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# ON PERMUTABILITY IN SEMIGROUP VARIETIES

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*Summary.* The paper contains characterizations of semigroup varieties whose semigroups with one generator (two generators) are permutable. Here all varieties of regular $*$-semigroups are described in which each semigroup with two generators is permutable.

*Keywords:* Permutable variety, semigroup, regular $*$-semigroup, semilattice.

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An algebra $A$ is called permutable if $\Phi \,.\, \Psi = \Phi \,.\, \Psi$ for each two congruences $\Phi$, $\Psi$ on $A$. A variety $\mathscr{V}$ is permutable if every $A \in \mathscr{V}$ has this property. I. Chajda [1] characterized varieties of algebra having permutable algebras with two generators. In my paper [2] all permutable varieties of semigroups are described. The aim of this note is to describe semigroup varieties having permutable semigroups with two generators.

By $W(i = j)$ we denote the variety of all semigroups satisfying the identity $i = j$.

**Theorem 1.** *The following conditions for a variety $\mathscr{V}$ of semigroups are equivalent:*

1. *$\mathscr{V}$ is permutable.*
2. *$\mathscr{V} \subseteq W(x^n y = y) \cap W(yx^n = y)$ for a positive integer $n$.*

Proof. See Theorem 2 of [2].

**Theorem 2.** *The following conditions for a variety $\mathscr{V}$ of semigroups are equivalent:*

1. *Each $S \in \mathscr{V}$ with one generator is permutable.*
2. *$\mathscr{V} \subseteq W(x = xx^n)$ or $\mathscr{V} \subseteq W(x^n = xx^n)$ for a positive integer $n$.*

Proof. $1 \Rightarrow 2$. Let $S \in \mathscr{V}$ and $a \in S$. By $\langle a \rangle$ we denote the subsemigroup of $S$ generated by $a$. Suppose that $\langle a \rangle$ is permutable. It follows from Theorem 6 and Theorem 13 of [3] that $a = aa^m$ or $a^m = aa^m$ for a positive integer $m$. In both cases $S$ contains an idempotent and so by Lemma 1 of [2] $\mathscr{V} \subseteq W(x^n x^n = x^n)$ for a positive integer $n$. By way of contradiction assume that $\mathscr{V} \nsubseteq W(x = xx^n)$ and $\mathscr{V} \nsubseteq W$

$(x^n = xx^n)$. Then $n \geqq 2$ and there exist $S \in \mathscr{V} \setminus W(x = xx^n)$ and $T \in \mathscr{V} \setminus W$ $(x^n = xx^n)$. Consequently, there are $a \in S$, $b \in T$ such that $a \neq aa^n$, $b^n \neq bb^n$. It is easy to show that according to Theorem 6 and Theorem 13 of [3], the subsemigroup $\langle(a, b)\rangle$ of $S \times T$ generated by $(a, b)$ is not permutable. Therefore $S \times T \notin \mathscr{V}$, which is a contradiction. Consequently $\mathscr{V} \subseteq W(x = xx^n)$ or $\mathscr{V} \subseteq W(x^n = xx^n)$.

$2' \Rightarrow 1$. This follows from Theorem 6 and Theorem 13 of [3].

**Theorem 3.** *The following conditions for a variety $\mathscr{V}$ of semigroups are equivalent:*

1. *Each $S \in \mathscr{V}$ with two generators is permutable.*
2. *$\mathscr{V} \subseteq W(x = xx^n) \cap W((xyx)^n = x^n)$ for a positive integer $n$.*

Before the proof we formulate the following

**Lemma.** $W(x = xx^n) \cap W((xyx)^n = x^n) = W(x = xx^n) \cap W((xyz)^n = (xz)^n)$.

Proof. We have $(xyz)^n = x^n(xyz)^n z^n = (xzx)^n (xyz)^n (zxz)^n = xzuxz$ and so $(xyz)^n = ((xyz)^n)^n = (xzuxz)^n = (xz)^n$.

Proof of Theorem 3. $1 \Rightarrow 2$. Suppose that every semigroup from $\mathscr{V}$ with two generators is permutable. By $\mathscr{Z}$ or $\mathscr{S}$ we denote the variety of all zero-semigroups or semilattices, respectively, i.e. $\mathscr{Z} = W(xy = uv)$ and $\mathscr{S} = W(xy = yx) \cap W(x^2 = x)$. It is well known that $\mathscr{Z}$ and $\mathscr{S}$ are minimal varieties in the lattice of all semigroup varieties. It follows from Theorem 6 and Theorem 13 of [3] that $\mathscr{Z} \cap \mathscr{V} = \mathscr{O} = W(x = y)$. According to Example 2 of [1] we have $\mathscr{S} \cap \mathscr{V} = \mathscr{O}$. By Lemma 3 of [2] we get $\mathscr{V} \subseteq W(x = xx^n) \cap W((xyx)^n = x^n)$ for a positive integer $n$.

$2 \Rightarrow 1$. Assume that

$$S \in W(x = xx^n) \cap W((xyx)^n = x^n) \tag{1}$$

for a positive integer $n$ and that $S$ has two generators $u$ and $v$. We can suppose that $n \geqq 2$. Evidently $S \in W(x^n x^n = x^n)$.

Put $e = u^n$ and $f = v^n$. It is clear that $e = e^2, f = f^2$ and

$$S = eS \cup fS = Se \cup Sf \tag{2}$$

Let $\Phi$ and $\Psi$ be two congruences on $S$. Suppose that $(a, b) \in \Phi \,.\, \Psi$. Then $(a, c) \in \Phi$ and $(c, b) \in \Psi$ for some $c \in S$.

Case 1. $a^n = b^n$. Then we put $d = a^n c a^n$. Using (1) it is easy to show that $(a, d) \in \Phi$, $(d, b) \in \Psi$ and $d^n = a^n$. Putting $h = bd^{n-1}b^n a = bd^{n-1}a$ we obtain $(a, h) = (bd^{n-1}db^{n-1}a, bd^{n-1}bb^{n-1}a) \in \Psi$ and $(h, b) = (ba^{n-1}ad^{n-1}a, ba^{n-1}dd^{n-1}a) \in \Phi$. Therefore $(a, b) \in \Psi.\Phi$.

Case 2. $a^n \neq b^n$. According to (1) and (2) we have the following eight possibilities.

Subcase 2.1. $a = ea$ and $b = eb$. Then we put $d = ec$ and so by (1) we have

$(a, d) \in \Phi$ and $(d, b) \in \Psi$ It follows from (1), Lemma and (2) that $d^n = a^n$ or $d^n = b^n$. Without loss of generality we can suppose that $d^n = a^n$. It follows from (1) that $a^n e = (ea)^n e = (eae)^n = e$ and so $a^n b = a^n eb = eb = b$. Putting $h = ad^{n-1}b = ad^{n-1} aa^{n-1}b$ we have $(a, h) = (ad^{n-1}d, ad^{n-1}b) \in \Psi$ and $(h, b) = (ad^{n-1}aa^{n-1}b, ad^{n-1}da^{n-1}b) \in \Phi$. Therefore $(a, b) \in \Psi . \Phi$.

Subcases 2.i ($i = 2, 3$ and 4). $a = fa$ and $b = fb$ ($a = ae$ and $b = be$, $a = af$ and $b = bf$, respectively). In an analogous manner it can be proved that $(a, b) \in \Psi . \Phi$.

Subcase 2.5. $a = eae$ and $b = fbf$. Then we have two possibilities.

Subcase 2.5.1. $c = ece$ or $c = fcf$. Without loss of generality we can suppose that $c = ece$. It follows from (1) that $c^n = e = a^n$ and so putting $h = bb^nc^{n-1}ab^n(b^nc^nb^n)^{n-1}$ we obtain $(a, h) = (cc^nc^{n-1}ac^n(c^nc^nc^n)^{n-1}, h) \in \Psi$ and $(h, b) = (h, bb^nc^{n-1}cb^n(b^nc^nb^n)^{n-1}) \in \Phi$. Therefore $(a, b) \in \Psi . \Phi$.

Subcase 2.5.2. $c = ecf$ or $c = fce$. Without loss of generality we can suppose that $c = ecf$. By Lemma we have $c^n = (ef)^n$ and so $c^n a = (ef)^n ea = (efe)^n a = ea = a$. Analogously we get $bc^n = b$. Putting $h = bc^{n-1}a$ we obtain $(a, h) = (cc^{n-1}a, bc^{n-1}a) \in \Psi$ and $(h, b) = (bc^{n-1}a, bc^{n-1}c) \in \Phi$. Therefore $(a, b) \in \Psi . \Phi$.

Subcase 2.6. $a = faf$ and $b = ebe$. Analogously we can show that $(a, b) \in \Psi . \Phi$.

Subcase 2.7. $a = eaf$ and $b = fbe$. According to Lemma we get $a^n = (ef)^n$ and $b^n = (fe)^n$. We have two possibilities.

Subcase 2.7.1. $c = ece$ or $c \in fcf$. Without loss of generality assume $c = ece$. By (1) we have $c^n = e$. Putting $h = bc^{n-1}a$ we obtain $(a, h) = (cc^{n-1}a, bc^{n-1}a) \in \Psi$ and $(h, b) = (bc^{n-1}a, bc^{n-1}c) \in \Phi$. Therefore $(a, b) \in \Psi . \Phi$.

Subcase 2.7.2. $c = ecf$ or $c = fce$. Without loss of generality assume that $c = ecf$. By Lemma we have $c^n = (ef)^n = a^n$ and $(c^nb^n)^n = e$. Putting $h = bc^{n-1}ab^n(c^nb^n)^{n-1}$ we obtain $(a, h) = (cc^{n-1}ac^n(c^nc^n)^{n-1}, h) \in \Psi$ and $(h, b) = (h, bc^{n-1}cb^n(c^nb^n)^{n-1}) \in \Phi$. Therefore $(a, b) \in \Psi . \Phi$.

Subcase 2.8. $a = fae$ and $b = ebf$. In an analogous manner it can be proved that $(a, b) \in \Psi . \Phi$.

We have proved that $\Phi . \Psi \subseteq \Psi . \Phi$. Analogously we can show that $\Psi . \Phi \subseteq \Phi . \Psi$ and so $S$ is a permutable semigroup.

Note 1. By a regular $*$-semigroup we shall mean (see [4]) an algebra $(S, \cdot, *)$ where $(S, \cdot)$ is a semigroup and $*$ is a unary operation on $S$ satisfying

$$(x^*)^* = x\,, \quad x = xx^*x \quad \text{and} \quad (xy)^* = y^*x^*\,.$$

By $W^*(i = j)$ we denote the variety of all regular $*$-semigroups satisfying the identity $i = j$. It follows from Theorem 1 of [5] and Theorem 1 of [6] that a variety $\mathscr{V}$ of regular $*$-semigroups is permutable if and only if $\mathscr{V} \subseteq W^*(xx^* = yy^*)$.

Now we shall show

**Theorem 4.** *The following conditions for a variety $\mathscr{V}$ of regular $*$-semigroups are equivalent*:

1. *$\mathscr{V}$ is permutable.*
2. *Each $S \in \mathscr{V}$ with two generators is permutable.*
3. *$\mathscr{V} \subseteq W^*(xx^* = yy^*)$.*

Proof. $1 \Rightarrow 2$. Evident.

$2 \Rightarrow 3$. Suppose that every regular $*$-semigroup with two generators from $\mathscr{V}$ is permutable. According to Lemma 4 of [5] it is sufficient to show that $S_2, S_4 \notin \mathscr{V}$, where $S_2$ is a two-element regular $*$-semigroup with the tables

| $\cdot$ | 1 | 0 |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 0 | 0 |

| $*$ | |
|---|---|
| 1 | 1 |
| 0 | 0 |

and $S_4$ is a four-element regular $*$-semigroup with the tables

| $\cdot$ | $e$ | $f$ | $ef$ | $fe$ |
|---|---|---|---|---|
| $e$ | $e$ | $ef$ | $ef$ | $e$ |
| $f$ | $fe$ | $f$ | $f$ | $fe$ |
| $ef$ | $e$ | $ef$ | $ef$ | $e$ |
| $fe$ | $fe$ | $f$ | $f$ | $fe$ |

| $*$ | |
|---|---|
| $e$ | $f$ |
| $f$ | $e$ |
| $ef$ | $ef$ |
| $fe$ | $fe$ |

By $\mathscr{T}$ we denote the variety of all semilattices with $* = \mathrm{id}$. It is easy to show that $\mathscr{T}$ is minimal in the lattice of all regular $*$-semigroup varieties. According to Example 2 of [1] we have $\mathscr{T} \cap \mathscr{V} = W^*(x = y)$. Evidently $S_2 \in \mathscr{T}$ and so $S_2 \notin \mathscr{V}$.

It is well known (see [7] and [8]) that an algebra $A$ has its congruence lattice $\mathrm{Con}(A)$ modular whenever $A$ is permutable. In the proof of Theorem 5 of [5] it is proved that the lattice $\mathrm{Con}(S_4 \times S_4)$ is not modular. Therefore the regular $*$-semigroup $S_4 \times S_4$ is not permutable. It is easy to show that $S_4 \times S_4$ is generated by $(e, e)$ and $(e, f)$. Consequently $S_4 \times S_4 \notin \mathscr{V}$ and so $S_4 \notin \mathscr{V}$.

$3 \Rightarrow 1$. See Note 1.

Note 2. The following problem remains open:

describe all varieties of regular $*$-semigroups in which each semigroup with one generator is permutable.

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## Souhrn


### O PERMUTABILITĚ VE VARIETÁCH POLOGRUP

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V práci jsou charakterizovány variety pologrup, v nichž jsou permutabilní pologrupy generované jedním resp. dvěma prvky. Zde se též popisují všechny variety regulárních $*$-pologrup, jejichž pologrupy generované dvěma prvky jsou permutabilní.



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