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A DESCRIPTIVE DEFINITION OF A BV INTEGRAL
IN THE REAL LINE

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Abstract. A descriptive characterization of a Riemann type integral, defined by BV partition of unity, is given and the result is used to prove a version of the controlled convergence theorem.

Keywords: pseudopartition, ACG° , WSL° -condition

MSC 1991: 26A39, 26A45

0. INTRODUCTION

J. Kurzweil, J. Mawhin and W. F. Pfeffer, to obtain an additive continuous integral for which a quite general formulation of Gauss-Green theorem holds, introduced in [5] a multidimensional integral (called \mathcal{I} -integral) defined via BV partitions of unity. In dimension one this integral falls properly in between the Lebesgue and Denjoy-Perron integrals and the integration by parts formula holds.

An integral satisfying quite the same properties but defined by using partitions with BV-sets or with *figures* (finite unions of intervals), was studied by W. F. Pfeffer in [7] and [9]. Descriptive characterizations for this integral are given in [3] and [8]. An application of the notion of absolute continuity given in [3] is contained in [1], where a version of the controlled convergence theorem for the one-dimensional Pfeffer-integral is proved.

It seems to us to be of interest to find a descriptive characterization even for the \mathcal{I} -integral. The aim of this paper is to solve this problem in the case of the

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one dimensional \mathcal{I} -integral and then to apply it to prove a controlled convergence theorem.

The main difficulty has been related to the fact that it is impossible to use the Saks-Henstock lemma, since it is not known whether it holds for the \mathcal{I} -integral. To solve our problem we have made use of a useful modification of the *Strong Lusin* condition introduced by P. Y. Lee in [6].

1. PRELIMINARES

The set of all real numbers is denoted by \mathbb{R} . If $E \subset \mathbb{R}$, then χ_E , $d(E)$, $\text{cl } E$ and $|E|$ denote the characteristic function, the diameter, the closure and the outer Lebesgue measure of E , respectively. Let $[a, b]$ be a fixed, non degenerate, compact interval of \mathbb{R} .

A *figure* of $[a, b]$ is a finite nonempty union of subintervals of $[a, b]$. A collection of figures is called *nonoverlapping* whenever the collection of their interiors is disjoint. The algebraic operations and convergence for functions on the same set are defined pointwise. The usual variation of a function ϑ over the interval $[a, b]$ is denoted $V(\vartheta, [a, b])$. Let θ be a function on \mathbb{R} , we set $S_\theta = \{x \in \mathbb{R} : \theta(x) \neq 0\}$. Given $\theta \in L^1(\mathbb{R})$ such that $S_\theta \subset (a, b)$ we set

$$\|\theta\| = \inf V(\vartheta, [a, b])$$

where the infimum is taken over all functions ϑ such that $S_\vartheta \subset (a, b)$ and $\vartheta = \theta$ almost everywhere with respect to the Lebesgue measure in \mathbb{R} (abbreviated as a.e.). The family of all nonnegative functions θ on $[a, b]$ for which θ and S_θ are bounded and $\|\theta\| < +\infty$ is denoted by $BV_+([a, b])$. The *regularity* of $\theta \in BV_+([a, b])$ at a point $x \in \mathbb{R}$ is the number

$$r(\theta, x) = \begin{cases} \frac{|\theta|_1}{d(S_\theta \cup \{x\})\|\theta\|} & \text{if } d(S_\theta \cup \{x\})\|\theta\| > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\theta|_1$ denotes the L^1 norm of θ . Let A be a figure of $[a, b]$, then the characteristic function χ_A of A belongs to $BV_+([a, b])$ and the symbols $\|A\| = \|\chi_A\|$ and $r(\chi_A, x) = r(\chi_A, x)$ coincide with those introduced in [2, Section 1].

A *partition* in $[a, b]$ is a collection $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ where A_1, \dots, A_p are nonoverlapping subfigures of $[a, b]$ and $x_i \in [a, b]$ for $i = 1, \dots, p$. In particular, P is called

- (i) *special* if A_1, \dots, A_p are intervals;
- (ii) *tight* if $x_i \in A_i$ for $i = 1, \dots, p$.

A *pseudopartition* in $[a, b]$ is a collection $Q = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ where $\theta_1, \dots, \theta_p$ are functions from $BV_+([a, b])$ such that $\sum_{i=1}^p \theta_i \leq \chi_{[a, b]}$ a.e. and $x_i \in [a, b]$ for $i = 1, \dots, p$. We say that a pseudopartition P is *anchored* in a set $E \subset [a, b]$ if $x_i \in E$ for $i = 1, \dots, p$. Let $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ be a partition in $[a, b]$, then $P^* = \{(\chi_{A_1}, x_1), \dots, (\chi_{A_p}, x_p)\}$ is a pseudopartition in $[a, b]$, called the pseudopartition in $[a, b]$ induced by P .

Let $\varepsilon > 0$ and let δ be a positive function on $[a, b]$. A pseudopartition $Q = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ in $[a, b]$ is called

- (i) a pseudopartition of $[a, b]$ if $\sum_{i=1}^p \theta_i = \chi_{[a, b]}$ a.e.;
- (ii) ε -regular if $r(\theta_i, x_i) > \varepsilon$, $i = 1, \dots, p$;
- (iii) δ -fine if $d(S_{\theta_i} \cup \{x_i\}) < \delta(x_i)$, $i = 1, \dots, p$.

A partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in $[a, b]$ is a partition of $[a, b]$, or ε -regular, or δ -fine whenever the pseudopartition P^* induced by P has the respective property.

For a given function f on $[a, b]$ and a pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ in $[a, b]$ we set $\sigma(f, P) = \sum_{i=1}^p f(x_i) \int_{[a, b]} \theta_i$, where the symbol \int is used to denote the Lebesgue integral.

Definition 1.1. (See [4].) A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be *integrable* in $[a, b]$ if there is a real number I with the following property: given $\varepsilon > 0$, we can find a positive function δ on $[a, b]$ such that

$$|\sigma(f, P) - I| < \varepsilon$$

for each ε -regular δ -fine pseudopartition P of $[a, b]$.

We denote by $\mathcal{I}([a, b])$ the family of all integrable functions in $[a, b]$ and set $\int_{[a, b]}^* f = I$. For each $f \in \mathcal{I}([a, b])$, the function $x \mapsto \int_{[a, x]}^* f$, defined on $[a, b]$, is called the *primitive* of f .

Let $\theta \in BV_+([a, b])$, then the distributional derivative $D\theta$ is a signed Borel measure in \mathbb{R} whose support is contained in $\text{cl } S_\theta$. For a bounded Borel function f on $[a, b]$, $\int_{[a, b]} f D\theta$ denotes the Lebesgue integral of f over $[a, b]$ with respect to $D\theta$.

Given a continuous function F on $[a, b]$ and a pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ in $[a, b]$ we define

$$\sum_P \int_{[a, b]} F D\theta = \sum_{i=1}^p \int_{[a, b]} F D\theta_i.$$

The following lemma was proved in [4, Lemma 3.1].

Lemma 1.2. *Let f be a bounded function on $[a, b]$ whose derivative $f'(x)$ exists at $x \in [a, b]$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\left| f'(x)|\theta|_1 + \int_{[a,b]} f D\theta \right| < \varepsilon |\theta|_1$$

for each $\theta \in \text{BV}_+([a, b])$ satisfying $d(S_\theta \cup \{x\}) < \delta$ and $r(\theta, x) > \varepsilon$.

Proposition 1.3. *Let $f \in \mathcal{I}([a, b])$. If $F(x) = \int_{[a,x]}^* f$ for each $x \in [a, b]$, then the function $F: [a, b] \rightarrow \mathbb{R}$ is continuous. In addition, for almost all $x \in [a, b]$, F is derivable at x and $F'(x) = f(x)$.*

Proof. Since $\mathcal{I}([a, b])$ is a subfamily of the family $\mathcal{R}_1^*(([a, b])$ introduced in [2, Section 3], the proposition follows from [8, Proposition 2.4]. \square

For each figure $A \subset [a, b]$ and for each function F defined on $[a, b]$ we set

$$F(A) = \sum_{h=1}^n [F(b_h) - F(a_h)],$$

where $[a_1, b_1], \dots, [a_n, b_n]$ are the connected components of A .

A function F (or a sequence $\{F_n\}$ of functions) is called AC^* (see [3]) (respectively uniformly AC^* (see [1])) on a set $E \subset [a, b]$ whenever for every $\varepsilon > 0$ there exist a positive number α and a positive function δ on E satisfying the condition

$$\sum_{i=1}^p |F(A_i)| < \varepsilon \quad \left(\sup_n \sum_{i=1}^p |F_n(A_i)| < \varepsilon \right)$$

for each tight ε -regular δ -fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in $[a, b]$ anchored in E with $\sum_{i=1}^p |A_i| < \alpha$. A function F (a sequence $\{F_n\}$) is called ACG^* (uniformly ACG^*) on a set $E \subset [a, b]$ whenever there are sets $E_n \subset E$, $n = 1, 2, \dots$ such that $E = \bigcup_{n=1}^{\infty} E_n$ and F is AC^* (uniformly AC^*) on each E_n .

2. CHARACTERIZATION OF PRIMITIVES

The following condition (denoted by WSL°) is a modification of the Strong Lusin condition, introduced by P. Y. Lee in [6].

Definition 2.1. Let $N \subset [a, b]$ be a set of measure zero. A continuous function F is said to satisfy condition WSL° on N if, given $\varepsilon > 0$, there exists a positive function δ on $[a, b]$ such that

$$\left| \sum_{x_i \in N} \int_{[a,b]} F D\theta_i \right| < \varepsilon$$

for each ε -regular δ -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ of $[a, b]$.

Proposition 2.2. *Let $f \in \mathcal{I}([a, b])$ and $F(x) = \int_{[a, x]}^* f$. Then F is continuous, derivable a.e. on $[a, b]$ and satisfies condition WSL $^\circ$ on $N = \{x: F'(x) \text{ does not exist}\}$.*

Proof. By Proposition 1.3, F is continuous and $F'(x) = f(x)$ a.e. in $[a, b]$. Then $|N| = 0$ and by [4, Corollary 2.10] we can assume $f(x) = 0$ on N and $f(x) = F'(x)$ elsewhere. By Lemma 1.2, for each $\varepsilon > 0$ and for each $x \in [a, b] \setminus N$ we can find a $\delta_0(x) > 0$ such that

$$\left| f(x)|\theta|_1 + \int_{[a, b]} FD\theta \right| < \frac{\varepsilon}{2(b-a)}|\theta|_1$$

for every $\theta \in BV_+([a, b])$ satisfying $d(S_\theta \cup \{x\}) < \delta_0(x)$ and $r(\theta, x) > \varepsilon$. Since $f \in \mathcal{I}([a, b])$, there is a positive function δ on $[a, b]$ ($\delta \leq \delta_0$) such that

$$|\sigma(f, P) - [F(b) - F(a)]| < \frac{\varepsilon}{2}$$

for each ε -regular δ -fine pseudopartition P of $[a, b]$.

Let $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ be an ε -regular δ -fine pseudopartition of $[a, b]$. Then

$$\sigma(f, P) = \sum_{i=1}^p f(x_i) \int_{[a, b]} \theta_i = \sum_{x_i \in [a, b] \setminus N} f(x_i) \int_{[a, b]} \theta_i$$

and

$$-[F(b) - F(a)] = \int_{[a, b]} FD\chi_{[a, b]} = \sum_{x_i \in N} \int_{[a, b]} FD\theta_i + \sum_{x_i \in [a, b] \setminus N} \int_{[a, b]} FD\theta_i.$$

Hence

$$\begin{aligned} \left| \sum_{x_i \in N} \int_{[a, b]} FD\theta_i \right| &\leq \left| \sum_{i=1}^p f(x_i) \int_{[a, b]} \theta_i + \int_{[a, b]} FD\chi_{[a, b]} \right| \\ &\quad + \left| \sum_{x_i \in [a, b] \setminus N} \left(f(x_i)|\theta_i|_1 + \int_{[a, b]} FD\theta_i \right) \right| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)} \sum_{x_i \in [a, b] \setminus N} |\theta_i|_1 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon. \end{aligned}$$

Thus the claim is proved. \square

Proposition 2.3. *A function f on $[a, b]$ belongs to $\mathcal{I}([a, b])$ if and only if there exists a continuous function F such that for almost all $x \in [a, b]$ F is derivable at x with*

$F'(x) = f(x)$ and satisfies condition WSL° on the set $N = \{x: F'(x) \text{ does not exist}\}$. In particular, $F(x) = \int_{[a,x]} f$.

P r o o f. The necessity is given in Proposition 2.2. Now suppose that there exists a function F on $[a, b]$ satisfying the hypotheses of the theorem. Then $|N| = 0$. Assume $f(x) = 0$ on N and $f(x) = F'(x)$ elsewhere. Since F satisfies condition WSL° on N , given $\varepsilon > 0$, there exists a positive function δ on $[a, b]$ such that

$$\left| \sum_{x_i \in N} \int_{[a,b]} FD\theta_i \right| < \frac{\varepsilon}{2}$$

for each ε -regular δ -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ of $[a, b]$. By Lemma 1.2, to each $x \in [a, b] \setminus N$ such a $\delta_0(x) > 0$ corresponds that

$$\left| f(x)|\theta_1 + \int_{[a,b]} FD\theta \right| < \frac{\varepsilon}{2(b-a)}|\theta_1|$$

for each $\theta \in BV_+([a, b])$ satisfying $d(S_\theta \cup \{x\}) < \delta_0(x)$ and $r(\theta, x) > \varepsilon$. Define

$$\delta^*(x) = \begin{cases} \min\{\delta(x), \delta_0(x)\} & \text{if } x \in [a, b] \setminus N, \\ \delta(x) & \text{if } x \in N. \end{cases}$$

Then for each ε -regular δ^* -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ of $[a, b]$ we have

$$\begin{aligned} & \left| \sum_{i=1}^p f(x_i) \int_{[a,b]} \theta_i - [F(b) - F(a)] \right| \\ &= \left| \sum_{i=1}^p f(x_i) \int_{[a,b]} \theta_i + \int_{[a,b]} FD\chi_{[a,b]} \right| \\ &\leq \left| \sum_{x_i \in N} \int_{[a,b]} FD\theta_i \right| + \sum_{x_i \in [a,b] \setminus N} \left| f(x_i) \int_{[a,b]} \theta_i + \int_{[a,b]} FD\theta_i \right| < \varepsilon. \end{aligned}$$

Hence $f \in \mathcal{I}([a, b])$. □

R e m a r k 2.4. Let $F: [a, b] \rightarrow \mathbb{R}$ be a continuous function. If F is differentiable a.e. on $[a, b]$ and satisfies condition WSL° on the set $N = \{x: F'(x) \text{ does not exist}\}$ then F is ACG^* on $[a, b]$.

Indeed, by the previous theorem $F' = f$ belongs to $\mathcal{I}([a, b])$, thus $f \in \mathcal{R}_t^+([a, b])$. By [3, Proposition 3.4] it follows that F is ACG^* on $[a, b]$.

Definition 2.5. Let F be a continuous function on $[a, b]$ and let $E \subset [a, b]$. The function F is called AC° on E if, given $\varepsilon > 0$, there exist a positive number α and a positive function δ on E such that

$$\sum_{i=1}^p \left| \int_{[\theta_i]} FD\theta_i \right| < \varepsilon$$

for each ε -regular δ -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ in $[a, b]$ anchored in E with $\sum_{i=1}^p |\theta_i|_1 < \alpha$. The function F is called ACG° on E if there are measurable sets $E_n \subset E$, $n = 1, 2, \dots$ such that $E = \bigcup_{i=1}^\infty E_n$ and F is AC° on each E_n .

Remark 2.6. If F is ACG° on $X \subset [a, b]$, then F is ACG^* on X . In particular, F is differentiable a.e. on X ([3], Corollary 3.3).

The following lemma is a straightforward modification of [3, Lemma 2.2].

Lemma 2.7. Let $X \subset [a, b]$ and let F be an ACG° function on X . If E is a subset of X of measure zero, given $\varepsilon > 0$, there exists a positive function δ on $[a, b]$ such that $\sum_{i=1}^p \left| \int_{[\theta_i]} FD\theta_i \right| < \varepsilon$ for each ε -regular δ -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ in $[a, b]$ anchored in E .

Proposition 2.8. Let F be a continuous function on $[a, b]$. Then F is differentiable a.e. in $[a, b]$ and satisfies condition WSL° on the set $N = \{x: F'(x) \text{ does not exist}\}$ if and only if there exists a set X with $|[a, b] \setminus X| = 0$ such that the function F is ACG° on X and satisfies condition WSL° on $[a, b] \setminus X$.

Proof. Assume first that F is differentiable a.e. in $[a, b]$ and satisfies condition WSL° on the set $N = \{x: F'(x) \text{ does not exist}\}$. We show that F is ACG° on $X = [a, b] \setminus N$. For $n = 1, 2, \dots$, let $E_n = \{x \notin N: n-1 \leq |F'(x)| < n\}$, then $X = \bigcup_{i=1}^\infty E_n$. By Lemma 1.2, for each $\varepsilon > 0$ and for each $x \in E_n$ there is a $\delta_n(x) > 0$ such that

$$\left| F'(x)|\theta|_1 + \int_{[\theta]} FD\theta \right| < \frac{\varepsilon}{2(b-a)}|\theta|_1$$

for all $\theta \in BV_+([a, b])$ satisfying $d(S_\theta \cup \{x\}) < \delta_n(x)$ and $r(\theta, x) > \varepsilon$. Now let $\alpha_n = \frac{\varepsilon}{2n}$. Then, for each ε -regular δ_n -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ in $[a, b]$ anchored in E_n with $\sum_{i=1}^p |\theta_i|_1 < \alpha_n$ it follows

$$\left| \sum_{i=1}^p \int_{[\theta_i]} FD\theta_i \right| \leq \sum_{i=1}^p \left| F'(x_i)|\theta_i|_1 + \int_{[\theta_i]} FD\theta_i \right| + \sum_{i=1}^p |F'(x_i)||\theta_i|_1 < \frac{\varepsilon}{2} + n\alpha_n = \varepsilon$$

Hence F is AC° on E_n . □

Conversely, let $T = [a, b] \setminus X$ and fix $\varepsilon > 0$. By Remark 2.6 F is differentiable a.e. on X . Let $N = \{x: F'(x) \text{ does not exist}\}$, then $N = N_1 \cup N_2$ where $N_1 \subset X$ and $N_2 \subset T$. By Lemma 2.7 there exists a positive function δ_1 on $[a, b]$ such that $\sum_P |\int_{[a, b]} FD\theta| < \frac{\varepsilon}{4}$ for each ε -regular δ_1 -fine pseudopartition P in $[a, b]$ anchored in N_1 .

Since F satisfies condition WSL² on T , there exists a positive function δ_0 ($\delta_0 \leq \delta_1$) such that

$$\left| \sum_{x_i \in T} \int_{[a, b]} FD\theta_i \right| < \frac{\varepsilon}{4}$$

for each ε -regular δ_0 -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ of $[a, b]$. Use Lemma 1.2 to find a positive function δ_2 in $T \setminus N_2$ such that

$$\left| F'(x)|\theta|_1 + \int_{[a, b]} FD\theta \right| < \frac{\varepsilon}{4(b-a)}|\theta|_1$$

for each $\theta \in BV_+([a, b])$ satisfying $d(S_\theta \cup \{x\}) < \delta_2(x)$ and $r(\theta, x) > \varepsilon$. For $n = 1, 2, \dots$, set $T_n = E_n \cap (T \setminus N_2)$, E_n being the sets defined above. Since $|T_n| = 0$ there exists an open set O_n such that $T_n \subset O_n$ and $|O_n| < \varepsilon/n2^{n+2}$. Now define a positive function δ on $[a, b]$ by setting

$$\delta(x) = \begin{cases} \min\{\delta_0(x), \delta_2(x), \varepsilon/n2^{n+2}\} & \text{if } x \in T_n, n = 1, 2, \dots, \\ \delta_0(x) & \text{elsewhere.} \end{cases}$$

Let $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ be an ε -regular δ -fine pseudopartition of $[a, b]$. It follows that

$$\begin{aligned} \left| \sum_{x_i \in N} \int_{[a, b]} FD\theta_i \right| &\leq \left| \sum_{x_i \in N_1} \int_{[a, b]} FD\theta_i \right| + \left| \sum_{x_i \in N_2} \int_{[a, b]} FD\theta_i \right| \\ &\leq \frac{\varepsilon}{4} + \left| \sum_{x_i \in N_2} \int_{[a, b]} FD\theta_i + \sum_{x_i \in T \setminus N_2} \int_{[a, b]} FD\theta_i \right| \\ &\quad + \left| \sum_{x_i \in T \setminus N_2} \int_{[a, b]} FD\theta_i \right| \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{x_i \in T \setminus N_2} \left| F'(x_i)|\theta_i|_1 + \int_{[a, b]} FD\theta_i \right| \\ &\quad + \sum_{x_i \in T \setminus N_2} |F'(x_i)||\theta_i|_1 \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \sum_{n=1}^{\infty} \sum_{x_i \in E_n} |F'(x_i)||\theta_i|_1 \leq \frac{3}{4}\varepsilon + \sum_{n=1}^{\infty} n \frac{\varepsilon}{n2^{n+2}} = \varepsilon. \end{aligned}$$

Combining Theorem 2.3 and Proposition 2.8 we get the following theorem.

Theorem 2.9. *A function f on $[a, b]$ belongs to $\mathcal{I}([a, b])$ if and only if there exist a subset X of $[a, b]$ and a continuous function $F: [a, b] \rightarrow \mathbb{R}$ such that*

- (i) $|[a, b] \setminus X| = 0$,
- (ii) F is ACG° on X ,
- (iii) F satisfies condition WSL° on $[a, b] \setminus X$,
- (iv) $F' = f$ a.e. on $[a, b]$.

In particular, $F(x) = \int_{[a, x]}^* f$.

It is interesting to point out that the Saks-Henstock lemma for the \mathcal{I} -integral has not been proved nor a counterexample has been produced. The validity of the Saks-Henstock lemma would allow us to improve the formulation of the above descriptive characterization. More precisely, in the formulation of condition WSL° the expression $|\sum_{\tau_i \in \mathcal{N}} \int_{[a, b]} F D\theta_i| < \varepsilon$ would be replaced by $\sum_{\tau_i \in \mathcal{N}} |\int_{[a, b]} F D\theta_i| < \varepsilon$. Thus in Proposition 2.3 the function F would satisfy such condition on every set of measure zero, moreover the statement of Theorem 2.9 would be:

A function f on $[a, b]$ belongs to $\mathcal{I}([a, b])$ if and only if there exists a continuous function F such that F is ACG° on $[a, b]$ and $F' = f$ a.e. on $[a, b]$.

3. CONTROLLED CONVERGENCE

In this section we give a definition of *uniform generalized absolute continuity* and use it to prove a controlled convergence theorem for sequences of \mathcal{I} -integrable functions.

Definition 3.1. Let $\{F_n\}$ be a sequence of functions defined on $[a, b]$. We say that $\{F_n\}$ is *uniformly AC°* on $E \subset [a, b]$ if, given $\varepsilon > 0$, there exist a positive function δ on $[a, b]$ and a positive number α such that

$$\sup_n \sum_{i=1}^p \left| \int_{[a, b]} F_n D\theta_i \right| < \varepsilon$$

for each ε -regular δ -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ in $[a, b]$ anchored in E with $\sum_{i=1}^p |\theta_i|_1 < \alpha$. A sequence $\{F_n\}$ of functions is said to be *uniformly ACG°* on E if there are disjoint sets $E_k \subset E$, $k = 1, 2, \dots$ such that $E = \bigcup_{k=1}^{\infty} E_k$ and every F_n is uniformly AC° on each E_k .

Definition 3.2. Let N be a set of measure zero. A sequence of functions $\{F_n\}$ defined on $[a, b]$ is said to satisfy *uniformly* condition WSL $^\circ$ on N if, given $\varepsilon > 0$, there exists a positive function δ on $[a, b]$ such that

$$\sup_n \left| \sum_{x_i \in N} \int_{[a, b]} F D\theta_i \right| < \varepsilon$$

for each ε -regular δ -fine pseudopartition $P = \{(\theta_1, x_1), \dots, (\theta_p, x_p)\}$ of $[a, b]$.

Lemma 3.3. Let $\{F_n\}$ be a sequence of functions and X a subset of $[a, b]$ such that

- (i) $|[a, b] \setminus X| = 0$,
- (ii) $\{F_n\}$ is uniformly ACG $^\circ$ on X ,
- (iii) $\{F_n\}$ satisfies uniformly condition WSL $^\circ$ on $[a, b] \setminus X$.

Then $\{F_n\}$ is uniformly ACG * on $[a, b]$.

Proof. Let $X = \bigcup_{k=1}^{\infty} E_k$, where the E_k 's are disjoint and the sequence $\{F_n\}$ is uniformly ACG $^\circ$ on each E_k . Clearly the sequence $\{F_n\}$ is uniformly AC * on E_k for $k = 1, 2, \dots$. We have to prove that the sequence $\{F_n\}$ is uniformly AC * on $[a, b] \setminus X$. Given $\varepsilon > 0$, there is a positive function δ on $[a, b]$ such that

$$\sup_n \left| \sum_{x_i \in [a, b] \setminus X} F_n(A_i) \right| = \sup_n \left| \sum_{x_i \in [a, b] \setminus X} \int_{[a, b]} F_n D\chi_{A_i} \right| < \frac{\varepsilon}{2}$$

for each ε -regular δ -fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ of $[a, b]$. Fix $n \geq 1$. By Theorem 2.9 the function f_n belongs to $\mathcal{I}([a, b])$, hence (see Remark 2.6) its primitive F_n is ACG * on $[a, b]$. Thus, by [3, Lemma 2.2] there is a positive function δ_n on $[a, b]$ ($\delta_n \leq \delta$) such that

$$\left| \sum_{i=1}^s F_n(A_i) \right| < \frac{\varepsilon}{2}$$

for each ε -regular δ_n -fine partition $\{(A_1, x_1), \dots, (A_s, x_s)\}$ in $[a, b]$ anchored in $[a, b] \setminus X$. Choose an ε -regular δ -fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ anchored in $[a, b] \setminus X$. By Cousin's lemma there exists a special and tight δ_n -fine partition $P_1 = \{(B_1, y_1), \dots, (B_r, y_r)\}$ of $[a, b] \setminus \cup P$. Then $P \cup P_1$ is an ε -regular δ -fine partition of $[a, b]$. Thus we obtain

$$\left| \sum_{i=1}^p F_n(A_i) \right| = \left| \sum_{i=1}^p F_n(A_i) + \sum_{y_r \in [a, b] \setminus X} F_n(B_j) \right| + \left| \sum_{y_r \in [a, b] \setminus X} F_n(B_j) \right| < \varepsilon.$$

Considering separately the subfigures A_i of P for which $F_n(A_i) \geq 0$ and those for which $F_n(A_i) < 0$ it follows that the inequality $|\sum_{i=1}^s F_n(A_i)| < \varepsilon$ can be replaced by $\sum_{i=1}^s |F_n(A_i)| < \varepsilon$. Thus we get

$$\sup_n \sum_{i=1}^s |F_n(A_i)| < 2\varepsilon$$

and this completes the proof. \square

Definition 3.4. A sequence $\{f_n\} \in \mathcal{I}([a, b])$ is called \mathcal{I} -control convergent to f on $[a, b]$ if $f_n \rightarrow f$ a.e. in $[a, b]$, $\{f_n\}$ is uniformly ACG $^\circ$ on X , where $[a, b] \setminus X$ is of measure zero, and $\{f_n\}$ satisfies uniformly condition WSL $^\circ$ on $[a, b] \setminus X$.

Theorem 3.5. If $\{f_n\} \in \mathcal{I}([a, b])$ is \mathcal{I} -control convergent to f on $[a, b]$, then $f \in \mathcal{I}([a, b])$ and

$$\lim_n \int_{[a, b]}^* f_n = \int_{[a, b]}^* f.$$

Proof. By Lemma 3.3 the primitives $F_n(x) = \int_{[a, x]}^* f_n$ of f_n are uniformly ACG $^\circ$. Thus by [1, Theorem 4.3] we get that $\lim_n \int_{[a, b]}^* f_n = (\mathcal{R}_t) \int_{[a, b]}^* f$ and $F(x) = (\mathcal{R}_t) \int_{[a, x]}^* f$ is ACG $^\circ$ on $[a, b]$. It remains to show that there exists a set X with $[a, b] \setminus X = 0$ such that F is ACG $^\circ$ on X and satisfies condition WSL $^\circ$ on $[a, b] \setminus X$. We note that the sequence $\{F_n\}$ is equicontinuous and since $F_n(a) = 0$, it is also equibounded. Then, by Ascoli's theorem, there is a subsequence $\{F_{n(j)}\}$ of $\{F_n\}$ that converges uniformly to F on $[a, b]$. Given $\varepsilon > 0$ and a fixed k , choose δ_k and δ on $[a, b]$ and α_k according to Definition 3.1 and Definition 3.2. Then the uniform convergence of $\{F_{n(j)}\}$ to F implies that

$$\sum_P \left| \int_{[a, b]} F D\theta_i \right| \leq \sup_{n(j)} \sum_P \left| \int_{[a, b]} F_{n(j)} D\theta_i \right| < \varepsilon$$

for each ε -regular δ_k -fine pseudopartition P in $[a, b]$ anchored in E_k with $\sum_{i=1}^p |\theta_i| < \alpha_k$ and also

$$\left| \sum_{x_i \in N} \int_{[a, b]} F D\theta_i \right| \leq \sup_{n(j)} \left| \sum_{x_i \in N} \int_{[a, b]} F_{n(j)} D\theta_i \right| < \varepsilon$$

for each ε -regular δ -fine pseudopartition P of $[a, b]$.

Hence F is ACG $^\circ$ on $X = \bigcup_{k=1}^{\infty} E_k$ with $[a, b] \setminus X = 0$ and F satisfies condition WSL $^\circ$ on $[a, b] \setminus X$. Thus by Theorem 2.9 we conclude that $f \in \mathcal{I}([a, b])$ and $F(x) = \int_{[a, x]}^* f$. \square

Remark 3.6. Let g be a function of bounded variation on $[a, b]$ and let $\{f_n\} \in \mathcal{I}([a, b])$ be \mathcal{I} -control convergent to f on $[a, b]$. Then, by the integration by parts formula [4, Proposition 3.3], we get

$$\lim_n \int_{[a,b]}^* f_n g = \lim_n \left[F_n(b)g(b) - \int_{[a,b]} F_n dg \right] = F(b)g(b) - \int_{[a,b]} F dg = \int_{[a,b]}^* fg.$$

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