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A MATCHING AND A HAMILTONIAN CYCLE OF THE FOURTH POWER OF A CONNECTED GRAPH

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Summary. The following result is proved: Let G be a connected graph of order $\geqslant 4$. Then for every matching M in G^4 there exists a hamiltonian cycle C of G^4 such that $E(C) \cap M = \emptyset$.

Keywords: power of a graph, matching, hamiltonian cycle

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Let G be a graph (in the sense of the book [1], for example) with a vertex set $V(G)$ and an edge set $E(G)$; note that the number $|V(G)|$ is referred to as the order of G. If n is a positive integer, then by the n-th power G^n of G we mean the graph G' such that $V(G') = V(G)$ and vertices u and v are adjacent in G' if and only if $1 \leqslant d_G(u, v) \leqslant n$, where d_G denotes the distance in G.

Chartrand, Polimeni and Stewart [2] and Sumner [6] have proved that if G is a connected graph of an even order, then G^2 has a 1-factor. As follows from Sekanina's paper [5], if G is a connected graph of order $\geqslant 3$, then G^3 has a hamiltonian cycle. The existence of 1-factors and/or a hamiltonian cycle of the fourth power of a connected graph was investigated in [3], [7], [4] and [8].

Let G be a connected graph of an even order $\geqslant 4$. The present author [3] proved that G^4 has a 3-factor each component of which is K_4 or $K_2 \times K_3$, where $\times$ denotes the cartesian product of graphs. Consequently, G^4 has tree mutually edge-disjoint 1-factors. Wisztová [7] proved that there exist a hamiltonian cycle C of G^3 and a 1-factor F of G^4 such that $E(F) \cap E(C) = \emptyset$. This result was improved by the present author [4] as follows: for any factor H of G^3 such that H contains no triangle and the maximum degree of H does not exceed 2, there exists a 1-factor F of G^4 such that $E(F) \cap E(H) = \emptyset$. Consequently, for every hamiltonian cycle C of G^3 there exists a 1-factor F of G^4 such that $E(F) \cap E(C) = \emptyset$.

Recently, Wisztová [8] has proved that if G is a connected graph of an order $\geqslant 4$ and M is a matching in G, then there exists a hamiltonian cycle C of G^4 such that $E(C) \cap M = \emptyset$. In the present paper the result obtained in [8] will be improved as follows: if G is a connected graph of an order $\geqslant 4$ and M is a matching in G^4, then there exists a hamiltonian cycle C of G^4 such that $E(C) \cap M = \emptyset$.

Before proving the main result of the paper we shall introduce some auxiliary notions and prove three lemmas.

If F_1 and F_2 are graphs, then we denote by $F_1 \cup F_2$ the graph F' with $V(F') = V(F_1) \cup V(F_2)$ and $E(F') = E(F_1) \cup E(F_2)$. If F is a graph and u and v are distinct vertices, then we denote by $F + uv$ the graph F'' with $V(F'') = V(F) \cup \{u, v\}$ and $E(F'') = E(F) \cup \{uv\}$. If H is a graph and W is a nonempty subset of $V(H)$, then we denote by $\langle W \rangle_H$ the subgraph of H induced by W.

An ordered pair (T, v), where T is a tree and $v \in V(T)$ will be referred to as a rooted tree. We say that rooted trees (T_1, v_1) and (T_2, v_2) are isomorphic if there exists an isomorphism f of T_1 onto T_2 such that $f(v_1) = v_2$.

Now, let $k \geqslant 1$ and $m \geqslant 1$ be integers, and let $u_0, \ldots, u_k, w_1, \ldots, w_m$ be mutually distinct vertices. We shall generalize some constructions used in [8]. By a Y_m-tree ($m \geqslant 5$) we mean a tree T such that

$$\begin{aligned}
&V(T) = \{w_1, \ldots, w_m\},\\
&\{w_j w_{j+1}\,;\ 1 \leqslant j \leqslant m-2\} \subseteq E(T), \text{ and}\\
&\text{either } w_{m-2}w_m \in E(T) \text{ or } w_{m-1}w_m \in E(T).
\end{aligned}$$

By a Y^*_m-tree ($m \geqslant 5$) we mean a tree isomorphic to a Y_m-tree. By an X_m-tree ($m \geqslant 5$) we mean a tree T' such that

$$\begin{aligned}
&V(T') = \{w_1, \ldots, w_m\},\\
&\{w_j w_{j+1}\,;\ 2 \leqslant j \leqslant m-2\} \subseteq E(T'),\\
&\text{either } w_1w_2 \in E(T') \text{ or } w_1w_3 \in E(T'), \text{ and}\\
&\text{either } w_{m-2}w_m \in E(T') \text{ or } w_{m-1}w_m \in E(T').
\end{aligned}$$

By an X^*_m-tree ($m \geqslant 5$) we mean a tree isomorphic to X_m-tree. By a $U_{k,m}$-tree we mean a rooted tree (T'', u_0) such that

$$\begin{aligned}
&V(T'') = \{u_k, \ldots, u_0, w_1, \ldots, w_m\},\\
&\{u_{i+1}u_i\,;\ 1 \leqslant i \leqslant k-2\} \cup \{u_1u_0,\ u_0w_1\} \cup \{w_jw_{j+1}\,;\ 1 \leqslant j \leqslant m-2\} \subseteq E(T'');\\
&\text{if } k = 2, \text{ then } u_2u_1 \in E(T''),\\
&\text{if } k \geqslant 3, \text{ then either } u_ku_{k-1} \in E(T'') \text{ or } u_ku_{k-2} \in E(T''),\\
&\text{if } m = 2, \text{ then } w_1w_2 \in E(T''), \text{ and}\\
&\text{if } m \geqslant 3, \text{ then either } w_{m-2}w_m \in E(T'') \text{ or } w_{m-1}w_m \in E(T'').
\end{aligned}$$

Finally, by a $U^*_{k,m}$-tree we mean a rooted tree isomorphic to $U_{k,m}$.

Lemma 1. *Let $m \geqslant 5$ be an integer, let T be a Y_m-tree, and let M be a matching in T^3. Then there exists a hamiltonian $w_1 - w_2$ path P of T^3 such that $E(P) \cap M = \emptyset$.*

Proof. We shall construct a hamiltonian $w_1 - w_2$ path P of T^3 such that $E(P) \cap M = \emptyset$.

First, let $m = 5$. We put

$$E(P) = \{w_1w_3,\ w_3w_4\ w_4w_5,\ w_5w_2\} \text{ if } w_3w_5 \in M,$$
$$E(P) = \{w_1w_4,\ w_4w_3\ w_3w_5,\ w_5w_2\} \text{ if } w_4w_5 \in M,$$
$$E(P) = \{w_1w_3,\ w_3w_5\ w_5w_4,\ w_4w_2\} \text{ if } (w_3w_5,\ w_4w_5 \notin M,\ w_2w_3 \in M) \text{ or } (w_2w_3,\ w_3w_5,\ w_4w_5 \notin M,\ w_1w_4 \in M), \text{ and}$$
$$E(P) = \{w_1w_4,\ w_4w_5\ w_5w_3,\ w_3w_2\} \text{ if } w_1w_4,\ w_2w_3,\ w_3w_5,\ w_4w_5 \notin M.$$

Now let $m = 6$. We put

$$E(P) = \{w_1w_3,\ w_3w_6,\ w_6w_5,\ w_5w_4,\ w_4w_2\} \text{ if } w_2w_3,\ w_4w_6 \in M,$$
$$E(P) = \{w_1w_4,\ w_4w_5,\ w_5w_6,\ w_6w_3,\ w_3, w_2\} \text{ if } w_2w_3 \notin M,\ w_4w_6 \in M,$$
$$E(P) = \{w_1w_3,\ w_3w_6,\ w_6w_4,\ w_4w_5,\ w_5w_2\} \text{ if } (w_2w_3 \in M,\ w_4w_6 \notin M,\ w_5w_6 \in M) \text{ or } (w_2w_3,\ w_4w_6 \notin M,\ w_1w_4,\ w_3w_5 \in M) \text{ or } (w_2w_3,\ w_4w_6 \notin M,\ w_1w_4 \in M,\ w_3w_5 \notin M,\ w_5w_6 \in M),$$
$$E(P) = \{w_1w_3,\ w_3w_4,\ w_4w_6,\ w_6w_5,\ w_5w_2\} \text{ if } (w_2w_3 \in M,\ w_4w_6 \notin M,\ w_5w_6 \notin M,\ w_1w_4 \in M) \text{ or } (w_2w_3,\ w_4w_6 \notin M,\ w_1w_4 \notin M,\ w_3w_5 \in M),$$
$$E(P) = \{w_1w_4,\ w_4w_3,\ w_3w_6,\ w_6w_5,\ w_5w_2\} \text{ if } w_2w_3 \in M,\ w_4w_6 \notin M,\ w_5w_6,\ w_1w_4 \notin M,$$
$$E(P) = \{w_1w_3,\ w_3w_5,\ w_5w_6,\ w_6w_4,\ w_4w_2\} \text{ if } w_2w_3,\ w_4w_6 \notin M,\ w_1w_4 \in M,\ w_3w_5,\ w_5w_6 \notin M,$$
$$E(P) = \{w_1w_4,\ w_4w_6,\ w_6w_3,\ w_3w_5,\ w_5w_2\} \text{ if } w_2w_3,\ w_4w_6 \notin M,\ w_1w_4,\ w_3w_5 \notin M,\ w_5w_6 \in M, \text{ and}$$
$$E(P) = \{w_1w_4,\ w_4w_6,\ w_6w_5,\ w_5w_3,\ w_3w_2\} \text{ if } w_2w_3,\ w_4w_6 \notin M,\ w_1w_4,\ w_3w_5,\ w_5w_6 \notin M.$$

Finally, let $m \geqslant 7$. We assume that for $m-2$ the statement of the lemma is proved. Denote $T_0 = T - w_1 - w_2$ and $M_0 = M \cap E((T_0)^3)$. According to our assumption, there exists a hamiltonian $w_3 - w_4$ path P_0 of $(T_0)^3$ such that $E(P_0) \cap M_0 = \emptyset$. We put

$$P + P_0 + w_1w_4 + w_2w_3 \quad \text{if } w_1w_3 \in M \text{ or } w_2w_4 \in M, \text{ and}$$
$$P + P_0 + w_1w_3 + w_2w_4 \quad \text{if } w_1w_3,\ w_2w_4 \notin M.$$

Thus, the proof of the lemma is complete. □

As immediately follows from Lemma 1, if $m \geqslant 5$ is an integer, T is a Y_m-tree, and M is a matching in T^4, then there exists a hamiltonian $w_1 - w_2$ path P of T^4 such that $E(P) \cap M = \emptyset$.

In the proof of the next lemma an idea from the proof of Lemma 3 in [8] will be used.

Lemma 2. *Let $m \geqslant 5$ be an integer, let T be an X_m-tree, and let M be a matching in T^4. Then there exists a hamiltonian cycle C of T^4 such that $E(C) \cap M = \emptyset$.*

P r o o f. Obviously, if $m = 5$ then $T^4 = K_5$, and if $m = 6$ then $T^4 = K_6 - e$ or K_6. Thus, we can see that if $m = 5$ or 6, the statement of the lemma holds.

Let $m \geqslant 7$. Denote $T_0 = T - w_1 - w_2$. Clearly, T_0 is a Y^*_{m-2}-tree. According to Lemma 1, there exists a hamiltonian $w_3 - w_4$ path P_0 of $(T_0)^3$ such that $E(P_0) \cap m = \emptyset$.

First, let $w_1w_2 \in M$. Obviously, there exists $w \in V(T_0 - w_3)$ such that $w_3w \in E(P_0)$. We put

$$C = P_0 - ww_3 + ww_2 + w_2w_3 + w_3w_1 + w_1w_4.$$

Now let $w_1w_2 \notin M$. We put

$$C = P_0 + w_3w_1 + w_1w_2 + w_2w_4 \quad \text{if } w_1w_4 \in M \text{ or } w_2w_3 \in M, \text{ and}$$
$$C = P_0 + w_3w_2 + w_2w_1 + w_1w_4 \quad \text{if } w_1w_4,\ w_2w_3 \notin M.$$

We can see that C is a hamiltonian cycle of T^4 such that $E(C) \cap M = \emptyset$. Thus, the proof of the lemma is complete. □

Lemma 3. *Let T be a tree of an order $n \geqslant 4$, and let M be a matching in T^4. Then there exists a hamiltonian cycle C of T^4 such that $E(C) \cap M = \emptyset$.*

P r o o f. We proceed by induction on n. If the diameter of T does not exceed four, then T^4 is a complete graph and thus the statement of the lemma holds. If T is an X^*_n-tree, then—according to Lemma 2—the statement of the lemma holds, too. We shall assume that the diameter of T is at least five and T is not a X^*_n-tree. This implies that $n \geqslant 7$. We distinguish the following cases and subcases:

1. Assume that there exist mutually distinct vertices v, v_1, v_2, v_3 such that vv_1, vv_2, $vv_3 \in E(T)$ and v_1, v_2 and v_3 are vertices of degree one in T. Obviously, there

exist distinct $g, h \in \{1, 2, 3\}$ such that $v_g v_h \notin M$. Without loss of generality, let $v_2v_3 \notin M$. Denote $T_0 = T - v_2 - v_3$. Since $|V(T_0)| = n - 2 \geqslant 5$, it follows from the induction hypothesis that there exists a hamiltonian cycle C_0 of $(T_0)^4$ such that $E(C_0) \cap (M - \{vv_2, vv_3\}) = \emptyset$. Since v_1 is a vertex of degree one in T_0, there exists $v_0 \in V(T_0 - v_1)$ such that $v_0v_1 \in E(C_0)$ and $d_T(v, v_0) \leqslant 3$. We put

$$C = C_0 - v_0v_1 + v_0v_2 + v_2v_3 + v_3v_1 \quad \text{if } v_1v_2 \in M \text{ or } v_0v_3 \in M,$$
$$C = C_0 - v_0v_1 + v_0v_3 + v_3v_2 + v_2v_1 \quad \text{if } v_1v_2,\ v_0v_3 \notin M.$$

Obviously, C is a hamiltonian cycle of T^4 and $E(C) \cap M = \emptyset$.

2. Assume that for every vertex v of T, at most two vertices adjacent to v have degree one. It is not difficult to see that there exist positive integers k and m, a vertex u of a degree$\geqslant 3$ in T and a subtree T' of T with the properties that $3 \leqslant k + m \leqslant n - 4$, $u \in V(T')$, the degree of u' in T' is equal to the degree of u' in T for each $u' \in V(T' - u)$, and (T', u) is a $U^*_{k,m}$-tree.

For the sake of simplicity we shall assume that (T', u) is a $U_{k,m}$-tree. Thus $u = u_0$ and $V(T_0) = \{u_k, \ldots, u_0, w_1, \ldots, w_m\}$. Without loss of generality we assume that

(1) $\quad k \geqslant 2$; if $m = 2$, then $k \leqslant 3$; if $m = 3$, then $k = 3$; if $m = 4$, then $k \leqslant 4$.

Denote $T_0 = T - w_1 - \ldots - w_m$ and $M_0 = M \cap E((T_0)^4)$. Since $5 \leqslant |V(T_0)| \leqslant n-1$, it follows from the induction hypothesis that there exists a hamiltonian cycle C_0 of $(T_0)^4$ such that $E(C_0) \cap M_0 = \emptyset$. We shall construct a hamiltonian cycle C of T^4 such that $E(C) \cap M = \emptyset$.

2.1. Let $m \neq 2, 3, 4$.

2.1.1. Assume that

(2) $\quad$ there exist mutually distinct $v_{11}, v_{12}, v_{21}, v_{22} \in V(T)$ such that $v_{i1}v_{i2} \in E(C_0)$, $d_T(u_0, v_{i1}) \leqslant d_T(u_0, v_{i2}) \leqslant 3$ and $d_T(u_0, v_{i1}) + d_T(u_0, v_{i2}) \leqslant 4$ for $i = 1$ and 2.

Without loss of generality we assume that $v_{12}w_1, v_{12}w_1 \notin M$.

2.1.1.1. Let $m = 1$. We put

$$C = C_0 - v_{11}v_{12} + v_{11}w_1 + w_1v_{12}.$$

2.1.1.2. Let $m \geqslant 5$. Obviously, $v_{11}w_2, v_{12}w_1 \in E(T^4)$ and if $d_T(v_{11}, w_2) = 4$, then $d_T(v_{12}, w_2) = 4$.

2.1.1.2.1. Assume that $v_{11}w_2 \notin M$ or $d_T(v_{11}, w_2) = 4$. According to Lemma 1 there exists a hamiltonian $w_1 - w_2$ path P of $\left(\langle\{w_1, \ldots, w_m\}\rangle_T\right)^4$. We put

$$C = (C_0 - v_{11}v_{12}) \cup P + v_{11}w_2 + w_1v_{12} \text{ if } v_{11}w_2 \notin M, \text{ and}$$
$$C = (C_0 - v_{11}v_{12}) \cup P + v_{11}w_1 + w_2v_{12} \text{ if } v_{11}w_2 \in M \text{ and } d_T(v_{11}, w_2) = 4.$$

2.1.1.2.2. Assume that $v_{11}w_2 \in M$ and $d_T(v_{11}, w_2) \leqslant 3$. Then $v_{11}w_3 \in E(T^4) - M$. Moreover, $w_1w_2,\ w_2w_3 \notin M$.

First, let $m = 5$. We put

$$C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_4 + w_4w_2 + w_2w_5 + w_5w_1 + w_1v_{12} \quad \text{if } w_4w_5 \in M,$$
$$C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_2 + w_2w_5 + w_5w_4 + w_4w_1 + w_1v_{12} \quad \text{if } w_4w_5 \notin M,\ w_1w_5 \in M, \text{ and}$$
$$C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_2 + w_2w_4 + w_4w_5 + w_5w_1 + w_1v_{12} \quad \text{if } w_4w_5,\ w_1w_5 \notin M.$$

Now let $m \geqslant 6$. According to Lemma 1 there exists a hamiltonian $w_2 - w_3$ path P' of $\left(\langle\{w_2, \ldots, w_m\}\rangle_T\right)^4$. We put

$$C = (C_0 - v_{11}v_{12}) \cup P' + v_{11}w_3 + w_2w_1 + w_1v_{12}.$$

2.1.2. Assume that (2) does not hold. According to (1), $k \geqslant 2$. It is not difficult to see that $k \geqslant 4$ and there exists $v \in V(T_0 - u_0 - \ldots - u_k)$ such that $d_T(u_0, v) \leqslant 3$ and $C_0 - u_1 - \ldots - u_k$ is an $u_0 - v$ hamiltonian path of $(T_0 - u_1 - \ldots - u_k)^4$. Moreover, we can see that if $k = 4$, then $u_0u_4 \in E(C_0)$ and therefore $u_0u_4 \notin M$.

2.1.2.1. Assume that $m = 1$.

2.1.2.1.1. Let $vw_1 \in M$. First, let $k = 4$. Recall that $u_0u_4 \notin M$. We put

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + u_0u_2 + u_2u_4 + u_4u_3 + u_3w_1 + w_1u_1 + u_1v \quad \text{if } u_2u_3 \in M,$$
$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + u_0u_4 + u_4u_2 + u_2u_3 + u_3w_1 + w_1u_1 + u_1v \quad \text{if } u_3u_4 \in M, \text{ and}$$
$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + u_0u_4 + u_4u_3 + u_3u_2 + u_2w_1 + w_1u_1 + u_1v \quad \text{if } u_2u_3,\ u_3u_4 \notin M.$$

Now let $k \geqslant 5$. As follows from Lemma 1, there exists a hamiltonian $u_1 - u_2$ path P of $\left(\langle\{u_1, \ldots, u_k\}\rangle_T\right)^4$. We put

$$C = (C_0 - u_1 - \ldots - u_k) \cup P + u_0w_1 + w_1u_2 + u_1v.$$

2.1.2.1.2. Let $vw_1 \notin M$. According to Lemma 1, there exists a hamiltonian $w_1 - u_0$ path P of $\left(\langle\{w_1,\ u_0, \ldots, u_k\}\rangle_T\right)^4$. We put

$$C = (C_0 - u_1 - \ldots - u_k) \cup P + w_1v.$$

2.1.2.2. Assume that $m \geqslant 5$.

2.1.2.2.1. Let $k = 4$. First, let $vw_1 \in M$ or $u_1w_2 \in M$. Then $vu_1 \notin M$. There exists a hamiltonian $u_0 - w_1$ path P of $\left(\langle\{u_0, w_1, \dots, w_m\}\rangle_T\right)^4$. Clearly, $u_1u_4 \notin M$ or $u_3w_1 \notin M$. We put

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + P + vu_1 + u_1u_3 + u_3u_4 + u_4u_2 + u_2w_1 \quad \text{if } u_2u_3 \in M,$$

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + P + vu_1 + u_1u_2 + u_2u_4 + u_4u_3 + u_3w_1 \quad \text{if } u_2u_3,\ u_3w_1 \notin M,\ u_1u_4 \in M,$$

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + P + vu_1 + u_1u_4 + u_4u_3 + u_3u_2 + u_2w_1 \quad \text{if } (u_2u_3,\ u_1u_4 \notin M,\ u_3w_1 \in M) \text{ or } (u_2u_3,\ u_1u_4,\ u_3w_1 \notin M,\ u_2u_4 \in M), \text{ and}$$

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) + P + vu_1 + u_1u_4 + u_4u_2 + u_2u_3 + u_3w_1 \quad \text{if } u_2u_3,\ u_1u_4,\ u_3w_1,\ u_2u_4 \notin M.$$

Now let vw_1, $u_1w_2 \notin M$. According to Lemma 1 there exist a hamiltonian $u_0 - u_1$ path P' of $\left(\langle\{u_0, \dots, u_4\}\rangle_T\right)^4$ and a hamiltonian $w_1 - w_2$ path P'' of $\left(\langle\{w_1, \dots, w_m\}\rangle_T\right)^4$. We put

$$C = (C_0 - u_1 - u_2 - u_3 - u_4) \cup P' \cup P'' + vw_1 + w_2u_1.$$

2.1.2.2.2. Let $k \geqslant 5$. According to Lemma 1 there exist hamiltonian $u_1 - u_2$ path P of $\left(\langle\{u_1, \dots, u_k\}\rangle_T\right)^4$ and a hamiltonian $w_1 - w_2$ path P' of $\left(\langle\{w_1, \dots, w_m\}\rangle_T\right)^4$. Obviously, $vw_1 \notin M$ or $vu_1 \notin M$. Without loss of generality we assume that $vw_1 \notin M$. We put

$$C = (C_0 - u_1 - \dots - u_k) \cup P \cup P' + u_0u_1 + u_2w_2 + w_1v \quad \text{if } u_0u_2 \in M \text{ or } u_1w_2 \in M, \text{ and}$$

$$C = (C_0 - u_1 - \dots - u_k) \cup P \cup P' + u_0u_2 + u_1w_2 + w_1v \quad \text{if } u_0u_2,\ u_1w_2 \notin M.$$

2.2. Let $m = 2$. According to (1), $k = 2$ or 3. It is easy to see that there exist u_1', $u_2' \in V(T_0)$ with the properties that $u_1' \neq u_1$, $u_2' \neq u_2$, u_1u_1', $u_2u_2' \in E(C_0)$, $u_1u_1' \neq u_2u_2'$, $d_T(u_0, u_1') \leqslant 3$ and $d_T(u_0, u_2') \leqslant 2$. We put

$$C = C_0 - u_1u_1' - u_2u_2' + u_1w_1 + w_1u_1' + u_2w_2 + w_2u_2' \quad \text{if } w_1w_2 \in M,$$

$$C = C_0 - u_2u_2' + u_2w_1 + w_1w_2 + w_2u_2' \text{ if } w_1w_2 \notin M \text{ and } (u_2'w_1 \in M \text{ or } w_2u_2 \in M), \text{ and}$$

$$C = C_0 - u_2u_2' + u_2'w_1 + w_1w_2 + w_2u_2 \quad \text{if } w_1w_2,\ u_2'w_1,\ w_2u_2 \notin M.$$

2.3. Let $m = 3$. According to (1), $k = 3$.

2.3.1. Assume that

(3) there exist $u_1' \in V(T_0 - u_1)$ such that $u_1u_1' \in E(C_0)$ and $d_T(u_0, u_1') \leqslant 2$.

We put

$$
\begin{aligned}
C &= C_0 - u_1u_1' + u_1w_3 + w_3w_2 + w_2w_1 + w_1u_1' && \text{if } w_1w_3 \in M,\\
C &= C_0 - u_1u_1' + u_1w_3 + w_3w_1 + w_1w_2 + w_2u_1' && \text{if } w_2w_3 \in M,\\
C &= C_0 - u_1u_1' + u_1w_1 + w_1w_3 + w_3w_2 + w_2u_1' && \text{if } w_1w_3,\ w_2w_3 \notin M,\\
& && \text{and } (u_1w_2 \in M \text{ or } w_1u_1' \in M), \text{ and}\\
C &= C_0 - u_1u_1' + u_1w_2 + w_2w_3 + w_3w_1 + w_1u_1' \\
& && \text{if } w_1w_3,\ w_2w_3,\ u_1w_2,\ w_1u_1' \notin M.
\end{aligned}
$$

2.3.2. Assume that (2) does not hold. Then there exist mutually distinct u_1', u_1'', $u_2' \in V(T_0 - u_1 - u_2)$ such that u_1u_1', u_1u_1'', $u_2u_2' \in E(C_0)$ and $d_T(u_0, u_2') \leqslant 2$. Clearly, $d_T(u_0, u_1') = 3 = d_T(u_0, u_1'')$. Obviously, $u_1'w_1 \notin M$ or $u_1''w_1 \notin M$. Without loss of generality we assume that $u_1'w_1 \notin M$. We put

$$
\begin{aligned}
C &= C_0 - u_1u_1' + u_1w_3 + w_3w_2 + w_2w_1 + w_1u_1' && \text{if } w_1w_3 \in M,\\
C &= C_0 - u_1u_1' - u_2u_2' + u_1w_3 + w_3w_1 + w_1u_1' + u_2w_2 + w_2u_2' && \text{if } w_2w_3 \in M,\\
C &= C_0 - u_2u_2' + u_2w_1 + w_1w_3 + w_3w_2 + w_2u_2' && \text{if } w_1w_3,\ w_2w_3 \notin M\\
& \quad \text{and } (u_2w_2 \in M \text{ or } u_2'w_1 \in M), \text{ and}\\
C &= C_0 - u_2u_2' + u_2w_2 + w_2w_3 + w_3w_1 + w_1u_2' && \text{if } w_1w_3,\ w_2w_3,\ u_2w_2,\ u_2'w_1 \notin M.
\end{aligned}
$$

2.4. Let $m = 4$. According to (1), $2 \leqslant k \leqslant 4$. Without loss of generality we assume that

(4) if $k = 4$ and $w_3w_4 \in M$, then $u_3u_4 \in M$.

2.4.1. Assume that

(5) there exist v_{11}, v_{12}, v_{21}, $v_{22} \in V(T_0)$ such that $v_{12} \neq v_{22}$, $v_{11} \neq v_{12} \neq v_{21}$, $v_{11} \neq v_{22} \neq v_{21}$, $v_{11}v_{12}$, $v_{21}v_{22} \in E(C_0)$, $d_T(u_0, v_{11}) \leqslant 1$, $d_T(u_0, v_{12}) \leqslant 3$, $d_T(u_0, v_{21}) \leqslant 1$ and $d_T(u_0, v_{22}) \leqslant 3$.

Obviously, $v_{12}w_1 \notin M$ or $v_{22}w_1 \notin M$. Without loss of generality we assume that $v_{12}w_1 \notin M$. We put

$$C = C_0 - v_{11}v_{12} + v_{11}w_2 + w_2w_3 + w_3w_4 + w_4w_1 + w_1v_{12}$$
$$\text{if } w_2w_4 \in M,$$
$$C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_2 + w_2w_4 + w_4w_1 + w_1v_{12}$$
$$\text{if } w_3w_4 \in M,$$
$$C = C_0 - v_{11}v_{12} + v_{11}w_3 + w_3w_4 + w_4w_2 + w_2w_1 + w_1v_{12}$$
$$\text{if } (w_2w_4,\ w_3w_4 \notin M,\ v_{11}w_2 \in M)$$
$$\text{or } (v_{11}w_2,\ w_2w_4,\ w_3w_4 \notin M,\ w_1w_3 \in M), \text{ and}$$
$$C = C_0 - v_{11}v_{12} + v_{11}w_2 + w_2w_4 + w_4w_3 + w_3w_1 + w_1v_{12}$$
$$\text{if } v_{11}w_2,\ w_1w_3,\ w_2w_4,\ w_3w_4 \notin M.$$

2.4.2. Assume that (5) does not hold. Then $k = 4$ and $u_1u_4 \in E(C_0)$ and $d_T(u_0, u_4) = 4$.

We first assume that $u_2u_3, u_2u_4 \in E(C_0)$. Then there exist $u_1', u_3' \in V(T_0 - u_1 - u_3)$ such that $u_1' \neq u_3'$, u_1u_1', $u_3u_3' \in E(C_0)$, $d_T(u_0, u_1') \leqslant 3$ and $d_T(u_0, u_3') \leqslant 1$, which contradicts (5).

Now we assume that $u_2u_3 \notin E(C_0)$ or $u_2u_4 \notin E(C_0)$. Then there exists $u_2' \in V(T_0 - u_2)$ such that $u_2u_2' \in E(C_0)$ and $d_T(u_0, u_2') \leqslant 2$.

2.4.2.1. Let $w_3w_4 \notin M$. Obviously, $u_2w_1 \notin M$ or $u_2'w_1 \notin M$. Without loss of generality we assume that $u_2w_1 \notin M$. We put

$$C = C_0 - u_2u_2' + u_2w_1 + w_1w_3 + w_3w_4 + w_4w_2 + w_2u_2'$$
$$\text{if } w_2w_3 \in M \text{ or } (w_1w_4 \in M,\ u_2'w_2 \notin M),$$
$$C = C_0 - u_2u_2' + u_2w_2 + w_2w_4 + w_4w_3 + w_3w_1 + w_1u_2'$$
$$\text{if } u_2'w_2,\ w_1w_4 \in M,$$
$$C = C_0 - u_2u_2' + u_2w_2 + w_2w_3 + w_3w_4 + w_4w_1 + w_1u_2'$$
$$\text{if } u_2'w_2 \in M,\ w_1w_4 \notin M, \text{ and}$$
$$C = C_0 - u_2u_2' + u_2w_1 + w_1w_4 + w_4w_3 + w_3w_2 + w_2u_2'$$
$$\text{if } u_2'w_2,\ w_1w_4,\ w_2w_3 \notin M.$$

2.4.2.2. Let $w_3w_4 \in M$. According to (4), $u_3u_4 \in M$. Therefore, $u_3u_4 \notin E(C_0)$. There exists $u_3'' \in V(T_0 - u_2 - u_3)$ such that $u_3u_3'' \in E(C_0)$. Since $u_3u_4 \notin E(C_0)$ and $d_T(u_0, u_3) = 3$, we have $d_T(u_0, u_3'') \leqslant 1$. We put $v_{11} = u_3''$ and $v_{12} = u_3$. Since $u_3u_4 \in M$, we have $v_{12}w_1 \notin M$. Thus we can construct C in the same way as in 2.4.1.

The proof of the lemma is complete. □

The following theorem is the main result of the present paper:

Theorem. *Let G be a connected graph of an order $\leqslant 4$. Then for every matching M in G^4 there exists a hamiltonian cycle C of G^4 such that $E(C) \cap M = \emptyset$.*

Proof. Consider an arbitrary spanning tree T of G. Denote $M_0 = M \cap E(T^4)$. Obviously, M_0 is a matching in T^4. According to Lemma 3, there exists a hamiltonian cycle C of T^4 such that $E(C) \cap M_0 = \emptyset$. Clearly, C is a hamiltonian cycle of G^4. Since $E(C) \subseteq E(T^4)$, we can see that $E(C) \cap M = \emptyset$, which completes the proof. □

As follows from [2] and [5], if G is a connected graph of an even order, then G^2 has a 1-factor. Combining this result with our theorem, we get the following corollary:

Corollary. *Let G be a connected graph of an even order $\geqslant 4$. Then there exist a 1-factor F of G^2 and hamiltonian cycle C of G^4 such that $E(C) \cap E(F) = \emptyset$.*

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Souhrn

PÁROVÁNÍ A HAMILTONOVSKÁ KRUŽNICE ČTVRTÉ MOCNINY SOUVISLÉHO GRAFU

LADISLAV NEBESKÝ

Nechť G je souvislý graf s alespoň čtyřmi uzly. V článku je dokázáno, že pro každé párování M v grafu G^4 existuje hamiltonovská kružnice grafu G^4, jejíž žádná hrana do M nepatří.

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