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ON DIRECTED GROUPS WITH ADDITIONAL OPERATIONS

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Summary. This paper deals with a question concerning d-ideals of d-groups which was proposed by V. M. Kopytov and Z. J. Dimitrov. We shall also investigate a class of d-groups which is in a certain sense near to the class of all lattice ordered groups.

Keywords: directed group, d-group, d-ideal, torsion class, radical class

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In this paper the notion of a d-group will be applied in the sense introduced in [6]. It is defined to be a directed group with two additional binary operations $\wedge$ and $\vee$ satisfying certain conditions. For elements x and y in a d-group G the element $x \wedge y$ need not be the infimum of the set $\{x, y\}$; similarly, $x \vee y$ is not, in general, the supremum of $\{x, y\}$.

Lattice ordered groups (where $\wedge$ and $\vee$ have the usual meaning) are particular cases of d-groups.

Homomorphisms of d-groups are defined in a natural way. A subset of a d-group which is the kernel of a homomorphism is called a d-ideal.

Let us consider the following conditions for a d-group G:

($*$) Each convex normal d-subgroup of G is a d-ideal.

($**$) Whenever x and y are elements of G such that $\sup\{x, y\}$ exists in G, then $x \vee y = \sup\{x, y\}$.

It is well-known that the condition analogous to ($*$) holds for lattice ordered groups. In [6] it was remarked that the question whether each d-group satisfies the condition ($*$) has been left open.

In the present paper it will be shown that the answer to this question is 'No'. In fact, we shall describe a rather large class $\mathscr{D}_1$ of d-groups which do not satisfy the condition ($*$).

Let $\mathscr{D}$ and $\mathscr{L}$ be the class of all d-groups and the class of all lattice ordered groups, respectively.

The condition (**) was dealt with in [6]; let us denote by $\mathscr{D}_0$ the class of all d-groups satisfying the condition (**).

We remark that lattice ordered groups can be characterized as d-groups G which satisfy the following condition:

($**_1$) Whenever x and y are elements of G, then $\sup\{x,y\}$ exists in G and $x \vee y = \sup\{x,y\}$.

In [6] it was proved that $\mathscr{L}$ is a subvariety of $\mathscr{D}$. The natural question arises whether $\mathscr{D}_0$ is a subvariety of $\mathscr{D}$. It will be proved that the answer is negative; moreover, it will be shown that $\mathscr{D}_0$ is neither a torsion class nor a radical class. (For definitions of these notions cf. Section 3 below.)

It will be shown that if X is a subvariety or $\mathscr{D}$ with $\mathscr{D}_0 \subseteq X$, then $X = \mathscr{D}$. In fact, we shall show that for each $G \in \mathscr{D}$ there exists H in $\mathscr{D}_0$ such that G is isomorphic to a d-subgroup of H.

Holland [2] proved that each variety of lattice ordered groups is a torsion class. We shall prove below that this result cannot be generalized to the case of d-groups.

1. Preliminaries

For the basic notions on ordered groups cf. [1] and [5]. In the present paper the group operation will be written additively.

By applying the conditions from [6] we recall the definition of a d-group.

Let G be a directed group. Assume that two binary commutative operations $\wedge$ and $\vee$ are defined on G. Consider the following conditions for these operations:

(a) If $x, y \in G$ and $y \leqslant y$, then $x \wedge y = x$ and $x \vee y = y$.

(b) If x_1 and x_2 are incomparable elements of G, then $x_1 \wedge x_2 < x_i < x_1 \vee x_2$ for $i = 1, 2$.

(H6) $(z + x + t) \vee (z + (x \vee y) + t) = z + (x \vee y) + t$ holds identically; and dually.

(H7) $-(x \vee y) = (-x) \wedge (-y)$ holds identically; and dually.

(D1) $x + (y \vee z) + t = (x + y + t) \vee (x + z + t)$ holds identically; and dually.

1.1. Definition. A directed group with two additional binary commutative operations $\wedge$ and $\vee$ satisfying the conditions (a), (b), (H6), (H7) and (D1) is called a d-group.

Another way of defining a d-group is as follows. Let G be a set with two binary operations $\wedge$ and $\vee$. Let us consider the following conditions for these operations.

(H1) $x \vee x = x, \quad x \wedge x = x.$

(H2) $x \vee y = y \vee x, \quad x \wedge y = y \wedge x$.

(H3) $x \vee (x \vee y) = x \vee y, \quad x \wedge (x \wedge y) = x \wedge y$.

(H4) $(x \vee y) \vee z = x \vee ((x \vee y) \vee z), \quad (x \wedge y) \wedge z = x \wedge ((x \wedge y) \wedge z)$.

(H5) $x \vee (x \wedge y) = x, \quad x \wedge (x \vee y) = x$.

1.2. Definition. Let G be a group and let $\wedge$, $\vee$ be binary operations on G. Suppose that the conditions (H1)–(H7) and (D1) are satisfied. Then the structure $(G; +, \wedge, \vee)$ is called a d-group.

The equivalence of the two definitions of a d-group formulated above is a consequence of [6], Theorem 2.2. (If we consider Definition 1.2 as basic, then we put $x \leqslant y$ if and only if $x \vee y = y$.)

In Definition 1.1 the condition (H6) can be cancelled (in fact, this condition follows from the conditions (a) and (b)).

Let us also remark that adopting Definition 1.2 we can view the class of all d-groups as a variety. This variety will be denoted by $\mathscr{D}$.

The meaning of $\mathscr{D}_0$ was defined above. All examples of d-groups which have been given in [6] belong to the class $\mathscr{D}_0$.

A subgroup H of a d-group G is called a d-subgroup if it is closed with respect to operations $\wedge$ and $\vee$.

Let us recall that the definition of a d-ideal of a d-group as given in [6] formally differs from that given above; [6], Section 5.2 yields:

1.3. Lemma. *A convex normal d-subgroup H of a d-group is a d-ideal of G if and only if it satisfies the following condition: $(***)$ whenever $x \in G$ and $h \in H$, then $-(x \vee 0) + ((x + h) \vee 0) \in H$.*

(In [6], the notion of a d-ideal was defined by means of the condition $(***)$.)

It is easy to verify that a d-group G belongs to $\mathscr{L}$ if and only if it satisfies the condition $(**_1)$ formulated above.

2. The class $\mathscr{D}_1$

For a partially ordered group A we denote by A^0 the partially ordered group which has the same underlying set as A, the same group operation and the trivial partial order (i.e., $a_1 \leqslant a_2$ iff $a_1 = a_2$).

The symbol $\circ$ stands for the operation of the lexicographic product of partially ordered groups.

Let A and B be nonzero abelian linearly ordered groups. Assume that A is divisible. Put $C = B \circ A^0$. The element $c = (0, a)$ of C will be denoted simply by a;

similarly, if $b \in B$, then we shall identify b with $(b, 0)$. Next, let f be a mapping of the set A^+ into C such that $f(0) = 0$ and $f(a) > 0$ whenever $a > 0$.

We define successively a binary operation $\vee$ on C as follows.

Let $a \in A$. We put $0 \vee a = f(a) + \frac{1}{2}a$ if $a \geqslant 0$, and $0 \vee a = \frac{1}{2}a + f(-a)$ otherwise. If $c = (b, a) \in C$ with $b \neq 0$, then we set $0 \vee c = c$ whenever $b > 0$, and $0 \vee c = 0$ otherwise.

Next, let $c_i \in C$, $i = 1, 2$. Suppose that $c_1 \neq 0$. In this case we put $c_1 \vee c_2 = c_1 + (0 \vee c_2 - c_1)$.

It is easy to verify that the operation $\vee$ on C is commutative and satisfies those parts of the conditions (a), (b) and (D1) which concern $\vee$.

Now we define the operation $\wedge$ on G by putting $c_1 \wedge c_2 = -((-c_1) \vee (-c_2))$ for each c_1, $c_2 \in C$. Then (H7) holds. Standard calculations show that the operation $\wedge$ is commutative and that also those parts of (a), (b) and (D1) which concern the operation $\wedge$ are valid. Denote $G = (C, +, \leqslant, \wedge, \vee)$. We have the following result.

2.1. Lemma. *Let G be constructed as above. Then G is a d-group.*

Let R be the additive group of all reals with the natural linear order and let D be an abelian linearly ordered divisible group. Put $A = D \circ R \circ R$, $B = A$ and $C = B \circ A^0$.

We define a mapping f of A^+ into C as follows. We set $f(0) = 0$. Let $0 \neq a = (d, r_1, r_2)$. We put $f(a) = (b_1, a_1)$, where $a_1 = 0$ and

$$b_1 = \begin{cases} a & \text{if } \ d > 0; \\ (0, r_1 + r_2, 0) & \text{if } \ d = 0 \quad \text{and} \quad r_1 > 0\,; \\ (0, 0, r_2) & \text{if } \ d = r_1 = 0 \quad \text{and} \quad r_2 \geqslant 0. \end{cases}$$

Now let G be as in 2.1. We denote by $\mathscr{D}_1$ the class of all d-groups G which can be constructed in the described way, where D runs over the class of all nonzero abelian divisible linearly ordered groups. For each cardinal k there exists D with the properties as above such that $\operatorname{card} D > k$. Therefore $\mathscr{D}_1$ is a proper class.

2.2. Theorem. *Let $G \in \mathscr{D}_1$. Then G fails to satisfy the condition* $(*)$.

Proof. Let H be the set of all $g = (b, a)$ with $b = (d, r_1, r_2)$, $a = (d', r_1', r_2')$, such that $d = d' = r_1 = r_1' = 0$. Then H is a convex normal d-subgroup of G. It suffices to verify that H fails to be a d-ideal. □

By way of contradiction, suppose that H is a d-ideal. Then H satisfies the condition $(***)$. Put $h = (b_1, a_1)$, $x = (b_2, a_2)$, where $b_1 = b_2 = 0$, $a_1 = (0, 0, 2)$ and

$a_2 = (0, 2, 0)$. We obtain that

$$0 \vee x = ((0,2,0),(0,1,0))$$
$$0 \vee (x + h) = ((0,4,0),(0,1,1)).$$

Hence

$$-(0 \vee x) + (0 \vee (x + h)) = ((0,2,0),(0,0,1)).$$

This element does not belong to H, which is a contradiction.

Let us remark that $\mathscr{D}_1$ is a subclass of $\mathscr{D}_0$.

3. The class $\mathscr{D}_0$

As usual, a nonempty subclass of $\mathscr{D}$ is called a variety in $\mathscr{D}$ if it is closed with respect to homomorphisms, d-subgroups and direct products. Theorem 4.3 in [6] implies that the class $\mathscr{L}$ of all lattice ordered groups is a variety in $\mathscr{D}$.

For a d-group G we denote by $c(G)$ the system of all convex d-subgroups of G; this system is partially ordered by inclusion. It is easy to verify that $c(G)$ is a complete lattice. The lattice operations in $c(G)$ will be denoted by $\wedge^c$ and $\vee^c$. If $\{G\}_{i \in I}$ is a nonempty subset of $c(G)$, then $\bigwedge\limits_{i \in I}^{c} G_i = \bigcap\limits_{i \in I} G_i$.

A subclass $\mathscr{D}'$ of $\mathscr{D}$ will be said to be closed with respect to joins of convex d-subgroups if, whenever $G \in \mathscr{D}$ and $\emptyset \neq \{G_i\}_{i \in I} \subseteq c(G) \cap \mathscr{D}'$, then $\bigvee\limits_{i \in I}^{c} G_i$ belongs to $\mathscr{D}'$.

For subclasses of $\mathscr{D}$ we can introduce also the notions of a torsion class and of a radical class analogously as in the case of lattice ordered groups (cf. [3] and [7]). Namely, a nonempty subclass of $\mathscr{D}$ will be called a radical class if it is closed with respect to isomorphisms, convex d-subgroups and joins of convex d-subgroups; if, moreover, it is closed also with respect to homomorphisms, then it will be called a torsion class. Radical classes of directed groups have been investigated in [4].

In this section we shall investigate the question whether the above mentioned conditions are satisfied by $\mathscr{D}_0$.

It is obvious that $\mathscr{D}_0$ is closed with respect to isomorphisms and with respect to direct products.

3.1. Lemma. *Let $G \in \mathscr{D}$. There exists H in $\mathscr{D}_0$ such that G is isomorphic to a d-subgroup of H.*

P r o o f. If G is linearly ordered, then obviously $G \in \mathscr{D}_0$. Assume that G fails to be linearly ordered. Let R be as above. Consider the lexicographic product $G \circ R$ of directed groups G and R. Let $h_1 = (g_i, r_i)$ $(i = 1, 2)$ be elements of $G \circ R$.

If these elements are comparable, then we set $h_1 \vee^0 h_2 = \sup\{h_1, h_2\}$ and $h_1 \wedge^0 h_2 = \inf\{h_1, h_2\}$.

Next assume that h_1 and h_2 are incomparable. We put

$$h_1 \vee^0 h_2 = (g_1 \vee g_2, h_1 \vee h_2), \qquad h_1 \wedge^0 h_2 = (g_1 \wedge g_2, h_1 \wedge h_2).$$

The directed group $G \circ R$ with the operations $\vee^0$ and $\wedge^0$ will be denoted by H. In the terminology of [6], H is a lexicographic product of the d-group G and the d-group R. Then H is a d-group (cf. [6], Section 5.3). Put $G' = \{(g, r) \in H : r = 0\}$. It is obvious that G' is a d-subgroup of H which is isomorphic to G. If h_1 and h_2 are incomparable elements of H, then $\sup\{h_1, h_2\}$ does not exist in H. Hence H belongs to $\mathscr{D}_0$. □

3.2. Corollary. *Let X be a subvariety of $\mathscr{D}$ such that $\mathscr{D}_0 \subseteq X$. Then $X = \mathscr{D}$.*

The fact that $\mathscr{D}_0 \neq \mathscr{D}$ yields

3.3. Corollary. *The collection $\mathscr{D}_0$ fails to be a subvariety of $\mathscr{D}$.*

3.4. *The collection $\mathscr{D}_0$ fails to be closed with respect to homomorphism.*

Example. There exists a d-group G which does not belong to $\mathscr{D}_0$. By applying this G, we construct H as in the proof of 3.1. Then $H \in \mathscr{D}_0$. For $h = (g, r)$ in H we put $\varphi(h) = g$. Then φ is a homomorphism of the d-group H onto the d-group G.

3.5. Corollary. *The collection $\mathscr{D}_0$ fails to be a torsion class of d-groups.*

3.6. *The collection $\mathscr{D}_0$ fails to be closed with respect to joins of convex d-subgroups.*

Example. Let A and B be nonzero abelian linearly ordered groups. Consider the direct product $A \times B$; the lattice operations in $A \times B$ will be denoted by $\vee$ and $\wedge$. Next, for h in $A \times B$, the absolute $|h|$ is defined in the usual way.

Let $h_i = (a_i, b_i)$ $(i = 1, 2)$ be elements of $A \times B$. We put $h_1 \vee^0 h_2 = \sup\{h_1, h_2\}$ if h_1 and h_2 are comparable, and

$$h_1 \vee^0 h_2 = h_1 \vee h_2 + |h_1 - h_2|$$

otherwise. Further, we set $h_1 \wedge^0 h_2 = -((-h_1) \vee (-h_2))$.

Let H be the directed group $A \times B$ with the two additional operations $\vee^0$ and $\wedge^0$. Then H is a d-group. Whenever h_1 and h_2 are incomparable elements of H, the relation $h_1 \vee^0 h = \sup\{h_1, h_2\}$ fails to be valid. Hence H does not belong to $\mathscr{D}_0$.

Put $A' = \{(a,b) \in H : b = 0\}$ and $B' = \{(a,b) \in H : a = 0\}$. Then A' and B' are convex d-subgroups of H which belong to $\mathcal{D}_0$ and $A' \vee^c B' = H$.

3.7. Corollary. *The collection $\mathcal{D}_0$ fails to be a radical class of d-groups.*

3.8. *A variety of d-groups need not be a torsion class of d-groups.*

Example. Let $\mathcal{L}$ be as above; then $\mathcal{L}$ is a variety of d-groups. Let A', B' and H be as in the example of 3.6. Then A' and B' are elements of $\mathcal{L}$, but H does not belong to $\mathcal{L}$. Since $A' \vee^c B' = H$, we infer that $\mathcal{L}$ is not closed with respect to joins of convex d-subgroups. Thus $\mathcal{L}$ is not a torsion class of d-groups.

The following question remains open: is $\mathcal{D}_0$ closed with respect to convex d-subgroups?

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