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ON RESULTS OF JAN MAŘÍK IN THE THEORY OF DERIVATIVES

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Summary. Results of Jan Mařík on the theory of derivatives of real functions are described.

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1. Introduction

We will describe in this note some interesting and simply formulated Mařík's results in the theory of derivatives. They concern mainly the theory of (ordinary) derivatives (of the first order) of real functions of one real variable.

Mařík started his research in this field around 1980 and from that time he worked mainly in the theory of derivatives, where he obtained many interesting and deep results.

He published 15 papers ([M3]–[M17]) and 6 preliminary communications on this subject. His research in the theory of derivatives concentrated mainly on two topics:

a) Representation of functions by derivatives.

b) Stability of the system of derivatives.

We will desribe some Mařík's contributions in these topics in parts 3 and 4 of the present paper. Here let us only note that the following questions are typical of these topics:

a) Which functions can be represented as a product of two (or of a finite number of) derivatives?

A talk with the same title was delivered on November 13, 1995 at a meeting commemorating the 75th birthday of the late Prof. Jan Mařík.

b) Which functions g have the property that fg (or $f \circ g$) is a derivative wheneve f is a derivative?

It is necessary to mention that Mařík worked, at least since the sixties, also i the theory of generalized derivatives of higher orders and the corresponding integral which are important in the theory of trigonometric series. Mařík did not publisl his research in this field, however he wrote several texts on this subject which wer known to many people and had significant influence on their research. For example, i the long article [CT] by G. E. Cross and B. S. Thomson the authors reproduce som Mařík's unpublished results from [M1]. Concerning the treatment of the Jame integral the authors write: "Some difficulties that arise in the James integral ar simplified by using a device due to Mařík". Mařík's ideas from [M1] were used als by B. S. Thomson in [T], cf. the note on p. 330.

Properties of generalized derivatives of higher orders are considered also in shor notes [M2], [M19] and in Mařík's joint work with H. Fejzic and C. E. Weil [M18].

In the last mentioned paper the authors considered the following problem:

(P) Let $H \subset \mathbb{R}$ be a closed set and let $f \colon H \to \mathbb{R}$ be n-times Peano differentiabl relative to H at each point of H. Is it possible to extend f to an n-times Peanc differentiable function defined on $\mathbb{R}$ whose Peano derivatives agree with those of f on H? (At isolated points of H the relative Peano derivatives can be prescribed in an arbitrary way.)

If $n = 1$, then the n-th Peano derivative is the ordinary derivative and it was shown by V. Jarník in [J] (for perfect sets H) and by J. Mařík [M9] (for arbitrary closed H) that in this case the answer is affirmative.

Z. Buczolich [Bu] proved that the answer to problem (P) for $n = 2$ is negative for some H and f.

The authors in [M18] proved that the answer is affirmative, whenever H satisfies a (rather complicated) condition which says, roughly speaking, that H is not too "small" at each of its point. For example, the desirable extension exists if H is the Cantor ternary set.

Note that this condition was relaxed in the recent work [BW].

Before we describe some Mařík's contributions in the theory of derivatives, we will shortly discuss some basic facts in this field.

2. Some basic facts in the theory of derivatives

In the following by derivative we always mean the (ordinary) first derivative of a real function defined on the real line.

Because of the fundamental importance of the notion of derivative, there arise two natural and very old questions:

(i) How "badly behaved" a derivative can be?

(ii) Is it possible to give a complete characterization of derivatives?

Concerning (i), there is a well-known classical observation that a derivative can be discontinuous at some points. For example, it is not difficult to show that the function $f(x) = \cos(1/x), f(0) = 0$ is a derivative. However, G. Darboux proved that each derivative has a continuity-like property, it has the Darboux (intermediate-value) property. Derivatives are close to continuous functions also in another sense—they belong to the first class of Baire, i.e. each derivative is a pointwise limit of a sequence of continuous functions. Consequently, each derivative is continuous at all points except a first category set, in particular at each point of a dense set.

Over one hundred years ago a natural question, whether there exists a differentiable function which is monotone on no interval (or equivalently, whether there exists a derivative which changes its sign on each interval), was discussed. U. Dini believed that such pathological functions exist; P. Du Bois-Reymond held the opposite view. Now it is well-known that such functions exist, but a construction of such a function using no theory or "trick" is very difficult; the first (very long) construction was published by A. Koepcke in 1887, but it was not quite correct. Some obviously correct constructions were published in 1915 by A. Denjoy; they were based on the notion of an approximately continuous function, which provides a very important tool in the theory of derivatives.

A function f is said to be *approximately continuous* if, for each $x \in \mathbb{R}$, there is a Lebesgue measurable set E such that x is a density point for E, i.e.

$$\lim_{h\to 0+} \frac{1}{2h}\lambda(E \cap (x-h, x+h)) = 1,$$

where λ is the Lebesgue measure, and $\lim\limits_{y\to x, y\in E} f(y) = f(x)$.

It is important that each bounded approximately continuous function is a derivative and approximately continuous functions can be constructed using topological methods.

These and other examples of "Koepcke derivatives" show that a derivative can be discontinuous at "many points", e.g. at all points of a set which is of positive Lebesgue measure in each interval. On the other hand, we have seen that the set of discontinuity points of a derivative must be small in the "topological sense"—it is always of the first category.

If we use the usual notation DB_1 for the system of all Darboux functions of the first class of Baire and $\triangle'$ for the system of all derivatives, we see that $\triangle' \subset DB_1$; but simple examples show that $\triangle' \neq DB_1$ (we can consider e.g. $g(x) = \cos(1/x)$, $g(0) = 1$).

Thus we have some answers to question (i). Concerning question (ii), it is now a generally accepted opinion that there exists no genuine characterization of $\triangle'$ (i.e. which is not a tautology).

However, Z. Zahorski [Z] in 1950 was able to characterize the level sets of bounded derivatives (i.e. the sets of the form $\{x\colon f'(x) > \alpha\}$, where f' is a bounded derivative). This research was completed by D. Preiss [P1] in 1982 who gave a complete characterization of level sets of arbitrary (finite) derivatives.

Among the most interesting results in the theory of derivatives we include also the Maximoff theorem which asserts that the classes $\triangle'$ and DB_1 are very close; namely for each $g \in DB_1$ there exists a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ such that $g \circ h \in \triangle'$. Thus we are able to give a simple characterization of functions of the form $f' \circ h$, where $f' \in \triangle'$ and h is a homeomorphism as above—such functions form exactly the class DB_1. The Maximoff theorem was published by I. Maximoff [Max] in 1940 (with a proof which is now generally considered incorrect) and proved by D. Preiss [P3] in 1979. It should be mentioned that the case when g is a semicontinuous Darboux function was proved by G. Choquet [Ch].

More information concerning the theory of derivatives can be found in A. Bruckner's monograph [Br2] and in survey articles by R. J. Fleissner [F1] and A. M. Bruckner, J. Mařík and C. E. Weil [M15].

3. Mařík's results on representation of functions by derivatives

W. Wilcosz in 1921 observed that the function $f(x) = \cos(1/x), f(0) = 0$ is a derivative, but the function f^2 is not. Consequently the system $\triangle'$ of all derivatives (which is obviously a linear space) is not closed under multiplication. Thus we can ask: Which functions can be represented as a product of two (or of a finite number of) derivatives? What is the algebra $\mathrm{Alg}(\triangle')$ generated by derivatives?

Some results in this direction were contained in the 1978 manuscript by S. J. Agronsky, R. Biskner and A. M. Bruckner, where the authors have proved that all elements of three important classes of functions (namely approximate derivatives, approximately continuous functions and B_1^* functions) belong to $\mathrm{Alg}(\triangle')$; moreover they can be written in the form

$$f = f_1' + f_2' f_3' f_4' + f_5' f_6' f_7' \quad (f_i' \in \Delta'). \tag{1}$$

J. Mařík, reading the preprint, brought new ideas to this research. As a result an essentially improved paper [M3] of four authors was published, which was the first Mařík's article in the theory of derivatives of the first order. In this article it is e.g. proved that the representation (1) can be improved to

$$f = g' + hk' \quad (\text{where } g', k' \in \Delta' \text{ and } h \text{ is a differentiable function}). \tag{2}$$

Moreover, f is of the form (2) if and only if there exists an open dense set $G \subset \mathbb{R}$ such that

(i) $f|G$ is a derivative (of a function defined on G) and

(ii) $f|(\mathbb{R} \setminus G)$ can be extended to a derivative on $\mathbb{R}$.

The article ends with a note that it can be true that the algebra $\mathrm{Alg}(\Delta')$ coincides with the whole Baire class one B_1 and perhaps each $f \in B_1$ is of the form

$$f = g' + h'k' \quad (g', h', k' \in \Delta').$$

This conjecture was proved by D. Preiss in [P2].

From the subsequent Mařík's papers [M4], [M8], [M11], [M12], [M13], [M17] which concern representation of functions by derivatives, we will mention several interesting results which can be easily formulated.

In [M11] A. M. Bruckner, J. Mařík and C. E. Weil proved that if $f \in B_1$ and $f(x) = 0$ for almost all x, then f can be represented in the form $f = g'h'$ (where $g', h' \in \Delta'$). Moreover, if the function f is non-negative, then the factors can be selected to be non-negative.

In the same paper the authors also solved the following problem:

Which characteristic functions C_A can be represented as a finite product of non-negative derivatives:

$$C_A = f_1' f_2' \cdots f_n', \quad (f_i' \geqslant 0)?$$

The necessary and sufficient condition they found for such a representation is that

(i) A is both an F_σ and a G_δ set (which is equivalent to the assumption that $C_A \in B_1$), and

(ii) each point x of the complement $B = \mathbb{R} \setminus A$ is a density point for B, i.e. $\lim_{h \to 0+} \frac{1}{2h} \lambda(B \cap (x - h, x + h)) = 1$, where λ is the Lebesgue measure on $\mathbb{R}$.

Moreover, if such a representation exists, then it exists also with $n = 2$ and with $f_1' = f_2' = 1$ on A.

If we do not demand that the derivatives f_i' be non-negative, we obtain a similar problem, which was solved by Mařík in [M12]. The answer is almost the same as the one above; the only difference is that the condition (ii) must be replaced by the (clearly weaker) assumption

(ii)* B is porous at none of its points.

Note that Mařík's results from [M12] were developed in a recent paper [Mal].

The following theorem is an interesting special case of results of J. Mařík and C. E. Weil [M8]:

Let u be a continuous and positive function on $(0, \infty)$. There exist non-negative numbers $q_2 \geqslant q_3 \geqslant q_4 \geqslant \ldots$ such that if u is extended to be constant on $(-\infty, 0]$ then this extended function can be expressed as the product of k derivatives if and only if $u(0) \geqslant q_k$.

Explicit values of the numbers q_k are given in [M8].

In the papers [M8], [M13] and [M17] also several theorems are proved, which illustrate the following rough idea:

If a well-behaved function is expressed algebraically in terms of several derivatives, then in some cases these derivatives are themselves also well-behaved.

For example, the following result is a special case of Theorem 7.8 of [M8]:

If the product of powers of several positive derivatives is approximately continuous, and if the corresponding exponents are positive, then all of the factors must be approximately continuous.

Note that within the class $b\triangle'$ of all bounded derivatives the class bC_{ap} of all bounded approximately continuous functions forms a subspace of "well-behaved" functions which is "small" in $b\triangle'$; it is nowhere dense when $b\triangle'$ is equipped with the supremum norm.

Some analogous results for sums of powers of derivatives is proved in [M13]. For example, the following result is a special case of Theorem 5.6:

If the sum of squares of several derivatives is bounded and approximately continuous, then all of these derivatives are approximately continuous.

4. Mařík's results on stability of the system of derivatives

Most of Mařík's results of this type concern multiplication and transformation (via an inner homeomorphism) of derivatives; they will be discussed below.

Note here that in [M17] also another problem concerning stability of the system of derivatives is considered. Namely the conditions under which the Euclidean norm

$\sqrt{f^2+g^2}$ (or a more general norm) of the pair (f,g) is derivative, if both f and g are derivatives, are investigated in that paper.

4.a Theory of multipliers.

Now consider functions on $[0,1]$ and use the following notation:

$\triangle'$ ... the system of all derivatives (on $[0,1]$),

$S\triangle'$ ... the system of all summable (integrable) derivatives,

$(\triangle')^+$ ... the system of all non-negative derivatives,

$b\triangle'$ ... the system of all bounded derivatives,

C_1 ... the system of all functions with a continuous derivative,

L ... the system of all Lebesgue functions, i.e. of all functions f such that each point $x \in [0,1]$ is a Lebesgue point of f,

C_{ap} ... the system of all approximately continuous functions,

bC_{ap} ... the system of all bounded approximately continuous functions.

Remember that x is a Lebesgue point of f if $\lim\limits_{h\to 0} \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,\mathrm{d}t = 0$. Further note that the class L forms an important subclass of derivatives ($L \subset \triangle'$) and that a bounded function belongs to L if and only if it is approximately continuous ($bL = bC_{ap}$).

It was mentioned in the preceding section that, if $g \in \triangle'$ is given and we multiply it by a function $f \in \triangle'$, then it is possible that $fg \notin \triangle'$. On the other hand, if $f \in C_1$, it is very easy to prove that then $fg \in \triangle'$ for each $g \in \triangle'$. It is natural to consider the problem whether only C_1 functions have this property or whether there exist some other functions with the same property (it can be e.g. shown that the assumption that f' exists is not sufficient). Thus we are led to the following definition:

Let $\Phi \subset \triangle'$ be a system of functions on $[0,1]$ and let f be a function on $[0,1]$. We say that f is a multiplier for Φ if $fg \in \triangle'$ for each $g \in \Phi$. The system of all multipliers for Φ is denoted by $M(\Phi)$. Thus we know that

$$\triangle' \not\subset M(\triangle'),\ C_1 \subset M(\triangle'),\ M(\triangle') \subset \triangle';$$

the last inclusion is obvious since $1 \in \triangle'$.

It is a rather confusing fact that a different notion of multipliers for Φ is also frequently used. By this different definition we say that f is a multiplier for Φ (and we will write $f \in M^*(\Phi)$) if $fg \in \Phi$ for each $g \in \Phi$.

The system $M(\triangle') = M^*(\triangle')$ was characterized by R. J. Fleissner [F2] in 1977. He described members of $M(\triangle')$ as those $f \in \triangle'$ which are of "distant bounded variation"; this notion he defined using the improper Stieltjes integral. In [F1] Fleissner posed the problem of characterization of $M^*(S\triangle')$.

Mařík [M5] reformulated Fleissner's result using the notion of variation instead of Stieltjes integral and, slightly changing the notion of variation, he obtained an analogous characterization of the system $M(S\Delta')$. Moreover, he proved that $M(S\Delta') = M^*(S\Delta')$ and thus solved Fleissner's problem.

Mařík's reformulation of Fleissner's result reads as follows:

$$\begin{aligned}
&f \in M(\Delta') \text{ if and only if } f \in \Delta',\\
&\limsup_{n\to\infty} \operatorname{Var}(f, [x + \frac{1}{n}, x + \frac{2}{n}]) < \infty \text{ for each } x \in [0,1) \text{ and}\\
&\limsup_{n\to\infty} \operatorname{Var}(f, [x - \frac{2}{n}, x - \frac{1}{n}]) < \infty \text{ for each } x \in (0,1],
\end{aligned}$$

where $\operatorname{Var}(f, I)$ is the total variation of f on the interval I.

(Note that we use here the reformulation from [M15]; in [M5] Mařík used intervals $[x + 2^{-n}, x + 2^{-n+1}]$ instead of $[x + \frac{1}{n}, x + \frac{2}{n}]$.)

In [M5] Mařík defined an auxiliary notion of a "variation-like" quantity

$$v(n, f, [a,b]) = \sup\left\{ \sum_{k=1}^{n} |f(y_k) - f(x_k)|,\ a \leqslant x_1 < y_1 \leqslant \ldots \leqslant x_n < y_n \leqslant b \right\}$$

and was able to prove the following theorem which is suprisingly similar to a reformulated Fleissner's characterization of $M(\Delta')$:

$$\begin{aligned}
&f \in M(S\Delta') = M^*(S\Delta') \text{ if and only if } f \in \Delta',\\
&\limsup_{n\to\infty} v(n, f, [x + \frac{1}{n}, x + \frac{2}{n}]) < \infty \text{ for each } x \in [0,1) \text{ and}\\
&\limsup_{n\to\infty} v(n, f, [x - \frac{2}{n}, x - \frac{1}{n}]) < \infty \text{ for each } x \in (0,1].
\end{aligned}$$

(Also here we present a reformulation (now from [M6]) of the original version.)

Using this characterization, Mařík was able in [M6] to prove the following remarkable results:

(I) If $f \in M(S\Delta')$, then the set D_f of points of discontinuity of f is a scattered set; i.e. each of its nonempty subsets has an isolated point (or equivalently: D_f is a countable G_δ set).

(II) There exists a continuous $f \in M(S\Delta')$ which is differentiable at no point of $[0,1]$.

Note also that in [M7] a (rather complicated) characterization of $M((\Delta')^+)$ is given. Thus also a characterization of $M^*((\Delta')^+)$ is obtained, since it clearly consists precisely from the nonnegative members of $M((\Delta')^+)$.

Further Mařík's results on multipliers are contained in [M10] and [M16].

In [M10] he proved that $M(L) = b\triangle'$.

In [M16] Mařík proved that $f \in M(C)$, where C is the set of all continuous functions on $[0,1]$, if and only if $f \in \triangle'$ and

$$\limsup_{y\to x, y\in[0,1]} \frac{1}{y-x}\int_x^y |f| < \infty \text{ for each } x \in [0,1].$$

Finally, note that in [M16] also a nice characterization of multipliers for the class $bC_{ap} = bL$ of all bounded approximately continuous functions on $[0,1]$ is given.

4.b Theory of transformers.

Denote by H the system of all increasing homeomorphisms $h\colon [0,1] \to [0,1]$. Let Φ be a system of functions on $[0,1]$ (we are interesting mainly in the case when Φ is a subclass of $\triangle'$) and let $h \in H$. We say that h is a transformer for Φ if $g \circ h \in \triangle'$ for each $g \in \Phi$. The system of all transformers for Φ is denoted by $T(\Phi)$.

We shall also define here the class $T^*(\Phi)$ as the set of all $h \in H$ such that $g \circ h \in \Phi$ for each $g \in \Phi$.

The problem of transformers was considered by G. Choquet in [Ch], where he stated (without a proof) a sufficient condition for $h \in H$ to be in $T(\triangle')$ and a characterization of $T(b\triangle')$.

A. M. Bruckner [Br1] in 1970 characterized the class $T^*(C_{ap})$. Note that obviously $T^*(C_{ap}) = T^*(bC_{ap})$ and it is easy to prove (cf. [M10]) that $T(bC_{ap}) = T^*(C_{ap})$.

M. Laczkovich and G. Petruska [LP] in 1978 proved a characterization of $T(\triangle')$.

Mařík [M10] proved a different characterization of $T(\triangle')$ and moreover, he found a very similar characterization of $T(L)$ and observed that $T(L) = T^*(L)$. The description of these nice and deep results follows.

Let $h \in H$ and let γ be an arbitrary function which is inbetween the lower and upper Dini derivatives of the inverse h^{-1}:

$$\underline{D}h^{-1}(x) \leqslant \gamma(x) \leqslant \overline{D}h^{-1}(x),\ \ x \in [0,1].$$

Then, if we put $[t,x] = [x,t]$ for $x < t$ and $\operatorname{Var}(\gamma, I) = \infty$ if $\gamma(x) = \infty$ for some $x \in I$, the following statements hold:

(A) $h \in T(\triangle')$ if and only if, for each $a \in [0,1]$,

$$\limsup_{x\to a, x\in[0,1]} \frac{1}{h^{-1}(x) - h^{-1}(a)}\int_a^x \operatorname{Var}(\gamma,[t,x])\,\mathrm{d}t < \infty$$

and

(B) $h \in T(L)$ if and only if, for each $a \in [0,1]$,

$$\limsup_{x \to a, x \in [0,1]} \frac{1}{h^{-1}(x) - h^{-1}(a)} \int_a^x \sup_{z \in [t,x]} \gamma(z)\,\mathrm{d}t < \infty.$$

In the same paper, using the above characterizations, Mařík proved that the obvious inclusions

$$T(\triangle') \subset T(L) \subset T(bC_{ap})$$

are proper.

He also constructed $h \in T(\triangle')$ such that $h' = \infty$ on a perfect subset of $[0,1]$.

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