r)} R_k^{m_i-|\alpha|} A_{ij}^{\alpha\beta}\big(x^k + R_k y, \delta_1(u_k(x^k + R_k y)), \delta_2(u_{kx^kR_k}(y))\big)\\ &\qquad \times D^\beta u^j_{kx^kR_k} \cdot D^\alpha \psi^i\,dy \to 0 \text{ for } k \to \infty. \end{aligned} \tag{2.15}$$

If $|\alpha| = m_i$, $i = 1, \ldots, N$ then $R_k^{m_i-|\alpha|} = 1$, $x^k + R_k y \to \bar{x}$ for $k \to \infty$. Because $H^{m_i-1,\infty}(\Omega) \circlearrowleft H^{m_i-1,p}(\Omega) \circlearrowleft\circlearrowleft C^{m_i-2}(\overline{\Omega})$, it is clear that $u_k^i \to P^i$ in $C^{m_i-2}(\overline{\Omega})$ (i.e. $D^\alpha u_k^i \rightrightarrows D^\alpha P^i$ on $\overline{\Omega}$, $|\alpha| \leqslant m_i - 2$, $i = 1, \ldots, N$) and $\delta_1(u_k(x^k + R_k y)) \to \delta_1(P(\bar{x}))$ in $B(0,r)$, $k \to \infty$.

From (2.5), (2.12) it follows that

$$\delta_2(u_{kx^kR_k}(y)) \to \delta_2(P(y)) \quad \text{a.e. in } B(0,r),\ k \to \infty.$$

Using (0.3), Lebesgue's dominated convergence theorem and (2.10) we obtain

$$\begin{aligned}(2.16)\qquad & \int_{B(0,r)} A_{ij}^{\alpha\beta}(x^k + R_k y, \delta_1(u_k(x^k + R_k y)), \delta_2(u_{kx^kR_k}(y))) \\ & \qquad \times D^\beta u_{kx^kR_k}^j(y) D^\alpha \psi^i(y)\,\mathrm{d}y \\ & \longrightarrow \int_{B(0,r)} A_{ij}^{\alpha\beta}(\bar{x}, \delta_1(P(\bar{x})), \delta_2(P(y))) D^\beta P^j(y) D^\alpha \psi^i(y)\,\mathrm{d}y\end{aligned}$$

for $k \to \infty$, $i, j = 1, \ldots, N$, $|\alpha| = m_i$, $|\beta| = m_j$.

The fact that $p_\alpha^i \geqslant 2$ ($i = 1, \ldots, N$, $|\alpha| \leqslant m_i$), the transformation $x = x^k + R_k y$ and the Hölder inequality imply

$$\begin{aligned}(2.17)\qquad & \left| \int_{B(0,r)} R_k^{m_i-|\alpha|+1} g_\alpha^i(x^k + R_k y) D^\alpha \psi^i(y)\,\mathrm{d}y \right| \\ & \qquad \leqslant c_7 R_k^{m_i-|\alpha|+1} \cdot R_k^{-n} \int_{B(x^k, rR_k)} |g_\alpha^i(x)|\,\mathrm{d}x \\ & \qquad \leqslant c_8 R_k^{(m_i-|\alpha|+1)(1-\frac{n}{p})} G \to 0 \text{ if } k \to \infty.\end{aligned}$$

From (2.15), (2.16) and (2.17) it follows that

$$(2.18)\qquad \sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{\mathbb{R}^n} A_{ij}^{\alpha\beta}(\bar{x}, \delta_1(P(\bar{x})), \delta_2(P(y))) D^\beta P^j(y) D^\alpha \psi^i(y)\,\mathrm{d}y = 0,$$
$$\psi \in [\mathcal{D}(\mathbb{R}^n)]^N.$$

The condition (L) and (2.13) imply that $P \in \underline{P^N_{m-1}}$. Using (2.5), (2.9), (2.11) and the transformation $x = x^k + R_k y$ we have

$$0 < \varepsilon \leqslant \liminf_{k\to\infty} R_k^{-n} \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} \int_{B(x^k,R_k)} |D^\alpha u_k^i(x) - (D^\alpha u_k^i)_{x^k R_k}|^2 dx$$
$$\leqslant \liminf_{k\to\infty} \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} \int_{B(0,1)} |D^\alpha u^i_{k x^k R_k}(y) - D^\alpha P^i|^2 dy = 0.$$

This implies that (1.57) holds uniformly with respect to $x^0 \in \overline{\Omega'}$ and $u \in [M]$. Lemma 1.56 implies the assertion of the theorem. □

By the standard method from [2], [4] we shall prove

Theorem 2.19. *Suppose that the system* (0.1) *has the property of regularity* (R). *Then it has Liouville's property* (L).

Proof. Let $x^0 \in \Omega$, $\xi \in \mathbf{R}^\tau$ and let u be a solution (in $\mathbf{R}^n$) to the system

(2.20)
$$\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{\mathbf{R}^n} A_{ij}^{\alpha\beta}(x^0,\xi,\delta_2(u(x)))D^\beta u^j(x) D^\alpha \varphi^i(x)\,dx = 0,$$
$$\varphi \in [\mathcal{D}(\mathbf{R}^n)]^N,$$

such that for $M > 0$

(2.21)
$$|D^\alpha u^i| \leqslant M, \quad |\alpha| \leqslant m_i - 1, \quad i = 1,\dots,N.$$

For $R > 0$ we define

$$u_R^i(y) = \frac{u^i(Ry)}{R^{m_i-1}}, \quad i = 1,\dots,N.$$

It is clear that

(2.22)
$$\begin{cases} D^\alpha u_R^i(y) = D^\alpha u^i(Ry), & |\alpha| = m_i - 1, \quad i = 1,\dots,N, \\ D^\alpha u_R^i(y) = R D^\alpha u^i(Ry), & |\alpha| = m_i, \quad i = 1,\dots,N. \end{cases}$$

Let $\varphi \in [\mathcal{D}(\mathbf{R}^n)]^N$.

Putting $\varphi(\frac{x}{R})$ as a test function in (2.20) and using the transformation $x = Ry$ we have

(2.23)
$$\sum_{i,j=1}^{N} \sum_{\substack{|\alpha|=m_i \\ |\beta|=m_j}} \int_{\mathbf{R}^n} A_{ij}^{\alpha\beta}(x^0,\xi,\delta_2(u_R(y)))D^\beta u_R^j(y) D^\alpha \varphi^i(y)\,dy = 0.$$

(2.21), (2.22) and the property (R) imply

$$|D^\alpha u^i_R(y) - D^\alpha u^i_R(0)| \leqslant c|y|^\mu, \qquad (2.24)$$
$$|\alpha| = m_i - 1,\ i = 1,\dots,N,\ R > 0,\ y \in \overline{B(0,\eta)},\ \eta > 0,\ \mu \in (0,1).$$

Let us choose $x \in \mathbf{R}^n$. Then there exists $R_0 > 0$ such that $y_R = \frac{x}{R} \in \overline{B(0,\eta)}$ for all $R \geqslant R_0$.

Using (2.24) and (2.22) we obtain

$$|D^\alpha u^i(x) - D^\alpha u^i(0)| \leqslant c\frac{|x|^\mu}{R^\mu},\ |\alpha| = m_i - 1, \quad i = 1,\dots,N,\ R \geqslant R_0. \qquad (2.25)$$

For R tending to infinity we have

$$D^\alpha u^i(x) = D^\alpha u^i(0) \quad \text{for all } x \in \mathbf{R}^n,\ |\alpha| = m_i - 1,\ i = 1,\dots,N.$$

This fact implies that $u \in P^N_{m-1}$. □

Remark 2.26. Using the method from [2], [4] we could prove that the system (0.1) has Liouville's property (L) for $n = 2$, i.e. for plane domains.

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