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A MODIFICATION OF THE MEDIAN OF A TREE

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Summary. The concept of median of a tree is modified, considering only distances from the terminal vertices instead of distances from all vertices.

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Let T be a finite tree with n vertices, let $V(T)$ be its vertex set. Let $d(x,y)$ denote the distance between vertices x, y of T, i.e. the length of the (unique) path in T connecting x and y.

The median of T was defined in [1]. It is a vertex x at which the functional

$$a(x) = \frac{1}{n} \sum_{y \in V(T)} d(x,y)$$

attains its minimum. In [2] it was proved that in every finite tree there is either exactly one median, or exactly two medians joint by an edge. In this paper we will modify the concept of median, considering only distances from the terminal vertices instead of distances form all vertices.

Let $K(T)$ be the set of all terminal vertices of T, i.e. the vertices of degree 1. We may consider the functional

$$b(x) = \frac{1}{|K(T)|} \sum_{y \in K(T)} d(x,y)$$

and look for its minimum. A vertex at which this minimum is attained will be called the K-median of T.

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Obviously $|K(T)|$ is a constant and therefore the minimum of $b(x)$ is attained at the same vertices as the minimum of

$$b_0(x) = |K(T)|\, b(x) = \sum_{y \in K(T)} d(x, y);$$

we shall use this simpler functional.

Lemma 1. *Let v_0 be a vertex of a finite tree T, let v_1, v_2 be two distinct vertices adjacent to v_0 in T. Then*

$$b_0(v_0) \leqslant \max\big(b_0(v_1),\ b_0(v_2)\big),$$

where the equality may occur only if the degree of v_0 in T is 2 and $b_0(v_0) = b_0(v_1) = b_0(v_2)$.

Proof. Let B_1 (or B_2) be the branch of T at v_0 which contains v_1 (or v_2, respectively). Let B_3 be the union of all branches of T at v_0 different from B_1 and B_2 (it may be empty). For $i \in \{1, 2, 3\}$ let t_i be the number of terminal vertices of T belonging to B_i. Let $x \in K(T)$. If x belongs to B_1, then

$$d(v_1, x) = d(v_0, x) - 1;$$

in the opposite case

$$d(v_1, x) = d(v_0, x) + 1.$$

Therefore

$$\begin{aligned} b_0(v_1) &= \sum_{x \in K(T)} d(v_1, x) \\ &= \sum_{x \in K(T) \cap V(B_1)} \big(d(v_0, x) - 1\big) \\ &\quad + \sum_{x \in K(T) \cap (V(B_2) \cup V(B_3))} \big(d(v_0, x) + 1\big) \\ &= b_0(v_0) - t_1 + t_2 + t_3. \end{aligned}$$

Analogously

$$b_0(v_2) = b_0(v_0) + t_1 - t_2 + t_3.$$

It is not possible that both the numbers $-t_1 + t_2 + t_3$, $t_1 - t_2 + t_3$ were negative, because then also their sum $2t_3$ would be negative, which is impossible. Therefore either $b_0(v_1) \geqslant b_0(v_0)$, to $b_0(v_2) \geqslant b_0(v_0)$ and thus

$$b_0(v_0) \leqslant \max\big(b_0(v_1), b_0(v_2)\big).$$

Suppose that

$$b_0(v_0) = \max\left(b_0(v_1), b_0(v_2)\right).$$

Without loss of generality we may suppose $b_0(v_1) \geqslant b_0(v_2)$ and therefore

$$b_0(v_1) = \max\left(b_0(v_1), b_0(v_2)\right).$$

We have $b_0(v_0) = b_0(v_1) = b_0(v_0) - t_1 + t_2 + t_3$, which implies $t_1 = t_2 + t_3$. On the other hand, we have

$$b_0(v_1) = b_0(v_0) - t_1 + t_2 + t_3 \geqslant b_0(v_0) + t_1 - t_2 + t_3 = b_0(v_2),$$

which implies $t_2 \geqslant t_1$ and, together with the preceding, $t_3 \leqslant 0$. As t_3 cannot be negative, we have $t_3 = 0$, therefore there are no branches of T at v_0 except B_1 and B_2, and the degree of v_0 in T is 2. Further $t_1 = t_2$ and thus

$$b_0(v_0) = b_0(v_1) = b_0(v_1) = b_0(v_2).$$

□

Theorem 1. *Let T be a finite tree. Then T has either exactly one K-median, or all K-medians of T form a path in T whose inner vertices (if any) have degree 2 in T.*

P r o o f. As T is finite, the minimum of $b_0(T)$ must be attained at least at one vertex. Now let v_1, v_2 be two non-adjacent K-medians of T. Let P be the path in T connecting v_1 and v_2. Let v_3 be the inner vertex of P such that $b_0(v_3)$ is the maximum value of $b_0(x)$ among all vertices of P. Obviously $b_0(v_3) \geqslant b_0(v_1) = b_0(v_2)$. Let v_4, v_5 be the vertices of P adjacent to v_3. According to Lemma 1 we have

$$b_0(v_3) \leqslant \max\left(b_0(v_4), b_0(v_5)\right).$$

However, the maximality of $b_0(v_3)$ implies that the degree of v_3 in T is 2 and $b_0(v_3) = b_0(v_4) = b_0(v_5)$, again according to Lemma 1. If $v_4 = v_1$ or $v_4 = v_2$, then $b_0(v_3) = b_0(v_1) = b_0(v_2)$. If not, we proceed in a similar way with v_4 instead of v_3; after a finite number of steps we obtain the above equality. As $b_0(v_1)$ is minimum and $b_0(v_3)$ is maximum, all vertices x of P have the same value of $b_0(x)$; all of them are K-medians of T and all inner vertices of P have the degree 2 in T. This implies the assertion. □

The following propositions are easy to prove.

Proposition 1. *All vertices of a finite tree T are its K-medians if and only if T is a path.*

Proposition 2. *For each vertex x of a tree T with n vertices the inequality $b_0(x) \geqslant n-1$ holds.*

Proposition 3. *If a tree T with n vertices contains exactly one vertex x of degree greater that 2, then this vertex x is its unique K-median and $b_0(x) = n-1$.*

Proposition 4. *Let a finite tree T have more than one K-median and be different from a path, let P be the path in T induced by all K-medians of T. By deleting all edges and inner vertices of P two trees with the same number of terminal vertices are obtained.*

The proof of this proposition follows from the proof of Lemma 1, where it was shown that $t_1 = t_2$.

Now we will prove a theorem concerning K-medians and medians.

Theorem 2. *For every positive integer h there exists a tree T which has exactly one K-median and exactly one median, the distance between these vertices in T being h.*

Proof. Let the vertex set of T be $V(T) = \{v_0, v_1, \ldots, v_{2h+2}, x, y\}$ and let its edges be v_0x, v_0y and v_iv_{i+1} for $i = 0, \ldots, 2h+1$. Then T has the unique K-median v_0 by Proposition 3, the unique median of T is v_h; the reader may verify it himself. Their distance is $d(v_0, v_h) = h$.

Analogously to the K-median, also the K-gravity center may be introduced. The gravity center [1] of a tree T with n vertices is the vertex of T at which the functional

$$f(x) = \frac{1}{n}\sqrt{\sum_{y \in V(t)} \left(d(x,y)\right)^2}$$

attains its minimum. Now the K-gravity center of T is a vertex at which the functional

$$g(x) = \frac{1}{|K(T)|}\sqrt{\sum_{y \in K(T)} \left(d(x,y)\right)^2}$$

attains its minimum. For the sake of simplicity instead of $g(x)$ we shall use

$$g_0(x) = \sum_{y \in K(T)} \left(d(x,y)\right)^2$$

which attains its minimum at the same vertices as $g(x)$ does. □

Lemma 2. *Let v_0 be a vertex of a finite tree T, let v_1, v_2 be two distinct vertices adjacent to v_0 in T. Then*

$$g_0(v_0) < \max\big(g_0(v_1), g_0(v_2)\big).$$

Proof. Let the notation be the same as in the proof of Lemma 1. Then

$$\begin{aligned} g_0(v_1) = & \sum_{x\in K(T)\cap V(B_1)} (d(v_0,x)-1)^2 + \sum_{x\in K(T)\cap(V(B_2)\cup V(B_3))} (d(v_0,x)+1)^2 \\ = & \sum_{x\in K(T)} (d(v_0,x))^2 - 2\sum_{x\in K(T)\cap V(B_1)} d(v_0,x) \\ & + 2 \sum_{x\in K(T)\cap(V(B_2)\cup V(B_3))} d(v_0,x) + t_1 + t_2 + t_3 \end{aligned}$$

If $g_0(v_0) < g_0(v_1)$, then the assertion holds. Suppose that $g_0(v_0) \geqslant g_0(v_1)$. Then

$$-2\sum_{x\in K(T)\cap V(B_1)} d(v_0,x) + 2\sum_{x\in K(T)\cap(V(B_1)\cup V(B_2))} d(v_0,x) + t_1 + t_2 + t_3 \leqslant 0,$$

which implies

$$2\sum_{x\in K(T)\cap(V(B_1)\cup V(B_2))} d(v_0,x) + t_1 + t_2 + t_3 \leqslant 2\sum_{x\in K(T)\cap V(B_1)} d(v_0,x)$$

and then $g_0(v_0) < g_0(v_2)$, because $t_1 + t_2 + t_3 \geqslant 2 > 0$. □

Theorem 3. *Let T be a finite tree. Then T has either exactly one K-gravity center, or exactly two K-gravity centers which are adjacent.*

Proof. Suppose that T has two non-adjacent K-gravity centers v_1, v_2. Let P be the path in T connecting v_1 and v_2. Let v_3 be the inner vertex of P such that $g_0(v_3)$ is the maximum value of $g_0(x)$ among all vertices of P. Let v_4, v_5 be the vertices of P adjacent to v_3. Then

$$g_0(v_3) < \max\big(g_0(v_4), g_0(v_5)\big)$$

according to Lemma 2. Hence either $g_0(v_3) < g_0(v_4)$, or $g_0(v_3) < g_0(v_5)$. Without loss of generality let $g_0(v_3) < g_0(v_4)$. The equalities $v_4 = v_1$ or $v_4 = v_2$ contradict the minimality of $g_0(v_1) = g_0(v_2)$. The hypothesis that v_4 is an inner vertex of P contradicts the maximality of $g_0(v_3)$ among the inner vertices of P. Hence two non-adjacent K-gravity centers cannot exist. As a tree does not contain triangles, this implies the assertion. □

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