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MEASURE AND TOPOLOGY: MAŘÍK SPACES

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Summary. This is an expository paper on Jan Mařík's result concerning an extension of a Baire measure to a Borel measure.

Keywords: Baire measure, Borel measure, topological space, normal space, countably paracompact space

AMS classification: 28C15

1. INTRODUCTION

A talk with the same title was delivered by the author at a meeting commemorating the 75th birthday of the late Professor Jan Mařík. Its main objective was to explain, in a way accessible to non-specialists in the field, the principal result of Mařík's paper [8] on the interplay of measure and topology. The result reads as follows:

Theorem. *Let X be a normal countably paracompact topological space. Then every (finite) Baire measure on X admits a unique extension to a regular Borel measure.*

Following [16], a completely regular Hausdorff topological space is called a *Mařík space*, if every finite Baire measure has an extension to a regular Borel measure. The theorem, using this terminology, says that the normal countably paracompact spaces are Mařík spaces.

This text is an extended version of the talk which was presented at the Faculty of Mathematics and Physics, Charles University, Prague, on November 13, 1995.

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We are going to summarize the relevant definitions and to give a reasonably self-contained proof of the above mentioned result. References to papers dealing with measure extension problems of this type are provided. An open problem is also presented in order to illustrate that the subject is still, after forty years, alive.

Measure and integral play a prominent rôle in Jan Mařík's scientific work since early years of his mathematical activity, as is clearly seen from the list of his publications. Papers dealing with the *surface integral* and *non-absolutely convergent integrals* are covered by the contributions by Josef Král and Štefan Schwabik published in the present volume. Of course, integrals appear also in Mařík's articles from *real analysis* described in Luděk Zajíček's contribution, also contained in this volume. There still remain at least eleven papers devoted to measure and/or integration: [4]–[14].

We would like to mention that papers [4], [5] and [11] show Jan Mařík's interest in integral representation theorems for positive functionals (see also [8], p. 445). Informally speaking, *Baire measures* usually arise in integral representations of positive linear functionals on the space of bounded continuous functions on a topological space. On the other hand, *Borel measures* appear in the Riesz representation theorem for compact (or locally compact) topological spaces; see e.g. [1]. Thus the question of extensions of a Baire measure to a reasonable Borel measure is quite natural in this context.

Given a Baire measure on a topological space (for definitions, see Sec. 4), one might try to attack the extension problem by constructing the corresponding outer measure and then to see whether the resulting σ-algebra of measurable sets contains all Borel sets. Another idea could be to move to the Čech-Stone compactification and to view the linear functional on bounded continuous functions (defined by integration with respect to the Baire measure in question) as a positive linear functional on the space of *extended* continuous functions on the compactification. Then the Riesz representation theorem provides a Borel measure on the compactification and the problem is whether its restriction to the original space coincides with the Baire measure. However, both ideas fail, see [16], p. 132. The former because the outer measure need not be, in general, even additive on Borel sets. The latter idea does not lead, in general, to an extension of the original measure. Mařík's idea which, as shown in Sec. 5, works on all normal countably paracompact spaces, is based on a *two-step regularization process* introduced in [8]. The main results of that paper appeared later in a shorter English version [10].

For this text, which also uses results from [2] and [3], we found convenient to consider *finite* measures only. Comments on a somewhat more general case investigated in [8] are included in Sec. 6.

2. Topology

In this section, we are going to summarize important definitions and results from [2] related to our exposition.

Let X be a topological space. A family $\{A_s; s \in S\}$ of subsets of X is said to be *locally finite*, if for every $x \in X$ there exists a neighborhood U of x such that $\{s \in S; U \cap A_s \neq \emptyset\}$ is finite.

A family $\{A_s; s \in S\}$ of subsets of X is called a *cover* of X, if $\bigcup\{A_s; s \in S\} = X$.

Let $\mathcal{A} = \{A_s; s \in S\}$ and $\mathcal{R} = \{R_t; t \in T\}$ be covers of X. We say that $\mathcal{R}$ is a *refinement* of $\mathcal{A}$, if for every $t \in T$ there exists $s \in S$ such that $R_t \subset A_s$.

The following definition goes back to C. H. Dowker and M. Katětov (1951):

A topological space is said to be *countably paracompact*, if it is Hausdorff and every countable open cover has a locally finite open refinement.

Let us mention that the class of countably paracompact spaces contains e.g. all compact spaces, Lindelöf spaces and metrizable spaces.

It turns out that *normal* countably paracompact spaces are of special interest for topologists as well as for analysts. They can be characterized e.g. by means of the following in-between type property: *For every couple of real functions f, g on X such that f is upper semicontinuous, g is lower semicontinuous and $f < g$ on X, there exists a continuous function h such that $f < h < g$* (C. H. Dowker, M. Katětov, 1951). Another characterization reads as follows: $X \times [0,1]$ is normal. Note that there are normal spaces which are not countably paracompact (Dowker spaces) and countably paracompact spaces which are not normal.

For our purpose, the following result is important; see [2], p. 317.

2.1. Theorem. *For every Hausdorff space X, the following conditions are equivalent:*

(i) *X is normal and countably paracompact;*

(ii) *For every countable open cover $\{U_n; n \in \mathbb{N}\}$ of the space X, there exists a closed cover $\{F_n; n \in \mathbb{N}\}$ such that $F_n \subset U_n$ whenever $n \in \mathbb{N}$.*

In Sec. 4, we will use the implication (i) $\Longrightarrow$ (ii) in an essential way, therefore we sketch here its p r o o f.

Let X be a normal countably paracompact space and $\{U_n; n \in \mathbb{N}\}$ a countable open cover of the space X. By definition, there exists a locally finite open refinement $\mathcal{V}$ of the cover $\{U_n; n \in \mathbb{N}\}$. Take now $V \in \mathcal{V}$ and choose $n(V) \in \mathbb{N}$ such that $V \subset U_{n(V)}$. Defining $V_n = \bigcup\{V \in \mathcal{V}; n(V) = n\}$, we arrive at a locally finite open cover $\{V_n; n \in \mathbb{N}\}$ with $V_n \subset U_n$ for every $n \in \mathbb{N}$. Now (ii) follows from a result on normal spaces (see [2], p. 44) which reads as follows:

Let $\{V_s; s \in S\}$ be an open cover of a normal space X such that, for every $x \in X$, the set $\{s \in S; x \in V_s\}$ is finite. Then there exists an open cover $\{W_s; s \in S\}$ of X such that $\overline{W}_s \subset U_s$ for every $s \in S$.

3. Special sets

Let X be a completely regular Hausdorff topological space. As usual, $C(X)$ stands for the space of continuous functions on $X, \mathcal{G}$ for the topology of X and $\mathcal{F}$ for the system of closed sets in X. Further we introduce *co-zero sets* and *zero sets* by

$$\mathcal{G}^* = \big\{\{f \neq 0\}; f \in C(X)\big\}, \quad \mathcal{F}^* = \big\{\{f = 0\}; f \in C(X)\big\}.$$

Clearly, $\mathcal{G}^* \subset \mathcal{G}$ and $\mathcal{F}^* \subset \mathcal{F}$. The following simple facts will be needed in what follows:

3.1. *If $Z, Y \in \mathcal{F}^*$, then $Z \cup Y \in \mathcal{F}^*$.*

Indeed, if $Z = \{f = 0\}$, $Y = \{g = 0\}$, then $Z \cup Y = \{f \cdot g = 0\}$.

3.2. *If $G_n \in \mathcal{G}^*$, $n \in \mathbb{N}$, then $\bigcup\limits_{n=1}^{\infty} G_n \in \mathcal{G}^*$.*

Indeed, suppose that $f_n \in C(X)$, $0 \leqslant f_n \leqslant 1$ and $G_n = \{f_n \neq 0\}$. Defining $f = \sum\limits_{n=1}^{\infty} 2^{-n} f_n$, we get $f \in C(X)$ such that $\bigcup\limits_{n=1}^{\infty} G_n = \{f \neq 0\}$.

3.3. *If $G \in \mathcal{G}^*$, then there exist $Z_n \in \mathcal{F}^*$ such that $Z_n \nearrow G$.*

Indeed, let $G = \{f \neq 0\}$ and $f_n = \min\big(|f| - (1/n), 0\big)$, $n \in \mathbb{N}$. Put $Z_n = \{f_n = 0\}$. Then $\{Z_n\}$ is an increasing sequence of sets from $\mathcal{F}^*$, $Z_n \subset G$ for every $n \in \mathbb{N}$. If $x \in G$, then $f_n(x) = 0$ for n large enough. We conclude that $Z_n \nearrow G$.

3.4. *Let X be a normal topological space, $F \in \mathcal{F}$, $G \in \mathcal{G}$ and $F \subset G$. Then there exists $Z \in \mathcal{F}^*$ with $F \subset Z \subset G$.*

Indeed, there exists $f \in C(X)$ such that $f_{|F} = 1$, $f_{|X \setminus G} = 0$ (Urysohn's lemma). Defining $g = \min(f - \frac{1}{2}, 0)$ and $Z = \{g = 0\}$, we arrive at $Z \in \mathcal{F}^*$ satisfying $F \subset Z \subset G$.

4. Measure

In Secs. 4 and 5, a *measure* (on a set X) always means a *finite* positive σ-additive set function defined on a σ-algebra of subsets of X.

Let us recall the following standard uniqueness extension **theorem**:

Let $\mathcal{E}$ be a family of subsets of X such that $E_1 \cap E_2 \in \mathcal{E}$ whenever E_1, $E_2 \in \mathcal{E}$ and $X \in \mathcal{E}$. Let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{E}$. If μ, ν are measures on $\mathcal{A}$ such that $\mu = \nu$ on $\mathcal{E}$, then $\mu = \nu$.

(For a more general version, see e.g. [1], p. 26.)

Assume now that X is a completely regular Hausdorff topological space. The smallest σ-algebra containing $\mathcal{G}$ will be denoted by $\mathcal{B}$ and the smallest σ-algebra containing $\mathcal{G}^*$ will be denoted by $\mathcal{B}^*$. Elements of $\mathcal{B}$ and $\mathcal{B}^*$ are called *Borel* and *Baire sets*, respectively. Of course, $\mathcal{B}^* \subset \mathcal{B}$ and $\mathcal{B}^* \neq \mathcal{B}$ in general; see [16], p. 108.

A measure defined on $\mathcal{B}$ will be called a *Borel measure* and a measure defined on $\mathcal{B}^*$ will be termed a *Baire measure.*

A Borel measure μ is said to be *regular*, if

$$\mu(A) = \sup\{\mu(F); F \subset A, F \in \mathcal{F}\}$$

whenever $A \in \mathcal{B}$. In view of our finiteness hypothesis, μ is obviously regular, if and only if

$$\mu(A) = \inf\{\mu(G); G \supset A, G \in \mathcal{G}\}$$

for every $A \in \mathcal{B}$ (one just passes to complements). Note that there are Borel measures which are not regular; see [1], p. 212. On the other hand, the Baire measures are always inner and outer regular in the following sense:

4.1. Proposition. *Let μ be a Baire measure on X. Then*

$$\mu(B) = \sup\{\mu(Z); Z \subset B, Z \in \mathcal{F}^*\} = \inf\{\mu(U); U \supset B, U \in \mathcal{G}^*\} \tag{4.1}$$

whenever $B \in \mathcal{B}^*$.

Proof. Let $B \in \mathcal{G}^*$. By 3.3, there exist $Z_n \in \mathcal{F}^*$ such that $Z_n \nearrow B$. Thus (4.1) holds.

Let us denote $\mathcal{A}^* = \{B \in \mathcal{B}^*;\ (4.1) \text{ holds}\}$. We have proved that $\mathcal{G}^* \subset \mathcal{A}^*$ and, obviously, $B \in \mathcal{A}^*$ implies $X \setminus B \in \mathcal{A}^*$. Suppose that $B_n \in \mathcal{A}^*$, $n \in \mathbb{N}$, and put $B = \bigcup_{n=1}^{\infty} B_n$.

Choose $\varepsilon > 0$. There exist $U_n \in \mathcal{G}^*$, $U_n \supset B_n$, such that $\mu(B_n) > \mu(U_n) - \varepsilon/2^n$. By 3.2, $U = \bigcup\limits_{n=1}^{\infty} U_n$ belongs to $\mathcal{G}^*$ and

$$\mu(U \setminus B) \leqslant \mu\Big(\bigcup_{n=1}^{\infty}(U_n \setminus B_n)\Big) \leqslant \sum_{n=1}^{\infty}\mu(U_n \setminus B_n) < \varepsilon.$$

It follows that

$$\mu(B) = \inf\{\mu(U); U \supset B, U \in \mathcal{G}^*\}.$$

Fix $c < \mu(B)$ and $k \in \mathbb{N}$ such that $\delta := \mu\Big(\bigcup\limits_{n=1}^{k} B_n\Big) - c > 0$. Choose $Z_n \in \mathcal{F}^*$, $Z_n \subset B_n$, such that $\mu(B_n \setminus Z_n) < \frac{\delta}{k}$. By 3.1, $Z = \bigcup\limits_{n=1}^{k} Z_n \in \mathcal{F}^*$ and

$$\mu\Big(\bigcup_{n=1}^{k} B_n \setminus Z\Big) \leqslant \mu\Big(\bigcup_{n=1}^{k}(B_n \setminus Z_n)\Big) \leqslant \sum_{n=1}^{k}\mu(B_n \setminus Z_n) < \delta.$$

We have

$$\mu(Z) > \mu\Big(\bigcup_{n=1}^{k} B_n\Big) - \delta = c,$$

thus

$$\mu(B) = \sup\{\mu(Z); Z \subset B, Z \in \mathcal{F}^*\}.$$

Hence $\mathcal{A}^* \subset \mathcal{B}^*$ is a σ-algebra containing $\mathcal{G}^*$, which gives $\mathcal{A}^* = \mathcal{B}^*$. □

In normal spaces, a Baire measure admits at most one extension to a regular Borel measure. This follows from the next proposition and the above quoted uniqueness theorem.

4.2. Proposition. *Let X be a normal topological space and ν a regular Borel measure on X. Then*

$$\nu(G) = \sup\{\nu(Z); Z \subset G, Z \in \mathcal{F}^*\}, \quad G \in \mathcal{G}.$$

Proof. This follows from the definition of regularity and from 3.4. □

The following easy result could be formulated in a quite abstract setting. We prefer the topological context in view of its application.

4.3. Proposition. *Let X be a topological space, let τ be a finite positive increasing set function defined on $\mathcal{G}$ such that $\tau(\emptyset) = 0$ and*

$$\tau\left(\bigcup_{n=1}^{\infty} G_n\right) \leqslant \sum_{n=1}^{\infty} \tau(G_n)$$

whenever $G_n \in \mathcal{G}$, $n \in \mathbb{N}$. For an arbitrary set $A \subset X$ define

$$\varphi(A) = \inf\left\{\tau(G); G \supset A, G \in \mathcal{G}\right\}.$$

Then φ is an outer measure and every set $E \subset X$ satisfying

$$\varphi(G \cap E) + \varphi(G \setminus E) \leqslant \varphi(G)$$

for every $G \in \mathcal{G}$ is φ-measurable.

Proof. Obviously, $\varphi(\emptyset) = 0$, φ is increasing and it is straightforward to verify that φ is σ-subadditive.

Let $E \subset X$ and let

$$\varphi(G \cap E) + \varphi(G \setminus E) \leqslant \varphi(G)$$

for every $G \in \mathcal{G}$. If $A \subset X$, $G \in \mathcal{G}$ and $A \subset G$, then

$$\varphi(A \cap E) + \varphi(A \setminus E) \leqslant \varphi(G \cap E) + \varphi(G \setminus E) \leqslant \varphi(G) = \tau(G).$$

Consequently,

$$\varphi(A \cap E) + \varphi(A \setminus E) \leqslant \varphi(A).$$

Since subadditivity of φ gives the converse inequality, E is φ-measurable. $\square$

(Professor Mařík used to say: *a measurable set is a sharp knife cutting each set additively.*)

5. Mařík's result

5.1. Theorem. *Let X be a normal countably paracompact topological space and let μ be a Baire measure on X. Then there exists a unique regular Borel measure ν such that $\nu_{|\mathcal{B}^*} = \mu$.*

Proof. *Uniqueness* follows from Proposition 4.2. As already mentioned, the *existence* is based on a *two-step regularization:* Starting with μ we produce a σ-subadditive set function on $\mathcal{G}$ which gives rise to an outer measure φ. We show that

$\varphi = \mu$ on $\mathcal{B}^*$ and that every Borel set is φ-measurable. We put $\nu = \varphi_{|\mathcal{B}}$; regularity will follow from the construction of φ. Here are the details.

Put

$$\tau(G) = \sup\{\mu(Z); Z \subset G, Z \in \mathcal{F}^*\}, \quad G \in \mathcal{G}.$$

Obviously, $\tau(\emptyset) = 0$ and τ is increasing. For $G \in \mathcal{G}^*$, $\tau(G) = \mu(G)$ by Proposition 4.1. We are going to establish σ-subadditivity of τ.

Let $G_n \in \mathcal{G}$ and $Z \in \mathcal{F}^*$, $Z \subset \bigcup_{n=1}^{\infty} G_n$. Then $\{X \setminus Z\} \cup \{G_n; n \in \mathbb{N}\}$ is a countable open cover of the space X. Since X is countably paracompact and normal, Theorem 2.1 guarantees the existence of closed sets F and F_n, $n \in \mathbb{N}$, such that $F \subset X \setminus Z$ and $F_n \subset G_n$, $n \in \mathbb{N}$, and $F \cup \bigcup_{n=1}^{\infty} F_n = X$. Then $Z \subset \bigcup_{n=1}^{\infty} F_n$ and, by 3.4, there exist $Z_n \in \mathcal{F}^*$ such that $F_n \subset Z_n \subset G_n$ for every $n \in \mathbb{N}$. Since $Z \subset \bigcup_{n=1}^{\infty} Z_n$, we get

$$\mu(Z) \leqslant \sum_{n=1}^{\infty} \mu(Z_n) \leqslant \sum_{n=1}^{\infty} \tau(G_n).$$

It follows that

$$\tau\Big(\bigcup_{n=1}^{\infty} G_n\Big) \leqslant \sum_{n=1}^{\infty} \tau(G_n).$$

Now we define, as in Proposition 4.3, the outer measure φ by

$$\varphi(A) = \inf\{\tau(G); G \supset A, G \in \mathcal{G}\}, \quad A \subset X.$$

Clearly, $\varphi(G) = \tau(G)$ for every $G \in \mathcal{G}$. We claim that every Borel set is φ-measurable. It suffices to show that every open set is φ-measurable.

So fix $E \in \mathcal{G}$ and take a "test set" $G \in \mathcal{G}$. It is sufficient to show that

$$\varphi(G \cap E) + \varphi(G \setminus E) \leqslant \varphi(G).$$

Fix $\varepsilon > 0$ and choose $Z \in \mathcal{F}^*$ such that $Z \subset G \cap E$ and $\tau(G \cap E) < \mu(Z) + \varepsilon$. Since $G \setminus Z \in \mathcal{G}$, there exists $Y \in \mathcal{F}^*$, $Y \subset G \setminus Z$, such that $\tau(G \setminus Z) < \mu(Y) + \varepsilon$. Taking into account that $G \cap E \in \mathcal{G}$, $G \setminus E \subset G \setminus Z$, $G \setminus Z \in \mathcal{G}$, $Z \cap Y = \emptyset$ and $Z \cup Y \in \mathcal{F}^*$, we get

$$\begin{aligned}\varphi(G \cap E) + \varphi(G \setminus E) &\leqslant \tau(G \cap E) + \varphi(G \setminus Z)\\ &= \tau(G \cap E) + \tau(G \setminus Z) < \mu(Z) + \mu(Y) + 2\varepsilon\\ &= \mu(Z \cup Y) + 2\varepsilon \leqslant \tau(G) + 2\varepsilon = \varphi(G) + 2\varepsilon.\end{aligned}$$

This proves that E is φ-measurable.

Define $\nu := \varphi_{|\mathcal{B}}$. Then ν is a Borel measure and, for every $B \in \mathcal{B}$,

$$\begin{aligned}\nu(B) = \varphi(B) &= \inf\{\tau(G); G \supset B, G \in \mathcal{G}\}\\ &= \inf\{\nu(G); G \supset B, G \in \mathcal{G}\}.\end{aligned}$$

It follows that ν is a regular Borel measure (see the text preceding Proposition 4.1). Since $\nu(G) = \varphi(G) = \tau(G) = \mu(G)$ for every $G \in \mathcal{G}^* \subset \mathcal{G}$, $G_1 \cap G_2 \in \mathcal{G}^*$ whenever G_1, $G_2 \in \mathcal{G}^*$ by 3.1, we conclude that $\nu = \mu$ on $\mathcal{B}^*$ (by the uniqueness theorem quoted in Sec. 4.), which completes the proof. □

6. Mařík's original paper

In this section, as in [8], the term measure means a positive σ-additive *extended real-valued* set function defined on a σ-algebra and vanishing on the empty set.

For a Baire measure μ on a topological space X, Jan Mařík introduces in [8] the system $\mathcal{P} = \mathcal{P}(\mu)$ of sets $B \subset X$, for which there exist sets $G_n \in \mathcal{G}^*$, $n \in \mathbb{N}$, of finite μ-measure covering B. He says that a measure μ has the *property* V_X, if μ is a Baire measure on X such that $B \in \mathcal{P}$ when $\mu(B) < \infty$. He provides an example of a Baire measure which does not have the property V_X.

For a Baire measure μ and an arbitrary set $A \subset X$, define

$$\underline{\mu}(A) = \sup\{\mu(F); F \subset A, F \in \mathcal{F}^*, \mu(F) < \infty\}.$$

The following important notion is introduced in [8]: A measure μ has the *property* W_X, if μ is a Baire measure and there exists a Borel measure ν having the following properties:

(1) $B \in \mathcal{B}^* \Longrightarrow \nu(B) = \mu(B)$;
(2) $G \in \mathcal{G} \cap \mathcal{P} \Longrightarrow \nu(G) = \underline{\mu}(G)$;
(3) $B \in \mathcal{B} \setminus \mathcal{P} \Longrightarrow \nu(B) = \infty$;
(4) $B \in \mathcal{B} \Longrightarrow \nu(B) = \inf\{\nu(G); G \supset B, G \in \mathcal{G}\}$.

The main results of [8] read as follows:

6.1. *Let X be a completely regular topological space and let a measure μ have the property V_X. Suppose that for every $F \in \mathcal{F}^*$ with $\mu(F) < \infty$ there are compact sets K_n, $n \in \mathbb{N}$, such that $\underline{\mu}\Big(F \setminus \bigcup_{n=1}^{\infty} K_n\Big) = 0$. Then μ has the property W_X.*

6.2. *Let X be a normal space, and let a measure μ have the property V_X. Suppose that, for every $F \in \mathcal{F}^*$ with $\mu(F) < \infty$, there exist pseudocompact sets A_n, $n \in \mathbb{N}$,* *such that* $\underline{\mu}\Big(F\setminus \bigcup\limits_{n=1}^{\infty} A_n\Big) = 0$. *Then* μ *has the property* W_X. (Recall that a topological space Y is said to be *pseudocompact*, if every continuous function on Y is bounded.)

6.3. *Let* X *be a normal countably paracompact space and let* μ *have the property* V_X. *Then* μ *also has the property* W_X.

A relation of the above results to results on integral representation of positive linear functionals on spaces of continuous functions is mentioned, in particular, in connection with papers of S. Kakutani (1941) and E. Hewitt (1950). An example of a completely regular space X and of a finite Baire measure μ on X which does not have the property W_X is constructed. However, on this space, there exist many extensions of μ to a Borel measure.

Let us mention that the journal received Mařík's paper [8] on November 25, 1955. The notion of countably paracompact spaces was recent. In fact, all topological results needed in the article were proved. For instance, Sec. 11 of [8] provides a proof of Urysohn's lemma, Sec. 20 a transfinite induction proof of the property of normal spaces stated at the end of our Sec. 2. Mařík's Sec. 21 contains the following result (cf. with Theorem 2.1):

Let X *be a normal space. Then the following conditions are equivalent:*

(1) *If* $G_n \in \mathcal{G}$, $G_n \nearrow X$, *then there are* $F_n \in \mathcal{F}$ *such that* $F_n \subset G_n$ *and* $F_n \nearrow X$;

(2) X *is countably paracompact.*

It seems that J. Mařík, although a non-topologist, came to this result independently. In fact, his final remark in [8] says:

This paper is directed also to readers who did not deal too much with topology. Therefore also known facts (e.g. in Sections 11 and 20) are proved. Professor M. Katětov pointed out to the author that also the theorem of Sec. 21 is known (references to Dowker's and Katětov's papers from 1951 follow).

7. Results and problems related to Mařík spaces

Various aspects of the Borel extension problem attracted the attention of specialists in topology or topological measure theory. A rather complete picture of the landscape can be obtained from [16], pp. 131–135 and from [15] where more than a dozen of papers closely related to [8] is quoted.

Let us state at least one result from [15]: *There exists a countably paracompact non-Mařík space.* In a letter of June 30, 1995, Professor Haruto Ohta kindly informed the author that Professor Stephen Watson (York University, Canada) can strengthen this result: *there exists a first countable countably paracompact non-Mařík space.* (Here *first countable* means that every point possesses a countable local base.)

In fact, [15] gives answers to several questions posed in [16] and proposes new seven unsolved problems. As a sample, we reproduce the following question:

Is the product of a Mařík space with a compact space a Mařík space? More generally, is the preimage of a Mařík space under an open, perfect map a Mařík space?

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