Włodzimierz M. Mikulski

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NATURAL DIFFERENTIAL OPERATORS BETWEEN SOME NATURAL BUNDLES

W. M. Mikulski, Kraków

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Summary. Let F and G be two natural bundles over n-manifolds. We prove that if F is of type (I) and G is of type (II), then any natural differential operator of F into G is of order 0. We give examples of natural bundles of type (I) or of type (II). As an application of the main theorem we determine all natural differential operators between some natural bundles.

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1. Natural bundles and bundle functors

Throughout the paper all manifolds are assumed to be paracompact, without boundary, second countable, finite dimensional and smooth, i.e. of class C^∞. Maps will be assumed to be C^∞.

Let Mf_n be the category of all n-dimensional manifolds and their embeddings, FMf_n the category of all fibered manifolds over n-manifolds and their morphisms, and let $B\colon FMf_n \to Mf_n$ be the base functor. Given a functor $F\colon Mf_n \to FMf_n$ satisfying $B\circ F = \mathrm{id}_{Mf_n}$, we denote by $p^F_M\colon FM \to M$ its value on an n-manifold M and by $F_xf\colon F_xM \to F_{f(x)}N$ the restriction of its value $Ff\colon FM \to FN$ for $f\colon M \to N$ to the fibres of FM over x and of FN over $f(x)$, $x\in M$.

The concept of a natural bundle over n-manifolds was introduced by A. Nijenhuis ([11]) as a modern approach to the classical theory of geometric objects. Using the results of D. B. A. Epstein and W. P. Thurston ([3]) we can formulate the definition of natural bundles as follows.

Definition 1.1. A natural bundle over n-manifolds is a functor $F\colon MF_n \to FMf_n$ satisfying $B \circ F = \mathrm{id}_{Mf_n}$ and the localization condition: if $i\colon U \to M$ is the inclusion of an open subset, then $Fi\colon FU \to (p_M^F)^{-1}(U)$ is a diffeomorphism.

If we replace (in Definition 1.1) Mf_n by the category Mf of all manifolds and all maps and FMf_n by the category FMf of all fibered manifolds and their morphisms we obtain the concept of bundle functors, [7]. The Weil functors T^A of A velocities ([10]) and the functors $T^{(r)}$ of linear higher order tangent bundles ([6]) are examples of bundle functors. Of course, the restriction of a bundle functor to Mf_n is a natural bundle. The classification of natural bundles can be obtained by the Palais-Terng theorem [12], see [5].

We say that a natural bundle (a bundle functor) F is linear if $F\colon Mf_n \to VMf_n$ ($F\colon Mf \to VMf$), where VMf_n (VMf) is the category of all vector bundles over n-manifolds (of all vector bundles, respectively) and their vector bundle morphisms.

Let M, N, P be manifolds. A parametrized system of smooth maps $f_p\colon M \to N$, $p \in P$ is said to be smoothly parametrized, if the resulting map $f\colon M \times P \to N$ is of class C^∞.

Proposition 1.1 ([3], [8]). *Every natural bundle $F\colon Mf_n \to FMf_n$ (every bundle functor $F\colon Mf \to FMf$) satisfies the regularity condition: if $f\colon M \times P \to N$ is a smoothly parametrized family of embeddings (of maps, respectively), then the family $\widetilde{Ff}\colon FM \times P \to FN$ defined by $(\widetilde{Ff})_p = F(f_p)$ is also smoothly parametrized.*

2. Natural differential operators

Let $F, G\colon Mf_n \to FMf_n$ be two natural bundles. Given a manifold $M \in Mf_n$ we denote by ΓFM the set of all locally defined smooth sections of the bundle $FM \to M$. Suppose that we have a family D of functions $D_M\colon \Gamma FM \to \Gamma GM$, $M \in Mf_n$ such that $\mathrm{Dom}\,(D_M(\sigma)) = \mathrm{Dom}(\sigma)$ for any $\sigma \in \Gamma FM$.

Definition 2.1 ([14]. A family D as above is called a natural differential operator of F into G if the following naturality condition is satisfied: for any $M \in Mf_n$, any $\sigma \in \Gamma FM$ and any embedding $\varphi\colon M \to N$ of two n-manifolds we have $D_N(\varphi_*^F\sigma) = \varphi_*^G D_M(\sigma)$, where $\varphi_*^F\colon \Gamma FM \to \Gamma FN$ is defined by $\varphi_*^F\sigma = F\varphi \circ \sigma \circ \varphi^{-1}$.

Definition 2.2. Let $r \geqslant 0$ be an integer or infinity. Let D be a natural differential operator of F into G. We say that D is of order r if for any $M \in Mf_n$, any $x \in M$ and any two sections $\sigma_1, \sigma_2 \in \Gamma FM$ defined on some open neighbourhoods of x the condition $j_x^r\sigma_1 = j_x^r\sigma_2$ implies $D_M(\sigma_1)(x) = D_M(\sigma_2)(x)$, and r is the smallest number with the above property.

J. Slovák has proved that every natural differential operator has locally finite order, [13]. In particular, every natural differential operator is of order $\leqslant \infty$.

In [2], D. B. A. Epstein has constructed such natural differential operator D of F into G and such sections σ_n, $\sigma \in \Gamma F\mathbf{R}^2$, $n = 1, 2, \ldots$, that $j_0^\infty \sigma_n \to j_0^\infty \sigma$ and $D_{\mathbf{R}^2}(\sigma_n)(0) \not\to D_{\mathbf{R}^2}(\sigma)(0)$ as $n \to \infty$. Therefore the following proposition is interesting.

Proposition 2.1 ([9]). *Let D be a natural differential operator of F into G, where F, G are natural bundles over n-manifolds. Suppose that σ_m, $\sigma \in \Gamma FM$, $m = 1, 2, \ldots$ are local sections defined on some open neighbourhoods of $x \in M$ such that $j_x^\infty \sigma_m \to j_x^\infty \sigma$ as $m \to \infty$. Suppose also that there exists a vector field X on M such that $X(x) \neq 0$ and $j_x^\infty(L_X\sigma) = j_x^\infty(0)$, where $L_X\sigma$ is the Lie derivative of σ with respect to X. Then $D_M(\sigma_m)(x) \to D_M(\sigma)(x)$ as $m \to \infty$.*

Remark. $L_X\sigma$: $\mathrm{Dom}(\sigma) \to TFM$ is defined as follows. For any $y \in \mathrm{Dom}(\sigma)$, $(L_X\sigma)(y)$ is the vector from $T_{\sigma(y)}FM$ given by a curve $t \to (\varphi_{-t})_*^F\sigma(y)$, where $\{\varphi_t\}$ is a local flow of X near y.

3. Natural bundles of type (I)

Definition 3.1. Let $F\colon \mathcal{M}f_n \to \mathcal{FM}f_n$ be a natural bundle. A section $\sigma_o \in \Gamma F\mathbf{R}^n$ is invariant with respect to the translations iff $(\tau_x)_*^F\sigma_o = \sigma_o$ for any $x \in \mathbf{R}^n$, where $\tau_x\colon \mathbf{R}^n \to \mathbf{R}^n$ is the translation by x. We say that F is of type (I) iff for any $\sigma \in \Gamma F\mathbf{R}^n$ defined on an open neighbourhood of $0 \in \mathbf{R}^n$ there exists $\sigma_o \in \Gamma F\mathbf{R}^n$ invariant with respect to the translations such that $j_0^\infty((t\,\mathrm{id})_*^F\sigma) \to j_0^\infty(\sigma_0)$ as $t \to \infty$, where $\mathrm{id}: \mathbf{R}^n \to \mathbf{R}^n$ is the identity map.

In this section we give some examples of natural bundles of type (I). We start with the following proposition.

Proposition 3.1. *Let $F\colon \mathcal{M}f_n \to \mathcal{FM}f_n$ be a natural bundle. Suppose that for every $v \in F_0\mathbf{R}^n$ there exists a chart (V, φ) on $F_0\mathbf{R}^n$ defined on an open neighbourhood of v onto an open neighbourhood of $0 \in \mathbf{R}^k$ such that*

(1) $F_0(t\,\mathrm{id})(V) \subset V$ for all sufficiently large $t \in \mathbf{R}$,

(2) $\varphi \circ F_0(t\,\mathrm{id}) \circ \varphi^{-1}$ is linear on $\varphi(V)$ for all sufficiently large t, and

(3) for every $w \in V$ there exists $\lim_{t\to\infty} F_0(t\,\mathrm{id})(w) \in V$.

Then F is of type (I).

Proof. Let us consider a section $\sigma \in \Gamma F\mathbf{R}^n$ defined on an open neighbourhood of $0 \in \mathbf{R}^n$. We can assume that $F_0\mathbf{R}^n$ is an open neighbourhood of $0 \in \mathbf{R}^k$ and that $F_0(t\,\mathrm{id})$ is linear for sufficiently large t. Moreover, we can assume that for every $w \in F_0\mathbf{R}^n$ there exists $\lim_{t\to\infty} F_0(t\,\mathrm{id})(w)$. Using the trivialization $\mathbf{R}^n \times F_0\mathbf{R}^n \ni (x, v) \to F\tau_x(v) \in F\mathbf{R}^n$ we identify σ with the map $\bar{\sigma}$: $\mathrm{Dom}(\sigma) \to F_0\mathbf{R}^n$. Then $(t\,\mathrm{id})^F_*\sigma$ is identified with $F_0(t\,\mathrm{id})\circ\bar{\sigma}\circ(\frac{1}{t}\,\mathrm{id})$. It is easy to see that $j_0^\infty\big(F_0(t\,\mathrm{id})\circ\bar{\sigma}\circ(\frac{1}{t}\,\mathrm{id})\big)$ tends to $j_0^\infty(x \to v_0)$ as $t \to \infty$, where $v_0 = \lim_{t\to\infty} F_0(t\,\mathrm{id})(\sigma(0))$. Therefore $j_0\big((t\,\mathrm{id})^F_*\sigma\big) \to j_0\sigma_0$ as $t \to \infty$, where $\sigma_0 \in \Gamma F\mathbf{R}^n$ is defined by $\sigma_0(y) = F\tau_y(v_0)$. □

We have the following obvious corollaries of Proposition 3.1.

Corollary 3.1. *Let F: $\mathcal{M}f_n \to \mathcal{FM}f_n$ be a natural bundle such that for every $t \in \mathbf{R}\setminus\{0\}$, $F_0(t\,\mathrm{id})$ is the identity map. Then F is of type* (I).

Corollary 3.2. *Let F: $\mathcal{M}f_n \to \mathcal{FM}f_n$ be a natural bundle. Suppose that there exists a diffeomorphism φ: $F_0\mathbf{R}^n \to \mathbf{R}^k$ such that $\varphi\circ F_0(t\,\mathrm{id})\circ\varphi^{-1}$ is linear for all $t \in \mathbf{R}\setminus\{0\}$. Then F is of type* (I) *if and only if for every $v \in F_0\mathbf{R}^n$ there exists $\lim_{t\to\infty} F(t\,\mathrm{id})(v)$.*

Corollary 3.3. *Let F: $\mathcal{M}f_n \to \mathcal{VM}f_n$ be a linear natural bundle. Then F is of type* (I) *if and only if for any $v \in F_0\mathbf{R}^n$ there exists $\lim_{t\to\infty} F(t\,\mathrm{id})(v)$.*

Example 3.1 (Grassmann natural bundles). Let $k \geqslant 1$ be an integer such that $k \leqslant n$. From Corollary 3.1 it follows that the Grassmann natural bundle G_k of k-planes tangent to n-manifolds ([4]) is of type (I).

Example 3.2 (natural bundles of connections of higher order). Let $r \geqslant 1$ be an integer. By Corollary 3.2, the natural bundle C_r of connections of order r on n-manifolds ([14]) is of type (I).

Example 3.3 (natural bundles of symmetric and antisymmetric covariant tensor fields). Let $p \geqslant 0$ be an integer. Using Corollary 3.3 we see that the natural bundle $\wedge^p T^*$ (S^pT^*) of p times covariant antisymmetric (symmetric, respectively) tensor fields is of type (I).

Example 3.4 (natural bundles of tensor fields). Let $p, q \geqslant 0$ be two integers. It follows from Corollary 3.3 that the natural bundle $T^{(p,q)}$ of q times covariant and p times contravariant tensor fields over n-manifolds is of type (I) if and only if $q \geqslant p$.

Example 3.5 (cotangent bundles of higher order). Let $r \geqslant 1$ be an integer. Let T^{r*}: $\mathcal{M}f_n \to \mathcal{VM}f_n$ be the linear natural bundle of cotangent bundles of order r defined as follows. Given $M \in \mathcal{M}f_n$, then $T^{r*}M = J^r(M,\mathbf{R})_0$ is the space of r-jets with target 0 of maps $M \to \mathbf{R}$. Then $T^{r*}M$ with the source projection is a vector bundle over M. Every embedding φ: $M \to N$ induces the map $T^{r*}\varphi$:

$T^{r*}M \to T^{r*}N$, $T^{r*}\varphi(j^r_x\gamma) = j^r_{\varphi(x)}(\gamma \circ \varphi^{-1})$. It is well known that $T^{(r)}|Mf_n$ is dual to T^{r*}, where $T^{(r)}: Mf \to VMf$ is the linear bundle functor of tangent bundles of order r, [6]. It follows from the following general lemma that T^{r*} is of type (I).

Lemma 3.1. *Let $E: Mf \to VMf$ be a linear bundle functor. Suppose that F: $Mf_n \to VMf_n$ is natural isomorphic to the dual natural bundle $(E|Mf_n)^*$. Then F is of type* (I).

Proof. One can assume that $F = (E|Mf_n)^*$. Consider an arbitrary $\omega \in F_0\mathbf{R}^n = (E_0\mathbf{R}^n)^*$. It follows from Proposition 1.1 that for every $v \in E_0\mathbf{R}^n$, $F(t\,\mathrm{id})(\omega)(v) = \omega(E(\frac{1}{t}\,\mathrm{id})(v)) \to \omega(E(0)(v))$ as $t \to \infty$. Therefore there exists $\lim_{t\to\infty} F(t\,\mathrm{id})(\omega)$, and we can apply Corollary 3.3. $\square$

Example 3.6 (fiber product of natural bundles of type (I)). Let $E, F: Mf_n \to FMf_n$ be two natural bundles of type (I). It is obvious that the fiber products $E \times_{()} F$ of E and F is of type (I).

Example 3.7 (natural closed subbundles of natural bundles of type (I)). Let E: $Mf_n \to FMf_n$ be a natural bundle of type (I). Let $F: Mf_n \to FMf_n$ be a natural closed subbundle of E, i.e. a natural bundle such that FM is a closed subbundle of EM for any $M \in Mf_n$, and $F\varphi = E\varphi|FM$ for any embedding $\varphi: M \to N$. It is easy to see that F is of type (I). For example, the natural bundle C^o_r of torsion free connections of order r on n-manifolds ([14]) is a natural closed subbundle of C_r. Hence C^o_r is of type (I).

4. Natural bundles of type (II)

Definition 4.1. Let $G: Mf_n \to FMf_n$ be a natural bundle. We say that G is of type (II) iff for every $v \in G_0\mathbf{R}^n$ either $G(t\,\mathrm{id})(v) = v$ for all $t \in \mathbf{R} \setminus \{0\}$ or $\lim_{t\to\infty} G(t\,\mathrm{id})(v)$ does not exist.

Example 4.1 (Grassmann natural bundles). The Grassmann natural bundle G_k of k-planes tangent to n-manifolds is of type (II).

Example 4.2 (natural bundles of tensor fields). Let $p, q \geqslant 0$ be integers. Then $T^{(p,q)}$ described in Example 3.4 is of type (II) iff $p \geqslant q$.

Example 4.3 (linear tangent bundles of high order). Let $r \geqslant 1$ be an integer. Let $T^{(r)}$ be the linear bundle functor of tangent bundles of order r, [6]. Then $T^{(r)}|Mf_n$ is of type (II) (see Lemma 4.1).

Example 4.4 (Weil functor). Let $T^A: \mathcal{M}f \to \mathcal{FM}f$ be the Weil functor of A velocities, [10]. Then (by Lemma 4.1) $T^A|\mathcal{M}f_n$ is of type (II).

Lemma 4.1. *Let $E: \mathcal{M}f \to \mathcal{FM}f$ be a bundle functor. Then $E|\mathcal{M}f_n$ is of type* (II).

Proof. Let $v \in E_0\mathbf{R}^n$ be a point. Suppose that $E(t\,\mathrm{id})(v) \to v_0$ as $t \to \infty$. In particular, $v_m = E(m\mathrm{id})(v) \to v_0$ as $m \to \infty$. Then it follows from Proposition 1.1 that $v = E(\frac{1}{m}\,\mathrm{id})(v_m) \to EO(v_0)$ as $m \to \infty$, i.e. $v = EO(v_0)$. Hence $E(t\,\mathrm{id})(v) = E(t\,\mathrm{id})EO(v_0) = EO(v_0) = v$ for all $t \in \mathbf{R} \setminus \{0\}$. □

Example 4.5 (natural subbundles). Let $E: \mathcal{M}f_n \to \mathcal{FM}f_n$ be a natural bundle of type (II). Let $G: \mathcal{M}f_n \to \mathcal{FM}f_n$ be a natural subbundle of E. Then G is of type (II).

Example 4.6 (natural bundles of non-vanishing tensor fields). Let $p, q \geqslant 0$ be integers. Then the natural bundle $T^{(p,q)\setminus o}: \mathcal{M}f_n \to \mathcal{FM}f_n$ of non-vanishing q times covariant and p times contravariant tensor fields is of type (II).

Example 4.7 (natural bundles of Riemannian metrics). The natural bundle $Riem: \mathcal{M}f_n \to \mathcal{FM}f_n$ of Riemannian metrics is of type (II).

Example 4.8 (fiber products of natural bundles of type (II)). Let $E, G: \mathcal{M}f_n \to \mathcal{FM}f_n$ be two natural bundles of type (II). It is obvious that the fiber product $E \times_{()} G$ is of type (II).

5. An order theorem

The main result of this paper is the following theorem.

Theorem 5.1. *Let $F, G: \mathcal{M}f_n \to \mathcal{FM}f_n$ be two natural bundles and let D be a natural differential operator of F into G. Suppose that F is of type* (I) *and G is of type* (II). *Then D is of order* 0. *Moreover, for any section $\sigma \in \Gamma F\mathbf{R}^n$ defined near $0 \in \mathbf{R}^n$ we have $G(t\,\mathrm{id})\big(D_{\mathbf{R}^n}(\sigma)(0)\big) = D_{\mathbf{R}^n}(\sigma)(0)$ for all $t \in \mathbf{R} \setminus \{0\}$.*

Remark. In [1], J. Dębecki has proved that any differential operator of $T^p \otimes T^{*p}$ into $T^q \otimes T^{*q}$ is of order 0. Thus Theorem 5.1 is a generalization of the above fact.

Proof. Consider a section $\sigma \in \Gamma F\mathbf{R}^n$ defined on a neighbourhood of 0. Let $\sigma_0 \in \Gamma F\mathbf{R}^n$ be a section invariant with respect to the translations such that $j_0^\infty\big((t\,\mathrm{id})_*^F\sigma\big) \to j_0\sigma_0$ as $t \to \infty$. Of course, $L_{\partial_1}\sigma_0 = 0$, where ∂_1 is the canonial vector field on $\mathbf{R}^n$. It follows from Proposition 2.1 that $G(t\,\mathrm{id})\big(D(\sigma)(0)\big) =$

$D((t\,\mathrm{id})^F_* \sigma)(0) \to D(\sigma_0)(0)$ as $t \to \infty$. Hence $G(t\,\mathrm{id})(D(\sigma)(0)) = D(\sigma)(0) = D(\sigma_0)(0)$ for all $t \in \mathbf{R} \setminus \{0\}$, for G is of type (II). Let $\sigma' \in \Gamma F\mathbf{R}^n$ be another section such that $\sigma'(0) = \sigma(0)$. Let $\sigma'_0 \in \Gamma F\mathbf{R}^n$ be invariant with respect to the translations such that $j^\infty_0((t\,\mathrm{id})^F_* \sigma') \to j_0\sigma'_0$ as $t \to \infty$. Then $\sigma_0(0) = \sigma'_0(0)$, i.e. $\sigma_0 = \sigma'_0$. Hence $D(\sigma)(0) = D(\sigma_0)(0) = D(\sigma'_0)(0) = D(\sigma')(0)$. Using the naturality condition we see that D is of order 0. □

6. Applications

We give the following two applications of Theorem 5.1.

(I) We determine all natural differential operators of a natural bundle of type (I) into the restriction of a bundle functor.

Let G be a bundle functor and P a fixed manifold such that $\mathrm{card}(P) = 1$. Let us denote by $G^P\colon Mf \to FMf$ the trivial bundle functor defined as follows. For every $M \in Mf$, $G^P M$ is the trivial bundle $M \times GP$ over M with the projection onto the first factor. For every map f we define $Gf = f \times \mathrm{id}_{GP}$. Let $i^G_M\colon G^P M \to GM$, $M \in Mf$, be the natural embedding given by $i^G_M(x, y) = G\tilde{x}(y)$, where $\tilde{x}\colon P \to \{x\} \subset M$. We prove the following corollary.

Corollary 6.1. *Let $F\colon Mf_n \to FMf_n$ be a natural bundle of type (I) and let G be a bundle functor. Then there exists a bijection between the set of all natural differential operators of F into $G^P|Mf_n$ and the set of all natural differential operators of F into $G|Mf_n$ given by $\{D_M\} \to \{i^G_M \circ D_M\}$. In particular, if G has the point property, i.e. $\mathrm{card}(GP) = 1$, [7], then every natural differential operator of F into $G|Mf_n$ is equal to the constant natural differential operator of F into $G|Mf_n$ given by $\Gamma FM \ni \sigma \to c^G_M|\,\mathrm{Dom}\,\sigma \in \Gamma GM$, where $c^G_M \in \Gamma GM$, $c^G_M(x) \in i^G_M(\{x\} \times GP) \subset G_x M$, $x \in M$.*

Proof. Of course, the function $\{D_M\} \to \{i^G_M \circ D_M\}$ is a well defined injection. We prove that the function is a surjection. Let $\tilde{D}$ be a natural differential operator of F into $G|Mf_n$. It follows from Lemma 4.1 that $G|Mf_n$ is of type (II). By Theorem 5.1, for every section $\sigma \in \Gamma F\mathbf{R}^n$ defined near 0 we have $G(t\,\mathrm{id})(\tilde{D}(\sigma)(0)) = \tilde{D}(\sigma)(0)$ for all $t \in \mathbf{R} \setminus \{0\}$, i.e. $\tilde{D}(\sigma)(0) \in \mathrm{Im}\, i^G_{\mathbf{R}^n}$ (see the proof of Lemma 4.1). Using the naturality condition we get that $\mathrm{Im}\,\tilde{D}(\sigma) \subset \mathrm{Im}\, i^G_M$ for any $\sigma \in \Gamma FM$. □

We say that a natural bundle $F\colon Mf_n \to FMf_n$ is transitive iff for every $v, w \in F_0\mathbf{R}^n$ there exists an embedding $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $F\varphi(v) = w$. If $F\colon Mf_n \to FMf_n$ is a transitive natural bundle of type (I) (for example, the Grassmann natural bundle G_k of k-planes tangent to n-manifolds or the natural bundle C^o_r of torsion free connections of order r on n-manifolds), then we can reformulate Corollary 6.1 as follows.

Corollary 6.1$^{(*)}$. *Let $F\colon Mf_n \to FMf_n$ be a transitive natural bundle of type* (I) *and let $G\colon Mf \to FMf$ be a bundle functor. Let $e \in \Gamma F\mathbf{R}^n$ be a fixed section with* $\mathrm{Dom}(e) = \mathbf{R}^n$. *Then there exists a bijection J between the set of all natural differential operators of F into $G|Mf_n$ and GP (P is as in Corollary 6.1) given by $J(D) = G\tilde{P}(D(e)(0))$, where $\tilde{P}\colon \mathbf{R}^n \to P$. The inverse bijection K is given by $K(p) = \{D^p_M\}$, where $D^p_M(\sigma)(x) = i^G_M(x,p)$ for every $\sigma \in \Gamma FM$, $x \in M$, $M \in Mf_n$.*

Proof. By Corollary 6.1 we can assume that $G = G^P$. Then $(0, J(D)) = D_{\mathbf{R}^n}(e)(0)$ and $D^p_M(\sigma)(x) = (x,p)$. We see that $(0, J \circ K(p)) = (0, J(\{D^p_M\})) = D^p_{\mathbf{R}^n}(e)(0) = (0,p)$ for every $p \in GP$. Therefore $J \circ K = \mathrm{id}$. It remains to show that $K \circ J = \mathrm{id}$. Let D be a natural differential operator of F into $G^P|Mf_n$. Let $q = J(D) \in GP$. We have to show that $D^q_M = D_M$ for every $M \in Mf_n$. Of course, $D^q_{\mathbf{R}^n}(e)(0) = (0,q) = D_{\mathbf{R}^n}(e)(0)$. Consider $\sigma \in \Gamma FM$ and $x \in \mathrm{Dom}(\sigma)$. Since F is transitive there exists an embedding $\varphi\colon \mathbf{R}^n \to M$ such that $F\varphi(e(0)) = \sigma(x)$. By Theorem 5.1, D and D^q are of order 0. Using the naturality condition we decude that $D^q_M(\sigma)(x) = D^q_M(\varphi_*(e))(x) = G^P\varphi(D^q_{\mathbf{R}^n}(e)(0)) = G^P\varphi(D_{\mathbf{R}^n}(e)(0)) = D_M(\sigma)(x)$ as well. □

(II) Using Theorem 5.1 we find all natural differential operators between Grassmann natural bundles.

Corollary 6.2. *Let $k, l \geqslant 1$ be two integers such that $k, l < n$. Let D be a natural differential operator of the Grassmann natural bundle G_k into the Grassmann natural bundle G_l over n-manifolds. Then $k = l$ and $D_M(\sigma) = \sigma$ for every $\sigma \in \Gamma G_k M$.*

Proof. We know that G_k is of type (I) and G_l is of type (II). Let $\sigma \in \Gamma G_k\mathbf{R}^n$ be a section defined near 0. Suppose that $D(\sigma)(0) \neq \sigma(0)$. Then there exists a linear isomorphism $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $d\varphi(\sigma(0)) = \sigma(0)$ and $d\varphi(D(\sigma)(0)) \neq D(\sigma)(0)$. On the other hand, it follows from Theorem 5.1 that D is of order 0, and then $D(\sigma)(0) = D(\varphi_*\sigma)(0) = (\varphi_* D(\sigma))(0)$. This is a contradiction. Hence $D(\sigma)(0) = \sigma(0)$. Using the naturality condition we deduce that $D_M(\sigma) = \sigma$ for every $\sigma \in \Gamma G_k M$. □

Remark. Of course, this list of applications of Theorem 5.1 is not complete. For example, we can use Theorem 5.1 to find all natural differential operators of G_k into $T^{(p,p)}$, of $T^{(p,p)}$ into $T^{(q,q)}$, of $T^{(p,p)}$ into G_k, of C_r into G_k e.t.c.

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Author's address: Institute of Mathematics, Jagiellonian University, Reymonta 4, Kraków, Poland.