Elena Wisztová

On a Hamiltonian cycle of the fourth power of a connected graph

Mathematica Bohemica, Vol. 116 (1991), No. 4, 385–390

Persistent URL: http://dml.cz/dmlcz/126033

ON A HAMILTONIAN CYCLE OF THE FOURTH POWER OF A CONNECTED GRAPH

ELENA WISZTOVÁ, Žilina

(Received November 21, 1989)

Summary. In this paper the following theorem is proved: Let G be a connected graph of order $p \geqq 4$ and let M be a matching in G. Then there exists a hamiltonian cycle C of G^4 such that $E(C) \cap M = \emptyset$.

Keywords: Powers of graphs, hamiltonian cycles, matchings in graphs.

AMS Classification: 05C.

By a graph we will mean a finite undirected graph with no loops or multiple edges (a graph in the sense of [1] and [2]). If G is a graph, then we denote by $V(G)$, $E(G)$, and $\Delta(G)$ the vertex set, the edge set, and the maximum degree of G, respectively. The number $|V(G)|$ is called the order of G. If $u, v, w \in V(G)$, then the degree of u in G and the distance between v and w in G will be denoted by $\deg_G u$ and $d_G(v, w)$, respectively.

If G is a graph and n is a positive integer, then the n-th power G^n of G is the graph defined as follows: $V(G^n) = V(G)$ and $E(G^n) = \{uv; u, v \in V(G) \text{ and } 1 \leqq d_G(u, v) \leqq n\}$.

We say that a graph F is a 1-factor of a graph G if F is a regular graph of degree one, and at the same time a spanning subgraph of G. A set $M \subseteq E(G)$ is called a matching in G if no two edges in M are incident with the same vertex.

We now mention some results concerning regular factors and hamiltonian properties of the fourth power of a connected graph.

Theorem A [3]. *If G is a connected graph of even order $\geqq 4$, then G^4 has a 3-factor F such that each component of F is a copy of K_4 or $K_3 \times K_2$.*

Theorem B [4]. *For every connected graph G of even order $\geqq 4$, G^4 has three mutually edge-disjoint 1-factors.*

Theorem C [7]. *Let G be a connected graph of even order $\geqq 4$. Then there exist a hamiltonian cycle C of G^3 and a 1-factor F of G^4 such that C and F are edge-disjoint.*

Theorem D [5]. *Let G be a connected graph of even order $\geqq 4$, and let H be* *a triangle-free subgraph of* G^3 *with* $\Delta(H) \leqq 2$. *Then there exists a* 1*-factor* F *of* G^4 *such that* $E(F) \cap E(H) = \emptyset$.

The following theorem is the main result of this note:

Theorem 1. *Let* G *be a connected graph of order* $p \geqq 4$ *and let* M *be a matching in* G. *Then there exists a hamiltonian cycle* C *of* G^4 *such that* $E(C) \cap M = \emptyset$.

To prove Theorem 1 we shall use five lemmas and two remarks. We say that an ordered pair (T', r') is a rooted tree if T' is a tree and $r' \in V(T')$. We say that rooted trees (T', r') and (T'', r'') are isomorphic if T' and T'' are isomorphic and there exists an isomorphism T' onto T'' which maps r' onto r''. Let T be a tree. Similarly as in [7], by a terminal subtree of T we mean a rooted tree (T', r') with the properties that T' is a subtree of T and for each $v \in V(T' - r')$, $\deg_{T'} v = \deg_T v$.

Let $m \geqq 0$ and $n \geqq 1$ be integers, and let $u_0, \ldots, u_m, w_1, \ldots, w_n$ be mutually distinct vertices. We denote by A_n the path with

$$V(A_n) = \{w_1, \ldots, w_n\} \quad \text{and} \quad E(A_n) = \{w_i w_{i+1};\ 1 \leqq i \leqq n-1\}\,.$$

Similarly, we denote by B_{mn} the path with

$$V(B_{mn}) = \{u_m, \ldots, u_0,\ w_1, \ldots, w_n\} \quad \text{and}$$
$$E(B_{mn}) = \{u_j u_{j-1};\ m \geqq j > 0\} \cup \{u_0 w_1\} \cup \{w_k w_{k+1};\ 1 \leqq k \leqq n-1\}.$$

Finally, we define the following rooted tree:

$$D_{mn} = (B_{mn}, u_0)\,.$$

Denote

$$\mathscr{D} = \{D_{11}, D_{14}, D_{21}, D_{22}, D_{23}, D_{24}, D_{31}, D_{33}, D_{34}, D_{44}, D_{05}\}\,,$$
$$\mathscr{D}' = \mathscr{D} - \{D_{05}\}\,.$$

Lemma 1. *Let* T *be a tree of order* $p \geqq 6$. *Then there exists a terminal subtree of* T *which is isomorphic to one of the elements of* $\mathscr{D}$.

Proof. Let δ denote the diameter of T. Obviously, there exists a terminal subtree (T_0, r_0) of T such that

$$d_{T_0}(r_0, v) \leqq 5 \quad \text{for every} \quad v \in V(T_0) \quad \text{and}$$
$$d_{T_0}(r_0, v') = \min(5, \delta) \quad \text{for at least one} \quad v' \in V(T_0)\,.$$

It is easy to see that there exists a terminal subtree (T', r') of T such that $V(T') \subseteq V(T_0)$, and (T', r') is isomorphic to one of the elements of $\mathscr{D}$.

If G is a graph, then we denote by $\mathscr{H}(G)$, $\bar{\mathscr{H}}(G)$ and $\mathscr{M}(G)$ the set of hamiltonian cycles of G, the set of hamiltonian paths of G and the set of matchings in G, respectively.

Lemma 2. *Let* $n \geqq 5$, *and let* M *be a matching in* A_n. *Then there exists a hamiltonian* $w_1 - w_2$ *path* P *of* $(A_n)^3$ *such that* $E(P) \cap M = \emptyset$.

Proof. If $n = 5$, then for a $i \in \{1, 2, 3\}$ matching $M_i \in \mathscr{M}(A_5)$ we determine $E(P_i)$:

$$M_1 = \{w_1w_2, w_3w_4\}, \quad E(P_1) = \{w_1w_3, w_3w_5, w_5w_4, w_4w_2\}.$$

$$M_2 = \{w_1w_2, w_4w_5\}, \quad E(P_2) = \{w_1w_4, w_4w_3, w_3w_5, w_5w_2\}.$$

$$M_3 = \{w_2w_3, w_4w_5\}, \quad E(P_3) = \{w_1w_4, w_4w_3, w_3w_5, w_5w_2\}.$$

The path P_i, $i \in \{1, 2, 3\}$ has the desired properties. For every matching $M' \in \mathscr{M}(A_5)$ there exists $i \in \{1, 2, 3\}$ such that $M' \subseteq M_i$.

Let $n \geqq 6$. Assume that for every tree A_m, where $5 \leqq m < n$, it is proved that for any matching $M^* \in \mathscr{M}(A_m)$ there exists a $w_1 - w_2$ path $P^* \in \mathscr{H}((A_m)^3)$ such that $E(P^*) \cap M^* = \emptyset$.

Denote

$$T_0 = T - w_1, \quad M_0 = M, \quad \text{if} \quad w_1w_2 \notin M \quad \text{and}$$

$$M_0 = M - \{w_1w_2\}, \quad \text{if} \quad w_1w_2 \in M.$$

Then $5 \leqq |V(T_0)| < n$, T_0 is isomorphic to A_{n-1} and $M_0 \in \mathscr{M}(T_0)$. It follows from the induction hypothesis that there exists a $w_2 - w_3$ path $P_0 \in \mathscr{H}((T_0)^3)$ such that $E(P_0) \cap M_0 = \emptyset$. We define

$$P = P_0 + w_1w_3.$$

Then $P \in \mathscr{H}((A_n)^3)$ has the desired properties.

Remark 1. Let M be a matching in A_4. Then there exists a hamiltonian $w_1 - w_3$ path P of $(A_4)^3$ such that $E(P) \cap M = \emptyset$.

Lemma 3. *Let $n \geqq 4$, and let M be a matching in A_n. Then there exists $C \in \mathscr{H}((A_n)^4)$ such that $E(C) \cap M = \emptyset$.*

Proof. Now we distinguish two cases and several subcases.

1. Assume that $n = 4$. From Remark 1 it follows that there exists a $w_1 - w_3$ path $P \in \mathscr{H}((A_4)^3)$ such that $E(P) \cap M = \emptyset$. We put

$$C = P + w_1w_3.$$

2. Assume that $n \geqq 5$. It follows from Lemma 2 that there exists a $w_1 - w_2$ path $P \in \mathscr{H}((A_n)^3)$ such that $E(P) \cap M = \emptyset$.

2.1. Let $w_1w_2 \notin M$. Then we put

$$C = P + w_1w_2.$$

2.2. $w_1w_2 \in M$.

2.2.1. Assume that $n \in \{5, 6\}$. For a matching $M_i \in \mathscr{M}(A_n)$ with $w_1w_2 \in M_i$ we will determine $E(C_i)$ for $i \in \{1, 2\}$. If $n = 5$, then

$$M_1 = \{w_1w_2, w_3w_4\}, \quad E(C_1) = \{w_1w_4, w_4w_5, w_5w_2, w_2w_3, w_3w_1\},$$

$$M_2 = \{w_1w_2, w_4w_5\}, \quad E(C_2) = \{w_1w_4, w_4w_2, w_2w_5, w_5w_3, w_3w_1\}.$$

If $n = 6$, then

$$M_1 = \{w_1w_2, w_3w_4, w_5w_6\},$$
$$E(C_1) = \{w_1w_3, w_3w_6, w_6w_2, w_2w_5, w_5w_4, w_4w_1\},$$
$$M_2 = \{w_1w_2, w_4w_5\},$$
$$E(C_2) = \{w_1w_3, w_3w_2, w_2w_5, w_5w_6, w_6w_4, w_4w_1\}.$$

For every matching $M' \in \mathscr{M}(A_n)$ with $w_1w_2 \in M'$ there exists $i \in \{1, 2\}$ such that $M' \subseteq M_i$.

2.2.2. Let $n \geqq 7$. Denote

$$T_0 = T - w_1 - w_2 \quad \text{and} \quad M_0 = M - \{w_1w_2\}.$$

Then $5 \leqq |V(T_0)| = n - 2$, T_0 is isomorphic to A_{n-2} and $M_0 \in \mathscr{M}(T_0)$. It follows from Lemma 2 that there exists a $w_3 - w_4$ path $P_0 \in \bar{\mathscr{H}}((T_0)^3)$ such that $E(P_0) \cap M_0 = \emptyset$. There exists $x \in \{w_5, w_6\}$ such that $w_3x \in E(P_0)$. We define

$$C = P_0 - w_3x + xw_2 + w_2w_3 + w_3w_1 + w_1w_4.$$

In each case $C \in \mathscr{H}((A_n)^4)$ has the desired properties.

Remark 2. Let $M = \{w_1w_2, w_2w_4, w_5w_6\}$ be the matching in A_6. It is easy to show that there exists no hamiltonian cycle C of $(A_6)^3$ such that $E(C) \cap M = \emptyset$. This means that value 4 of the power in Lemma 3 is the best possible.

Lemma 4. *Let T be a tree of order $p \geqq 4$ and let M be a matching in T. Then there exists a hamiltonian cycle C of T^4 such that $E(C) \cap M = \emptyset$.*

Proof. If $p \in \{4, 5\}$, then T is isomorphic to one of the 5 trees presented in the list in [2], p. 233. It is easy to show that the statement of the lemma is correct.

Let $p \geqq 6$. Assume that for every tree T^* of order p^*, where $5 \leqq p^* < p$, it is proved that for any matching $M^* \in \mathscr{M}(T^*)$ there exists a hamiltonian cycle $C^* \in \mathscr{H}((T^*)^4)$ such that $E(C^*) \cap M^* = \emptyset$.

If T is isomorphic to A_p then the result follows from Lemma 3. We shall assume that T is not isomorphic to A_p. It follows from Lemma 1 that T has a terminal subtree isomorphic to one of the elements of $\mathscr{D}$. Now we shall distinguish two cases and several subcases.

1. Assume that T has a terminal subtree isomorphic to one of the elements of $\mathscr{D}'$. Consider such a terminal subtree (T_1, r_1) that (T_1, r_1) is isomorphic to one of the elements of $\mathscr{D}'$ and that for every terminal subtree (T', r') of T which is isomorphic to one of the elements of $\mathscr{D}'$, $|V(T_1)| \leqq |V(T')|$. For the sake of simplicity we will assume that $(T_1, r_1) \in \mathscr{D}'$. Then $r_1 = u_0$ and there exist $m \geqq 1$ and $n \geqq 1$ such that $V(T_1) = \{u_m, \ldots, u_0, w_1, \ldots, w_n\}$. Denote

$$M_1 = M \cap (\{u_0w_1\} \cup \{w_kw_{k+1}, 1 \leqq k \leqq n - 1\}).$$

Moreover, we denote

$$T_0 = T - w_1 - \dots - w_n, \quad M_0 = M - M_1,$$
$$\text{if} \quad (T_1, u_0) \in \mathscr{D}' - \{D_{22}\},$$
$$T_0 = T - w_2, \quad M_0 = M - \{w_1 w_2\}, \quad \text{if} \quad (T_1, u_0) = D_{22}.$$

Then $5 \leqq |V(T_0)| < p$ and $M_0 \in \mathscr{M}(T_0)$. It follows from the induction hypothesis that there exists $C_0 \in \mathscr{H}((T_0)^4)$ such that $E(C_0) \cap M_0 = \emptyset$.

1.1. Let $(T_1, u_0) \in \{D_{11}, D_{21}, D_{31}\}$. There exists $x \in V(T_0)$ such that $x \neq u_0$ and $xu_1 \in E(C_0)$. Then $d_T(x, w_1) \leqq 4$. We define

$$C = C_0 - u_1 x + u_1 w_1 + w_1 x.$$

1.2. Let $(T_1, u_0) \in \{D_{14}, D_{24}, D_{34}, D_{44}\}$. Then $T - V(T_0) = A_4$. It follows from Remark 1 that there exists a $w_1 - w_3$ path $P \in \mathscr{H}((A_4)^3)$ such that $E(P) \cap M = \emptyset$. There exists $x \in V(T_0)$ such that $xu_1 \in E(C_0)$, and if $(T_1, u_0) = D_{44}$, then $x \neq u_4$. Hence $d_T(x, w_1) \leqq 4$. We define

$$C = (C_0 - u_1 x + u_1 w_3 + x w_1) \cup P \quad \text{if} \quad x \neq u_0 \quad \text{and}$$
$$C = (C_0 - u_1 x + u_1 w_1 + x w_3) \cup P \quad \text{if} \quad x = u_0.$$

1.3. Let $(T_1, u_0) \in \{D_{23}, D_{33}\}$. There exist $x, y \in V(T_0)$ such that $u_1 x, u_2 y \in E(C_0)$, $u_1 x \neq u_2 y$, and if $(T_1, u_0) = D_{33}$, then $y \neq u_3$. Then $d_T(w_1 x) \leqq 4$ and $d_T(w_2 y) \leqq 4$. We define

$$C = C_0 - u_1 x - u_2 y + u_1 w_3 + w_3 w_1 + w_1 x + u_2 w_2 + y w_2$$
$$\text{if} \quad x \neq u_0 \quad \text{and}$$
$$C = C_0 - u_1 x - u_2 y + u_1 w_1 + w_1 w_3 + w_3 x + u_2 w_2 + y w_2$$
$$\text{if} \quad x = u_0.$$

1.4. Let $(T_1, u_0) = D_{22}$. There exists $x \in V(T_0)$ such that $u_2 x \in E(C_0)$ and $x \neq w_1$. Then $d_T(w_2, x) \leqq 4$. We define

$$C = C_0 - u_2 x + u_2 w_2 + x w_2.$$

We can see that in each subcase C has the desired properties.

2. Assume that T contains no terminal subtree isomorphic to an element of $\mathscr{D}'$. It follows from Lemma 1 that there exists $n \geqq 5$ and a terminal subtree (T_2, r_2) of T such that (T_2, r_2) is isomorphic to D_{0n} and $\deg_T r_2 \geqq 3$. For the sake of simplicity we will assume that $(T_2, r_2) = D_{0n}$, thus $r_2 = u_0$ and $V(T_2) = \{u_0, w_1, w_2, \dots, w_n\}$. Denote

$$M_2 = M \cap E(T_2).$$

Then $M_2 \in \mathscr{M}(T_2)$. As follows from Lemma 2, there exists a hamiltonian $w_1 - w_2$ path $P \in \mathscr{H}((T_2 - u_0)^3)$ such that $E(P) \cap M_2 = \emptyset$. Further, we denote

$$T_0 = T - w_1 - \dots - w_n \quad \text{and} \quad M_0 = M - M_2.$$

Then $M_0 \in \mathcal{M}(T_0)$. Since T is isomorphic to no A_p and T contains no terminal subtree isomorphic to an element of $\mathcal{D}'$, we have $5 < |V(T_0)| < p$. It follows from the induction hypothesis that there exists $C_0 \in \mathcal{H}((T_0)^4)$ such that $E(C_0) \cap M_0 = \emptyset$. Since $\deg_{T_0} u_0 \geqq 2$, there exist $x, y \in (V(T_0) - \{u_0\})$ such that $xy \in E(C_0)$ and $d_T(u_0, x) + d_T(u_0, y) \leqq 4$. Without loss of generality we may assume that $d_T(u_0, x) \leqq$ $\leqq d_T(u_0, y)$. We define

$$C = (C_0 - xy + xw_2 + yw_1) \cup P,$$

then $C \in \mathcal{H}(T^4)$ and $E(C) \cap M = \emptyset$.

Thus the proof of Lemma 4 is complete.

Lemma 5. ([6] *p.* 63.) *Let G be a connected graph and let L be a subgraph of G which contains no cycle. Then there exists a spanning tree T of G such that L is a subgraph of T.*

Proof of Theorem 1. Let G be a graph satisfying the conditions of Theorem 1 and let M be an arbitrary matching in G. As follows from Lemma 5, there exists a spanning tree T of G such that M is a matching in T. According to Lemma 4, T^4 has a hamiltonian cycle C such that $E(C) \cap M = \emptyset$. Thus G^4 also has a hamiltonian cycle C such that $E(C) \cap M = \emptyset$.

References

[1] *M. Behzad, G. Chartrand, L. Lesniak-Foster:* Graphs & Digraphs. Prindle. Weber & Schmidt, Boston 1979.

[2] *F. Harary:* Graph Theory. Addison-Wesley, Reading, Mass., 1969.

[3] *L. Nebeský:* On the existence of a 3-factor in the fourth power of a graph. Čas. pěst. mat. *105* (1980), 204—207.

[4] *L. Nebeský:* Edge-disjoint 1-factors in powers of connected graphs. Czech. Math. J. *34* (*109*) (1984), 499—505.

[5] *L. Nebeský:* On a 1-factor of the fourth power of a connected graph. Čas. pěst. mat. *113* (1988), 415—420.

[6] *J. Sedláček:* Introduction into the Graph Theory (Czech). Academia nakl. ČSAV, Praha 1981.

[7] *E. Wisztová:* A hamiltonian cycle and a 1-factor in the fourth power of a graph. Čas. pěst. mat. *110* (1985), 403—412.

Súhrn

O HAMILTONOVSKEJ KRUŽNICI V ŠTVRTEJ MOCNINE SÚVISLÉHO GRAFU

ELENA WISZTOVÁ

V článku je dokázaná nasledovná veta: Nech G je súvislý graf s p vrcholmi, kde $p \geqq 4$ a nech M je párenie v grafe G. Potom v G^4 existuje hamiltonovská kružnica C taká, že $E(C) \cap M = \emptyset$.

Author's address: Vysoká škola dopravy a spojov, Hurbanova 15, 010 26 Žilina.