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# GENERALIZED BOUNDARY VALUE PROBLEMS WITH LINEAR GROWTH

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*Abstract.* It is shown that for a given system of linearly independent linear continuous functionals $l_i\colon C^{n-1} \to \mathbb{R}$, $i = 1, \dots, n$, the set of all $n$-th order linear differential equations such that the Green function for the corresponding generalized boundary value problem (BVP for short) exists is open and dense in the space of all $n$-th order linear differential equations. Then the generic properties of the set of all solutions to nonlinear BVP-s are investigated in the case when the nonlinearity in the differential equation has a linear majorant. A periodic BVP is also studied.

*Keywords*: generic properties, periodic boundary value problem

*MSC 1991*: 34B15

## 1. INTRODUCTION

B. Rudolf in [14] has shown that for a given system of linearly independent linear continuous functionals $l_i\colon C^n([a,b],\mathbb{R}) \to \mathbb{R}$, $i = 1, \dots, n$, there exists a linear differential equation

$$(1) \qquad (L(x) \equiv) x^{(n)} + \sum_{k=1}^{n} p_k(t) x^{(n-k)} = 0, \quad a \leqslant t \leqslant b$$

such that the BVP (1),

$$(2) \qquad l_i(x) = 0, \ i = 1, \dots, n$$

has only the trivial solution. This result also holds when the functionals $l_i$ are given in the space $C^{n-1}([a,b],\mathbb{R})$. In this paper we will prove that the set $S$ of all differential equations (1) such that (1), (2) has only the trivial solution is open and dense in the space of all $n$-th order linear differential equations and we will derive some consequences of that result. Besides the classical existence theorems the generic properties of the set of all solutions to nonlinear BVP-s having a linear majorant will also be studied. The main tool for showing these properties will be a priori estimates like Leray-Schauder estimations. Finally, a special case will be investigated, namely the periodic BVP.

Throughout the paper we will assume that $n \geqslant 1, -\infty < a < b < \infty$, $p_k \in C([a,b],\mathbb{R})$, $k = 1,\ldots,n$, $l_i\colon C^{n-1}([a,b],\mathbb{R}) \to \mathbb{R}$ is a linear continuous functional, $i = 1,\ldots,n$ and the functionals $l_i$, $i = 1,\ldots,n$, are linearly independent.

Let $C^0 = (C([a,b],\mathbb{R}), \|.\|_0)$ be a Banach space with the sup-norm $\|.\|_0$ and let the topology in $C^l = C^l([a,b],\mathbb{R})$ be given by the norm $\|.\|_l$, whereby

$$\|x\|_l = \max_{k=0,\ldots,l} \|x^{(k)}\|_0, \quad l = 1,\ldots,n.$$

Further let $C_n = C^0 \times \ldots \times C^0$ ($n$ times) be the product space with the norm $\|(x_1,\ldots,x_n)\| = \sum\limits_{k=1}^{n} \|x_k\|_0$. Then $C_n$ is a Banach space and the equation (1) can be represented by the $n$-tuple $(p_1,\ldots,p_n)$.

## 2. Regular case

We will start with the following definition.

**Definition 1.** The BVP (1), (2) will be called regular if and only if it has only the trivial solution.

**Theorem 1.** *Let a system of linearly independent linear continuous functionals $l_i$, $i = 1,\ldots,n$, be given. Then the set $S$ of all $n$-tuples $(p_1,\ldots,p_n) \in C_n$ such that the BVP (1), (2) is regular is nonempty, open and dense in the space $C_n$.*

Proof. By the Rudolf theorem, [14], $S \neq \emptyset$. Suppose that there exists a sequence of nontrivial solutions $y_m$ to the BVP-s

$$(L_m(x) \equiv) x^{(n)} + \sum_{k=1}^{n} p_{k,m}(t) x^{(n-k)} = 0, \quad a \leqslant t \leqslant b \tag{$1_m$}$$

$$l_i(x) = 0, \quad i = 1,\ldots,n \tag{2}$$

where

$$\|(p_{1,m},\ldots,p_{n,m}) - (p_1,\ldots,p_n)\| \to 0 \quad \text{as} \quad m \to \infty. \tag{3}$$

Here $p_{k,m} \in C^0$, $k = 1, \dots, n$, $m = 1, 2, \dots$.

The solutions $y_m$ can be normalized by the condition

$$(y_m(a))^2 + (y'_m(a))^2 + \dots + (y_m^{(n-1)}(a))^2 = 1. \tag{4}$$

By Corollary 4.1, [5], (3) and (4) imply that there is a subsequence $y_{m_l}$ of $y_m$ and a solution $y$ of (1) such that $\|(y_{m_l}, \dots, y_{m_l}^{(n-1)}) - (y, \dots, y^{(n-1)})\| \to 0$ as $l \to \infty$. Hence $y$ is a nontrivial solution of (1), (2) and thus $C_n \setminus S$ is closed. This implies that $S$ is open in $C_n$.

Now we prove that $S$ is dense in the space $C_n$. Again by the Rudolf theorem, there is a differential equation

$$x^{(n)} + \sum_{k=1}^{n} r_k(t) x^{(n-k)} = 0 \tag{5}$$

such that the BVP (5), (2) is regular. Denote by $A\colon D(A) \subset C^{n-1} \to C$ the operator

$$Ax = \sum_{k=1}^{n} (p_k(t) - r_k(t)) x^{(n-k)}$$

where

$$D(A) = \{x \in C^{n-1} : l_i(x) = 0,\ i = 1, \dots, n\}.$$

Then the linear operator $L - A\colon D(L) \subset C^n \to C$ which is defined on

$$D(L) = \{x \in C^n : l_i(x) = 0, \quad i = 1, \dots, n\} \tag{6}$$

is onto and one-to-one. By Lemma 4 ([18], p. 512), its inverse mapping $(L - A)^{-1}\colon C \to C^n$ is continuous and as a mapping from $C$ to $C^{n-1}$ it is completely continuous. Since the equation $L(x) = 0$ is equivalent to the equation

$$x = (L - A)^{-1}(-A(x)) \tag{7}$$

and the operator $K = -(L - A)^{-1} \circ A\colon D(A) \subset C^{n-1} \to C^{n-1}$ is completely continuous, either (7) has only the trivial solution or 1 is an eigenvalue of $K$. In the latter case there is an $\varepsilon > 0$ such that the equation $\lambda x = (L - A)^{-1} \circ (-Ax)$ as well as $Lx = (1 - \frac{1}{\lambda})Ax$ has only the trivial solution for all $\lambda \in (1 - \varepsilon, 1) \cup (1, 1 + \varepsilon)$. This means that for these $\lambda$ the BVP $L(x) + (\frac{1}{\lambda} - 1)A(x) = 0$, (2) is regular. □

Consider now a sequence of generalized boundary conditions

$$(2_j) \qquad l_i^{(j)}(x) = 0, \quad i = 1, \dots, n,$$

$j = 1, 2, \dots$, where

$$l_i^{(j)} \colon C^{n-1} \to \mathbb{R}$$

is a linear continuous functional, $i = 1, \dots, n$, $j = 1, 2, \dots$ and for each $j = 1, 2, \dots$ the functionals $l_i^{(j)}$, $i = 1, \dots, n$ are linearly independent.

Denote by $S_j$ the set of all $n$-tuples $(p_1, \dots, p_n) \in C_n$ such that the BVP (1), $(2_j)$ is regular. On the basis of the Baire theorem ([13], Theorem 2.2), Theorem 1 implies that the set $\bigcap_{j=1}^{\infty} S_j$ is dense in $C_n$ and hence the following corollary holds.

**Corollary 1.** *If a sequence of the boundary conditions* $(2_j)$, $j = 1, 2, \dots$ *is given, then the set of all $n$-tuples* $(p_1, \dots, p_n) \in C_n$ *for which the BVP-s* (1), $(2_j)$, $j = 1, 2, \dots$ *are regular is dense in the space* $C_n$.

Let $\{t_j\}_{j=0}^{\infty} \subset [a, b]$ be an injective sequence of points in $[a, b]$. A point $t_j$ is conjugate to $t_0$ for the equation

$$(8) \qquad x'' + p_1(t)x' + p_2(t)x = 0$$

([8], p. 216) if and only if the BVP (8),

$$x(t_0) = 0, \quad x(t_j) = 0$$

is not regular. Hence, by Corollary 1, the set of all pairs $(p_1, p_2)$ such that all conjugate points to $t_0$ in $[a, b]$ for the equation (8) (when they exist) are different from $\{t_j\}_{j=1}^{\infty}$ is dense in $C_2$.

Now we introduce the notions which are well-known in the coincidence theory developed by J. Mawhin (see e.g. [7]). According to (1), the operator $L \colon D(L) \subset C^{n-1} \to C^0$ is defined by

$$(9) \qquad L(x) = x^{(n)} + \sum_{k=1}^{n} p_k(t)x^{(n-k)}, \quad x \in D(L)$$

where $D(L)$ is determined by (6). By the Rudolf theorem $L$ is a Fredholm mapping of index zero. Hence, there exist linear continuous projectors $P \colon C^{n-1} \to C^{n-1}$, $Q \colon C^0 \to C^0$ such that

$$R(P) = N(L), \quad N(Q) = R(L)$$

and

$$C^{n-1} = N(L) \oplus N(P), \quad C^0 = R(Q) \oplus R(L)$$

as topological direct sums. Further, the restriction $L_P$ of $L$ to $D(L) \cap N(P)$ is one-to-one and onto $R(L)$ so that its algebraic inverse $K_P\colon R(L) \to D(L) \cap N(P)$ is well defined.

We shall distinguish two cases: Either (1), (2) is regular or not. When the BVP (1), (2) is regular, then $P(x) \equiv 0$, $x \in C^{n-1}$, $Q(y) \equiv 0$, $y \in C^0$ and $C^{n-1} = N(P)$, $C^0 = R(L)$. In this case $L_P = L$, $K_P = L^{-1}$. The operator $L^{-1}\colon C^0 \to D(L) \subset C^{n-1}$ is constructed with help of the Green function $G = G(t,s)$, $a \leqslant t, s \leqslant b$ for the BVP (1), (2) given in [18]. $L^{-1}$ is defined in Lemma 1, ([18], p. 510) by the relation

$$L^{-1}(x)(t) = \int_a^b G(t,s)x(s)\,\mathrm{d}s, \quad a \leqslant t \leqslant b, \quad x \in C^0. \tag{10}$$

Further, for each $k \in \{1, \dots, n-1\}$ (if $n \geqslant 2$) we have

$$(L^{-1}(x))^{(k)}(t) = \int_a^b \frac{\partial^k G(t,s)}{\partial t^k} x(s)\,\mathrm{d}s, \quad a \leqslant t \leqslant b, \quad x \in C^0. \tag{11}$$

By Lemma 3 ([18], p. 512), the function $\varphi(t) = \int_a^b |G(t,s)|\,\mathrm{d}s$, $a \leqslant t \leqslant b$, is continuous on $[a,b]$ and hence, by the result in [6], p. 187,

$$\|L^{-1}\| = \max_{a \leqslant t \leqslant b} \int_a^b |G(t,s)|\,\mathrm{d}s. \tag{12}$$

A similar relation holds for the norm $\|L_k^{-1}\|$ of the linear operator standing on the right-hand side of (11). Hence

$$\|L_k^{-1}\| = \max_{a \leqslant t \leqslant b} \int_a^b \left| \frac{\partial^k G(t,s)}{\partial t^k} \right| \mathrm{d}s. \tag{13}$$

For brevity, in what follows we will write $\|L_0^{-1}\|$ instead of $\|L^{-1}\|$.

Let $f = f(t, x_1, \dots, x_{p+1}) \in C([a,b] \times \mathbb{R}^{p+1}, \mathbb{R}), g \in C([a,b], \mathbb{R})$, where $0 \leqslant p \leqslant n-1$. The following existence lemma holds for the nonlinear perturbation of the regular BVP.

**Lemma 1.** *If the BVP (1), (2) is regular and if there exist positive constants $c_1, \dots, c_{p+1}, d$ such that*

$$\delta = \sum_{k=1}^{p+1} c_k \|L_{k-1}^{-1}\| < 1, \tag{14}$$

$$|f(t, x_1, \dots, x_{p+1})| \leqslant \sum_{k=1}^{p+1} c_k |x_k| + d, \quad a \leqslant t \leqslant b, \quad x_1, \dots, x_{p+1} \in \mathbb{R} \tag{15}$$

*and*

$$(16) \qquad |g(t)| \leqslant M, \quad a \leqslant t \leqslant b,$$

*then each possible solution $x$ of the BVP (2),*

$$(17) \qquad L(x) = f(t, x, \dots, x^{(p)}) + g(t)$$

*satisfies a priori estimates*

$$(18) \qquad \big(\min_{k=1,\dots,p+1} c_k\big)\|x\|_p \leqslant \sum_{k=1}^{p+1} c_k \|x^{(k-1)}\|_0 \leqslant \frac{\delta}{1-\delta}(d+M),$$

*and for each $g \in C^0$ there exists a solution of the BVP (17), (2).*

Proof. By means of Lemmas 5 and 6 in [18], p. 516, the problem (17), (2) is equivalent to the fixed point problem for the operator

$$(19) \quad T(x)(t) = \int_a^b G(t,s)[f(s, x(s), \dots, x^{(p)}(s)) + g(s)]\,\mathrm{d}s, \quad a \leqslant t \leqslant b, \quad x \in C^p.$$

This operator as a mapping from $C^p$ to $C^p$ is completely continuous. By the Leray-Schauder principle the existence of a fixed point of $T$ will be proved if it is shown that the set of all possible solutions of the family of equations

$$(20) \qquad x = \lambda T(x), \quad 0 \leqslant \lambda \leqslant 1$$

is a priori bounded (in the norm $\|.\|_p$) independently of $\lambda$.

Let $\lambda \in [0,1]$, $x$ be a possible solution of (20) and let $k \in \{1, \dots, p+1\}$. Then

$$x^{(k-1)}(t) = \lambda \int_a^b \frac{\partial^{k-1} G(t,s)}{\partial t^{k-1}}[f(s, x(s), \dots, x^{(p)}(s)) + g(s)]\,\mathrm{d}s, \quad a \leqslant t \leqslant b,$$

and by (12), (13), (15), (16) we have

$$\|x^{(k-1)}\|_0 \leqslant \|L^{-1}_{(k-1)}\| \Big(\sum_{j=1}^{p+1} c_j \|x^{(j-1)}\|_0 + d + M\Big).$$

Hence

$$\sum_{k=1}^{p+1} c_k \|x^{(k-1)}\|_0 \leqslant \delta \Big(\sum_{k=1}^{p+1} c_k \|x^{(k-1)}\|_0 + d + M\Big)$$

and thus (18) is true. □

E x a m p l e. Let $0 < \eta < 1$. Consider the Green function for the problem

$$x'' = 0,$$

$$x(0) = 0, \quad x(1) - x(\eta) = 0. \tag{21}$$

Since

$$x(t) = t(1-\eta)^{-1}\left[(\eta - 1)\int_0^\eta g(s)\,\mathrm{d}s + \int_\eta^1 [sg(s) - g(s)]\,\mathrm{d}s\right] + \int_0^t (t-s)g(s)\,\mathrm{d}s,$$

$0 \leqslant t \leqslant 1$, is the unique solution of the BVP $x'' = g(t)$, (21), the Green function $G = G(t,s)$ for that problem is determined by

$$G(t,s) = G_1(t,s) + K(t,s) \quad 0 \leqslant t, s \leqslant 1$$

where

$$G_1(t,s) = \begin{cases} -t & 0 \leqslant s \leqslant \eta \\ -\frac{1-s}{1-\eta}t & \eta \leqslant s \leqslant 1 \end{cases},$$

$$K(t,s) = \begin{cases} t - s & 0 \leqslant s \leqslant t \leqslant 1 \\ 0 & 0 \leqslant t < s \leqslant 1 \end{cases}$$

and hence $G(t,s) < 0$ for $0 < s < 1$, $0 < t \leqslant 1$, $G(t,0) = G(t,1) = 0$, $0 \leqslant t \leqslant 1$, $G(0,s) = 0$, $0 \leqslant s \leqslant 1$.

Therefore the solution $x_0(t) = -\frac{1}{2}(1+\eta)t + \frac{t^2}{2}$, $0 \leqslant t \leqslant 1$ of the problem $x'' = 1$, (21) satisfies $|x_0(t)| = \int_0^1 |G(t,s)|\,\mathrm{d}s$, $0 \leqslant t \leqslant 1$ and hence

$$\|L_0^{-1}\| = \max_{0 \leqslant t \leqslant 1} |x_0(t)| = \tfrac{1}{8}(1+\eta)^2. \tag{22}$$

Since $\frac{\partial G(t,s)}{\partial t} \leqslant 0$ for $0 \leqslant t \leqslant \eta$, $0 \leqslant s \leqslant 1$, we have

$$\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| \mathrm{d}s = |x_0'(t)| \leqslant |x_0'(0)| = \tfrac{1}{2}(1+\eta)$$

in $[0,\eta]$. A direct calculation gives

$$\int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right| \mathrm{d}s = \frac{1}{2(1-\eta)}[2t^2 + \eta^2 + 1 - 2t(1+\eta)] \leqslant \tfrac{1}{2}(1-\eta), \quad \eta \leqslant t \leqslant 1.$$

Thus

$$\|L_1^{-1}\| = \tfrac{1}{2}(1+\eta). \tag{23}$$

By means of (22), (23), Lemma 1 gives an existence statement for the nonlinear BVP (21),

$$x'' = f(t, x, x'), \tag{24}$$

which completes the statements in [4].

Remark 1. Estimations of $\|L_k^{-1}\|$ for a BVP of the third and of the fourth order can be found in [16], Lemma 3 and in [10], Lemma 4, respectively.

The existence and uniqueness of a solution to (17), (2) is guaranteed by

**Theorem 2.** *If the BVP (1), (2) is regular and if there exist constants $c_1, \ldots, c_{p+1}$ such that (14) is true and*

$$|f(t,x_1,\ldots,x_{p+1}) - f(t,y_1,\ldots,y_{p+1})| \leqslant \sum_{k=1}^{p+1} c_k|x_k - y_k|,$$

$$a \leqslant t \leqslant b, \quad x_1,\ldots,x_{p+1},y_1,\ldots,y_{p+1} \in \mathbb{R}, \tag{25}$$

*then the BVP (17), (2) has a unique solution.*

Proof. Consider the space $C_1^p = C^p([a,b],\mathbb{R})$ provided with the norm $\|x\|_{p,1} = \sum\limits_{k=1}^{p+1} c_k\|x^{(k-1)}\|_0$. With respect to inequalities

$$\big(\min_{k=1,\ldots,p+1} c_k\big)\|x\|_p \leqslant \|x\|_{p,1} \leqslant \left(\sum_{k=1}^{p+1} c_k\right)\|x\|_p$$

the norms $\|.\|_p, \|\cdot\|_{p,1}$ are equivalent and hence $C_1^p$ is a Banach space. Consider the operator $T\colon C_1^p \to C_1^p$ given by (19). In view of (25), we have

$$\|T^{(k-1)}(x) - T^{(k-1)}(y)\|_0 \leqslant \|L_{k-1}^{-1}\| \sum_{k=1}^{p+1} c_k\|x^{(k-1)} - y^{(k-1)}\|_0$$

for each $k = 1,\ldots,p+1$ and $x, y \in C_1^p$. Thus

$$\|T(x) - T(y)\|_{p,1} \leqslant \delta\|x - y\|_{p,1},$$

which means that $T$ is a strict contraction on $C_1^p$. The result follows by the Banach fixed point theorem. □

Remark 2. In the case of homogeneous boundary conditions, Theorem 2 generalizes Theorem 1.1.1 in [1].

Corollary 1 and Lemma 1 imply

**Theorem 3.** *If a sequence of the boundary conditions $(2_j)$, $j = 1, 2, \ldots$, is given and the function $f$ satisfies the condition*

$$\lim_{|x_1|+\ldots+|x_{p+1}|\to\infty} \frac{|f(t,x_1,\ldots,x_{p+1})|}{|x_1| + \ldots + |x_{p+1}|} = 0$$

*uniformly in $t \in [a, b]$, then there exists a sequence of $n$-tuples $(p_{1,m}, \dots, p_{n,m}) \in C_n$, $m = 1, 2, \dots$, such that* (3) *is fulfilled and the BVP* ($2_j$),

$$L_m(x) = f(t, x, \dots, x^{(p)}) + g(t) \tag{$17_m$}$$

*has a solution $x_{m,j}$ for each $m = 1, 2, \dots$ and each $j = 1, 2, \dots$.*

Another sufficient condition for the existence of a solution to (17), (2) is given in the next lemma.

**Lemma 2.** *Let there exist a sequence of $n$-tuples $(p_{1,m}, \dots, p_{n,m}) \in C_n$, $m = 1, 2, \dots$ such that* (3) *is satisfied and the BVP* ($17_m$), (2) *has a solution $x_m$ for each $m = 1, 2, \dots$. Let the sequence $\{x_m\}$ be bounded in $C^{n-1}$. Then there exists a subsequence $\{x_{m_l}\}$ of $\{x_m\}$ and a solution $x$ of the problem* (17), (2) *such that $x_{m_l} \to x$ as $l \to \infty$ in $C^n$.*

Proof. By ($17_m$), the sequence $\{x_m\}$ is even bounded in $C^n$. Hence, by the Ascoli theorem, there exists a subsequence $\{x_{m_l}\}$ of the sequence $\{x_m\}$ and a function $x \in C^{n-1}$ such that $x_{m_l} \to x$ as $l \to \infty$ in $C^{n-1}$. In view of (3) and ($17_m$) the sequence $x_{m_l}^{(n)}$ is uniformly convergent on $[a, b]$ and hence $x \in C^n$ and $x_{m_l} \to x$ as $l \to \infty$ in $C^n$. Thus $x$ is a solution of (17), (2). □

Now we prove generic properties of the set of all solutions to (17), (2). To that aim we need the definition of the range of bifurcation $R_b$ of the BVP (17), (2) (see Definition 4.1 in [20], p. 29). For the sake of completeness we will give it here.

First we introduce the Banach space $X_0 = (D(L), \|\cdot\|_n)$ ([20], p. 28). Then the set $R_b$ of all $g \in C^0$ with the property that there is a solution $x$ of the BVP (17), (2) and a sequence $g_k \to g$ as $k \to \infty$ such that the BVP (17), (2) for $g = g_k$ has at least two different solutions $x_k$, $z_k$ for each $k$ and $x_k \to x$, $z_k \to x$ in $X_0$ for $k \to \infty$ is called the range of bifurcation of the BVP (17), (2).

By Lemma 1, Theorems 4.1 and 4.2 in [20], pp. 31–32, we get the following theorem.

**Theorem 4.** *If the problem* (1), (2) *is regular and there exist positive constants $c_1, \dots, c_{p+1}, d$ such that* (14), (15) *are true, then the following statements hold:*

1. *For each $g \in C^0$ the set $S_g$ of all solutions of the BVP* (17), (2) *is nonempty and compact.*
2. *If $C^0 \setminus R_b \neq \emptyset$, then each component of that set is nonempty, open and hence a region. The number $n_g$ of solutions of the BVP* (17), (2) *is finite and constant on each component of the set $C^0 \setminus R_b$.*
3. *If $R_b = \emptyset$, then the problem* (17), (2) *has a unique solution for each $g \in C^0$ and this solution continuously depends on $g$ as a mapping from $C^0$ onto $X_0$.*

4. *If* $\frac{\partial f}{\partial x_i} \in C([a,b] \times \mathbb{R}^{p+1}, \mathbb{R})$, $i = 1, \dots, p+1$, *then the open set* $C^0 \setminus R_b$ *is dense in* $C_0$ *and hence,* $R_b$ *is nowhere dense in* $C^0$.

Proof. Since (1), (2) is regular, the operator $L$ given by (9) satisfies the assumption (H.1) of Theorem 4.1. Since $f$ is continuous, (H.2) is satisfied and in the case that $\frac{\partial f}{\partial x_i}$ are continuous, (H.4) is fulfilled. By the apriori estimates (18) it follows that also (H.3) holds. Then Lemma 1 and Theorems 4.1, 4.2 imply the statements. □

Remark 3. We see that under the assumptions of Theorem 4, uniqueness of the BVP (17), (2) implies correctness of that BVP, that is the existence, uniqueness and continuous dependence of the solution $x$ of the BVP (17), (2) on $g$.

## 3. Non regular case

This case is more complicated as the previous one. Now we apply the results of [17] and [20]. Denote by $F$ the Nemitskij operator $F\colon C^p \to C^0$ which is defined by

$$F(x) = f \circ x, \quad x \in C^p \tag{26}$$

The properties of the operator $L$ and $F$ are given by the following lemma.

**Lemma 3.** *The following statements hold:*

(i) *For each integer* $k$, $0 \leqslant k \leqslant n-1$, *the operator* $L\colon D(L) \subset C^k \to C^0$ *is a linear Fredholm operator of index zero.*

(ii) *If there exists a continuous linear operator* $A\colon D(L) \subset C^r \to C^0$ *with* $0 \leqslant r \leqslant n-1$ *such that* $L - A\colon D(L) \subset C^r \to C^0$ *is one-to-one, then* $L - A$ *is onto, the inverse operator* $(L-A)^{-1}\colon C^0 \to D(L) \subset C^{n-1}$ *is completely continuous and* $(L-A)^{-1}$ *as a mapping from* $C^0$ *into* $C^n$ *is continuous.*

(iii) *The operator* $K_P\colon R(L) \subset C^0 \to D(L) \cap N(P) \subset C^{n-1}$ *is completely continuous.*

(iv) $F + g\colon C^p \to C^0$ *is continuous and maps bounded sets in* $C^p$ *into bounded sets in* $C^0$.

(v) $F + g\colon C^p \to C^0$ *is L-completely continuous.*

Proof. (i) By the Rudolf theorem, there exists a continuous linear operator $A\colon C^r \to C^0$ with $0 \leqslant r \leqslant n-1$ such that $L - A\colon D(L) \subset C^r \to C^0$ is one-to-one. Then by Lemmas 1 and 4, [18], the operator $L - A\colon X_0 \to C^0$ is a homeomorphism of $X_0$ onto $C^0$ and $A\colon X_0 \to C^0$ is a linear completely continuous operator. Nikoľskij theorem ([20], p. 21) implies that $L\colon X_0 \to C^0$ is a linear Fredholm operator of index zero. The same is true about $L\colon D(L) \subset C^k \to C^0$.

(ii), (iii) If $L - A\colon D(L) \subset C^r \to C^0$ is one-to-one, then it is onto, and by Lemma 4, [18], $(L-A)^{-1}\colon C^0 \to C^{n-1}$ is completely continuous and $(L-A)^{-1}\colon C^0 \to C^n$ is continuous and hence, by Remark 1 and Lemma 1 in [17], p. 555, the operators $L\colon D(L) \subset C^r \to C^0$ and $L_P\colon D(L) \cap N(P) \subset C^r \to C^{n-1}$ are closed and $K_P\colon R(L) \subset C^0 \to C^r$ is completely continuous. Since $A$ is continuous also as a mapping from $C^{n-1}$ to $C^0$, $K_P\colon R(L) \subset C^0 \to C^{n-1}$ is completely continuous, too.

(iv) The statement follows from the continuity of the functions $f$ and $g$.

(v) Let $E \subset C^p$ be a bounded set. Then by (iii) and (iv) the mappings $Q \circ (F+g)$, $K_P \circ (I-Q) \circ (F+g)$ are continuous on $E$ and the sets $Q \circ (F+g)(E)$ and $K_P \circ (I-Q) \circ (F+g)(E)$ are relatively compact in $C^0$ and in $C^p$, respectively. This implies the statement. □

Remark 4. By (iii), the statements (iv), (v) also hold for the restriction $F+g\colon C^k \to C^0$, $p < k \leqslant n-1$.

On the basis of Theorem 3 ([17], p. 561), the following lemmma holds which is analogous to Lemma 1.

**Lemma 4.** *Suppose that the BVP (1), (2) is not regular and the following assumptions hold:*

(a) $R(L) \cap N(L) = \{0\}$;

(b) *there exists a continuous linear operator* $A\colon C^r \to C^0$ *with* $0 \leqslant r \leqslant n-1$ *such that* $L - A\colon D(L) \subset C^r \to C^0$ *is one-to-one;*

(c) *there exist constants* $c_1, \dots, c_p$, $d > 0$ *such that (15) is true.*
*Let* $d_1 > 0$ *and let* $s = \max(p, r)$.

(d) *The constant* $c = \sum_{k=1}^{p+1} c_k$ *satisfies*

$$c < \frac{1}{\|K_P\|} \frac{d_1}{1+d_1} \tag{27}$$

*where* $\|K_P\|$ *is the norm of* $K_P\colon R(L) \subset C^0 \to C^s$.
*Let* $\varepsilon = \pm 1$.

(e) *There exists an* $R_1 > 0$ *with the following property:*

$$\varepsilon F(\bar{x} + \tilde{x}) + \varepsilon g + k\bar{x} \notin R(L)$$

*for all* $x = \bar{x} + \tilde{x} \in D(L)$, $\bar{x} \in N(L)$, $\tilde{x} \in N(P)$, $k \in \mathbb{R}$ *such that*

$$\|\bar{x}\|_s \geqslant R_1, \quad \|\tilde{x}\|_s \leqslant d_1 \|\bar{x}\|_s \quad \text{and} \quad k > 0.$$

*Then the problem* (17), (2) *has a solution. Moreover, if* $g$ *satisfies* (16) *and*

(f) *there exists an* $R_3 \geqslant R_2$ *where*

$$R_2 = \frac{\|K_P\|(d+M)}{d_1(1-\|K_P\|c) - \|K_P\|c}$$

*such that*

$$F(\bar{x}+\tilde{x}) + g \notin R(L)$$

*for all* $x = \bar{x}+\tilde{x} \in D(L)$, $\bar{x} \in N(L)$, $\tilde{x} \in N(P)$ *satisfying*

$$\|\bar{x}\|_s \geqslant R_3, \quad \|\tilde{x}\|_s \leqslant d_1\|\bar{x}\|_s,$$

*then any solution* $x$ *of the BVP* (17), (2) *satisfies the inequalities*

$$\|\bar{x}\|_s < R_3 \tag{28}$$

*and*

$$\|\tilde{x}\|_s \leqslant \frac{\|K_P\|c}{1-\|K_P\|c}\|\bar{x}\|_s + \frac{\|K_P\|(d+M)}{1-\|K_P\|c}. \tag{29}$$

P r o o f. On the basis of Lemma 3, the operator $L\colon D(L) \subset C^s \to C^0$ satisfies the assumptions ($L_1$), ($L_2'$) and ($L_3$) in [17], pp. 554–555 with $X = C^s$, $Z = C^0$, and by the assumption (a) of this lemma ($L_4$) is satisfied, too. Lemma 3 with Remark 4 also implies that $F+g\colon C^s \to C^0$ is continuous, maps bounded sets in $C^s$ into bounded sets in $C^0$ and is $L$-completely continuous. By virtue of (15), the assumption ($F_5$) of Theorem 3 in [17], p. 561, is satisfied. Then (27) together with the assumption (e) imply that also ($F_6$) is fulfilled. By the just mentioned Theorem 3 the existence statement follows.

Now we prove the a priori estimates (28) and (29). By Lemma 1 ([17], p. 555), and by (15), (16) any solution $x = \bar{x}+\tilde{x}$ of (17), (2), $\bar{x} \in N(L)$, $\tilde{x} \in N(P)$, satisfies the inequalities

$$\|\tilde{x}\|_s \leqslant \|K_P\|\|L(x)\|_0 \leqslant \|K_P\|\|F(x)+g\|_0 \leqslant \|K_P\|c\|\tilde{x}\|_s + \|K_P\|c\|\bar{x}\|_s + \|K_P\|(d+M)$$

and hence (29) is true. Since (27) is equivalent to $\|K_P\|c/(1-\|K_P\|c) < d_1$, the right-hand side of (29) is less than or equal to $d_1\|\bar{x}\|_s$ if and only if

$$\|\bar{x}\|_s \geqslant R_2.$$

Hence for $\|\bar{x}\|_s \geqslant R_2$ we have

$$\|\tilde{x}\|_s \leqslant d_1\|\bar{x}\|_s. \tag{30}$$

By the assumption (f) the solution $x$ of $L(x) = F(x)+g$ cannot satisfy $\|\bar{x}\|_s \geqslant R_3$, (30) and thus (28) and (29) are true. □

By virtue of Lemma 4 the proof of the following theorem is similar to that of Theorem 4.

**Theorem 5.** *If the problem* (1), (2) *is not regular, the assumptions* (a)–(d) *of Lemma* 4 *hold, further for each* $g \in C^0$ *the assumption* (e) *of that lemma is satisfied and for each* $M > 0$ *and each* $g \in C^0$ *satisfying* (16) *the assumption* (f) *of Lemma* 4 *holds, then all statements of Theorem* 4 *hold.*

Consider the BVP (2),

$$L(x) = f(t, x) + g(t) + h(t, x, x', \dots, x^{(p)}) \tag{17'}$$

where $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$, $g \in C([a, b])$, $h \in C([a, b] \times \mathbb{R}^{p+1}, \mathbb{R})$, $0 \leqslant p \leqslant n - 1$. We show that under simple assumptions on $N(L)$, $R(L)$ Theorem 5 implies the following theorem.

**Theorem 6.** *Assume that the following conditions are satisfied:*

(i)

$$N(L) = \{x \in D(L) : x \text{ is a constant on } [a, b]\},$$
$$R(L) = \{y \in C^0 : \int_a^b y(x)\,\mathrm{d}x = 0\};$$

(ii) *there exists a continuous linear operator* $A: C^r \to C^0$ *with* $0 \leqslant r \leqslant n - 1$ *such that* $L - A: D(L) \subset C^r \to C^0$ *is one-to-one;*

(iii) *there exist constants* $c, d, \delta > 0$ *such that*

$$|f(t, x_1)| \leqslant c|x_1| + d, \ |h(t, x_1, \dots, x_{p+1})| \leqslant \delta \text{ for } a \leqslant t \leqslant b,\ x_1, \dots, x_{p+1} \in \mathbb{R};$$

(iv)

$$2c\|K_P\| < 1 \tag{31}$$

*where* $\|K_P\|$ *is the norm of* $K_P: R(L) \subset C^0 \to C^s$, $s = \max(p, r)$;

(v) *for* $\varepsilon = 1$ *or* $\varepsilon = -1$

$$\lim_{x \to \infty} \varepsilon f(t, x) = \infty, \quad \lim_{x \to -\infty} \varepsilon f(t, x) = -\infty \text{ uniformly in } t \in [a, b]. \tag{32}$$

*Then all statements of Theorem* 4 *hold for the BVP* (17′), (2).

Proof. We shall show that all assumptions of Theorem 5 concerning the BVP (17′), (2) hold and by this theorem Theorem 6 follows. The assumptions (i)–(iii) imply that the problem (1), (2) is not regular and the assumptions (a)–(c) of Lemma 4 are satisfied where instead of $d$ we have $d+\delta$. In view of (31) there exists a $d_1$ such that

$$0 < d_1 < 1 \tag{33}$$

and (27) is true. Hence the assumption (d) of Lemma 4 is also fulfilled.

Now with respect to (i) we can choose the projector $P(x) = \frac{1}{b-a}\int_a^b x(t)\,\mathrm{d}t$, $x \in C^s$. Then $P(x) = \bar{x}$ for each $x \in D(L)$ and $x(t) = \bar{x} + \tilde{x}(t)$, $a \leqslant t \leqslant b$. On the basis of (33), the condition $\|\tilde{x}\|_s \leqslant d_1\|\bar{x}\|_s$ implies that $|\tilde{x}(t)| \leqslant d_1|\bar{x}|$, $a \leqslant t \leqslant b$ and hence

(34) $x(t) \geqslant (1-d_1)\bar{x}$ for $\bar{x} > 0$ and $x(t) \leqslant (1-d_1)\bar{x}$ for $\bar{x} < 0$, $a \leqslant t \leqslant b$, $x \in D(L)$.

Suppose further that $g$ satisfies (16). Since $F\colon C^p \to C^0$ is now determined by the relation

$$F(x) = f \circ x + h \circ x \tag{26'}$$

and $\varepsilon F(\bar{x} + \tilde{x}) + \varepsilon g + k\bar{x} \in R(L)$ if and only if

$$\int_a^b \varepsilon\big[f(t, \bar{x} + \tilde{x}(t)) + g(t) + h(t, \bar{x} + \tilde{x}(t), \ldots, \tilde{x}^{(p)}(t))\big]\,\mathrm{d}t + k\bar{x}(b-a) = 0,$$

on the basis of (34) we get that both conditions (e) and (f) of Lemma 4 will be satisfied if for all sufficiently great $|\bar{x}|$ and $k \geqslant 0$ we have

$$\operatorname{sign}\big[\varepsilon f(t, \bar{x} + \tilde{x}(t)) + \varepsilon g + \varepsilon h(t, x(t), \ldots, x^{(p)}(t)) + k\bar{x}\big] = \operatorname{sign}\bar{x}.$$

This follows by the boundedness of $g, h$ and (32). The proof is complete. □

If $L(x) = x^{(n)}$ and the conditions (2) are of the form

$$x^{(i)}(a) - x^{(i)}(b) = 0, \quad i = 0, \ldots, n-1, \tag{35}$$

then the condition (i) is satisfied and $A = cI\colon C^0 \to C^0$ where $c \neq 0$ is sufficiently small and $I$ is the identity in $C^0$. Hence $r = 0$ and thus $s = p$ in conditions (ii) and (iv), respectively.

**Corollary 2.** *If the conditions* (iii), (iv) *(with $s = p$) and* (v) *of Theorem* 6 *are satisfied, then all statements of Theorem* 4 *hold for the BVP* (35),

$$x^{(n)} = f(t, x) + g(t) + h(t, x, \ldots, x^{(p)}). \tag{36}$$

Remark 5. The operator $K_P$ for certain periodic BVP-s is constructed in [9], [11].

Consider a special case of (36), (35), namely the BVP (35),

$$x^{(n)} = f(x) + g(t) + h(t,x) \tag{37}$$

where $f \in C(\mathbb{R}, \mathbb{R})$. Similarly as in [2], the results will depend on the fact whether $n$ is odd or even. We will denote the scalar product and the norm in $L^2([a,b],\mathbb{R})$ by $(.,.)$ and $\|\cdot\|_{L^2}$, respectively.

**Lemma 5.** *Suppose that $n = 2m+1$, $n \geqslant 3$, $\varepsilon = 1$ or $\varepsilon = -1$, $f$ satisfies the condition*

$$\lim_{x\to\infty} \varepsilon f(x) = \infty, \quad \lim_{x\to-\infty} \varepsilon f(x) = -\infty, \tag{32'}$$

*$g$ fulfils (16) and $h$ satisfies*

$$|h(t,x_1)| \leqslant \delta \text{ for } a \leqslant t \leqslant b, \quad x_1 \in \mathbb{R} \tag{38}$$

*with a $\delta > 0$. Then the following statements hold:*

(1) *There exists a constant $R > 0$ such that each possible solution $x$ of the BVP (37), (35) where $x(t) = \bar{x} + \tilde{x}(t)$, $a \leqslant t \leqslant b$, $\bar{x} = \frac{1}{b-a}\int_a^b x(t)\,\mathrm{d}t$, satisfies the inequalities*

$$|\bar{x}| \leqslant R, \tag{39}$$

$$\|\tilde{x}\|_0 \leqslant \frac{1}{3}\left(\frac{b-a}{2\pi}\right)^{n-2}\left(\frac{b-a}{2}\right)^2 (M+\delta). \tag{40}$$

(2) *For each $c_1 > 0$ sufficiently small there exists an $R_1 > 0$ such that all possible solutions $x(t) = \bar{x} + \tilde{x}(t)$, $a \leqslant t \leqslant b$, of (35),*

$$x^{(n)} - (1-\mu)\varepsilon c_1 x = \mu[f(x) + h(t,x)], \quad 0 < \mu < 1 \tag{41}$$

*satisfy the inequalities*

$$|\bar{x}| \leqslant R_1 \tag{39'}$$

$$\|\tilde{x}\|_0 \leqslant \frac{1}{3}\left(\frac{b-a}{2\pi}\right)^{n-2}\left(\frac{b-a}{2}\right)^2 \delta. \tag{40'}$$

P r o o f. 1. If $x$ is a possible solution of (37), (35), then, similarly as in Lemma 1, [2], we get

$$(-1)^{\frac{n-1}{2}}\left\|x^{\left(\frac{n+1}{2}\right)}\right\|_{L^2}^2=(x^{(n)},x')=\big(f(x(.)),x'\big)+(g,x')+\big(h(.,x(.)),x'\big)$$

and in view of (35), (16), (38) we have

$$\left\|x^{\left(\frac{n+1}{2}\right)}\right\|_{L^2}^2\leqslant(M+\delta)(b-a)\|x'\|_0. \tag{42}$$

By Sobolev and Wirtinger inequalities ([12], pp. 216–217),

$$\|x'\|_0\leqslant 3^{-1/2}\frac{(b-a)^{1/2}}{2}\left(\frac{b-a}{2\pi}\right)^{\frac{n-3}{2}}\left\|x^{\left(\frac{n+1}{2}\right)}\right\|_{L^2}. \tag{43}$$

(42) and (43) imply that

$$\left\|x^{\left(\frac{n+1}{2}\right)}\right\|_{L^2}\leqslant\frac{b-a}{2}\left(\frac{b-a}{3}\right)^{1/2}\left(\frac{b-a}{2\pi}\right)^{\frac{n-3}{2}}(M+\delta)$$

and thus

$$\|x'\|_{L^2}\leqslant\frac{b-a}{2}\left(\frac{b-a}{3}\right)^{1/2}\left(\frac{b-a}{2\pi}\right)^{n-2}(M+\delta),$$

which implies (40).

If (39) were not true, there would exist solutions $x_k(t)=\bar{x}_k+\tilde{x}_k(t)$, $a\leqslant t\leqslant b$, $k=1,2,\dots$ of (37), (35) such that either $\lim\limits_{k\to\infty}\bar{x}_k=\infty$ or $\lim\limits_{k\to\infty}\bar{x}_k=-\infty$. Only the first case will be considered. Then in view of (16), (38), (40) and the first condition in (38′) $f(x_k(t))+g(t)+h(t,x_k(t))$ would be of constant sign for all sufficiently great $k$ and hence, $x_k^{(n-1)}(b)-x_k^{(n-1)}(a)\neq 0$. This contradiction with (35) proves (39). We remark that the contradiction is also attained in the case when $x_k$ are solutions of $x^{(n)}=f(x)+g_k(t)+h(t,x,\dots,x^{(p)})$ and all $g_k$ satisfy $|g_k(t)|\leqslant M$, $a\leqslant t\leqslant b$.

2. If we start with (41) instead of (37) and proceed in the same way as above, we come to the inequality

$$\left\|x^{\left(\frac{n+1}{2}\right)}\right\|_{L^2}^2\leqslant\delta(b-a)\|x'\|_0$$

which now replaces (42). This inequality leads to (40′).

If there existed solutions $x_k(t)=\bar{x}_k+\tilde{x}_k(t)$, $a\leqslant t\leqslant b$, $k=1,2,\dots$ of $x^{(n)}=(1-\mu_k)\varepsilon c_1x+\mu_k[f(x)+h(t,x)]$ with $\lim\limits_{k\to\infty}\bar{x}_k=\infty$, then with respect to (38), (40′) and the first condition in (32′), the functions $(1-\mu_k)\varepsilon c_1x_k(t)+\mu_k\big[f(x_k(t))+h(t,x_k(t)\big]$ would be of constant sign for all sufficiently great $k$. This again contradicts (35) and thus (39′) is true. Similarly we proceed in the case when $\lim\limits_{k\to\infty}\bar{x}_k=-\infty$. □

**Lemma 6.** *Suppose that $n = 2m$, $n \geqslant 2$, $\varepsilon = 1$ or $\varepsilon = -1$, $f$ satisfies (32′) and the following condition:*

*There exists $\beta$, $0 \leqslant \beta < \left(\frac{2\pi}{b-a}\right)^n$, such that*

$$(-1)^{n/2}(f(v) - f(w))(v - w) \leqslant \beta(v - w)^2, \quad v, w \in \mathbb{R}, \tag{44}$$

*$g$ fulfils (16) and $h$ satisfies (38). Then the following statements hold:*

(1) *There exists a constant $R > 0$ such that each possible solution $x$ of the BVP (37), (35) where $x(t) = \bar{x} + \tilde{x}(t)$, $a \leqslant t \leqslant b$, $\bar{x} = \frac{1}{b-a}\int_a^b x(t)\,\mathrm{d}t$, satisfies the inequalities*

$$|\bar{x}| \leqslant R, \tag{45}$$

$$\|\tilde{x}\|_0 \leqslant \Big(1 - \beta\Big(\frac{b-a}{2\pi}\Big)^n\Big)^{-1}\frac{1}{3}\Big(\frac{b-a}{2}\Big)^2\Big(\frac{b-a}{2\pi}\Big)^{n-2}(M + \delta). \tag{46}$$

(2) *For each $c_1$, $0 < c_1 < \beta$, $c_1$ sufficiently small, there exists an $R_1 > 0$ such that all possible solutions $x(t) = \bar{x} + \tilde{x}(t)$, $a \leqslant t \leqslant b$ of (41), (35) satisfy the inequalities*

$$|\bar{x}| \leqslant R_1, \tag{45′}$$

$$\|\tilde{x}\|_0 \leqslant \Big(1 - \beta\Big(\frac{b-a}{2\pi}\Big)^n\Big)^{-1}\frac{1}{3}\Big(\frac{b-a}{2}\Big)^2\Big(\frac{b-a}{2\pi}\Big)^{n-2}\delta. \tag{46′}$$

P r o o f. 1. If $x$ is a possible solution of the BVP (37), (35), then, similarly as in the proof of Lemma 2, [2], we get that

$$\begin{aligned}\big\|x^{(\frac{n}{2})}\big\|_{L^2}^2 &= (-1)^{\frac{n}{2}}(x^{(n)}, \tilde{x})\\ &= (-1)^{\frac{n}{2}}\big(f(x(.)) - f(\bar{x}), x - \bar{x}\big) + (-1)^{\frac{n}{2}}(g, \tilde{x}) + (-1)^{\frac{n}{2}}\big(h(., x(.)), \tilde{x}\big).\end{aligned}$$

Then by (44), (16), (38)

$$\big\|x^{(\frac{n}{2})}\big\|_{L^2}^2 \leqslant \beta\|\tilde{x}\|_{L^2}^2 + (b-a)(M + \delta)\|\tilde{x}\|_0. \tag{47}$$

Again Sobolev and Wirtinger inequalities imply that

$$\|\tilde{x}\|_{L^2} \leqslant \left(\frac{b-a}{2\pi}\right)^{\frac{n}{2}}\big\|x^{(\frac{n}{2})}\big\|_{L^2}, \tag{48}$$

$$\|\tilde{x}\|_0 \leqslant \frac{1}{2}\left(\frac{b-a}{3}\right)^{\frac{1}{2}}\left(\frac{b-a}{2\pi}\right)^{\frac{n-2}{2}}\big\|x^{(\frac{n}{2})}\big\|_{L^2}. \tag{49}$$

From (47), (48) and (49) we get

$$\|x^{(\frac{n}{2})}\|_{L^2} \leqslant \left(1-\beta\left(\frac{b-a}{2\pi}\right)^n\right)^{-1} \frac{b-a}{2}\left(\frac{b-a}{3}\right)^{\frac{1}{2}}\left(\frac{b-a}{2\pi}\right)^{\frac{n-2}{2}} (M+\delta)$$

and further

$$\|x'\|_{L^2} \leqslant \left(1-\beta\left(\frac{b-a}{2\pi}\right)^n\right)^{-1} \frac{b-a}{2}\left(\frac{b-a}{3}\right)^{\frac{1}{2}}\left(\frac{b-a}{2\pi}\right)^{n-2} (M+\delta),$$

which implies (46).

If (45) were not true, then similarly as in the proof of statement 1 in Lemma 5, the existence of solutions $x_k(t)=\bar{x}_k+\tilde{x}_k(t)$, $a\leqslant t\leqslant b$, $k=1,2,\ldots$ of (37), (35) with the property $\lim\limits_{k\to\infty}\bar{x}_k=\infty$ or $\lim\limits_{k\to\infty}\bar{x}_k=-\infty$ in view of (46), (16), (38) would imply that $f(x_k(t))+g(t)+h(t,x_k(t))$ is of constant sign for all sufficiently great $k$ and this would lead to a contradiction with (35). Thus (45) is proved.

2. Since $0<c_1<\beta$ and $(x,\bar{x})=(\tilde{x},\tilde{x})$, each solution $x(t)=\bar{x}+\tilde{x}(t)$, $a\leqslant t\leqslant b$ of (41), (35) satisfies the inequality

$$\begin{aligned}\|x^{(\frac{n}{2})}\|_{L^2}^2 &\leqslant [(1-\mu)c_1+\mu\beta]\|\tilde{x}\|_{L^2}^2+(b-a)\delta\|\tilde{x}\|_0\\ &\leqslant \beta\|\tilde{x}\|_{L^2}^2+(b-a)\delta\|\tilde{x}\|_0\end{aligned}$$

which replaces (47). Therefore (46) with $M=0$ implies (46′). The inequality (45′) can be proved in the same way as (39′) has been proved. □

Remark 6. It is clear that the condition (44) is equivalent to the following condition: If $n=4m$ ($n=4m+2$), $m\geqslant 1$, then the function $F(x)=f(x)-\beta x$ ($F(x)=f(x)+\beta x$) is nonincreasing (nondecreasing) in $\mathbb{R}$.

**Theorem 7.** *Suppose that $n\geqslant 2$, $\varepsilon=1$ or $\varepsilon=-1$, $f$ satisfies (32′), $h$ fulfils (38) and when $n=2m$, there exists $\beta$, $0\leqslant\beta<\left(\frac{2\pi}{b-a}\right)^n$ such that (44) is satisfied. Then the statements 1–3 of Theorem 4 hold for the BVP (37), (35). Moreover, if $f\in C^1(\mathbb{R},\mathbb{R})$ and $\frac{\partial h}{\partial x}\in C([a,b]\times\mathbb{R},\mathbb{R})$, then also statement 4 of Theorem 4 holds for that BVP.*

Proof. Proceeding in a similar way as in the proof of Theorem 4 we see that the assumptions (H.1), (H.2) and in the case that $f'$, $\frac{\partial h}{\partial x}$, are continuous, also (H.4) of Theorems 4.1 and 4.2, [20], are satisfied. Since the BVP $x^{(n)}-\varepsilon c_1 x=0$, (35), is regular for all sufficiently small $c_1>0$, the a priori estimates given in Lemmas 5 and 6 imply that also (H.3) and (H.5) are fulfilled. Then the result follows by Theorems 4.1 and 4.2 as well as by Corollary 4.2 in [20]. □

By Remark 6, Theorem 3 in [19] and Theorem 5.2 in [20], Theorem 7 implies the following corollary.

**Corollary 3.** *Suppose that $n \geqslant 2$, $\varepsilon = 1$ or $\varepsilon = -1$, $f$ satisfies (32′), $h$ fulfils (38) with a positive constant $\delta$ and that the following conditions are true:*

(a) *If $n = 2m+1$, then the function $f(.) + h(t,.)$ is either nonincreasing in $\mathbb{R}$ or nondecreasing in $\mathbb{R}$ for every $t \in [a,b]$.*

(b) *If $n = 4m$, then the function $f(.)+h(t,.)$ is nonincreasing in $\mathbb{R}$ for every $t \in [a,b]$ and there exists a $\beta$, $0 \leqslant \beta < \left(\frac{2\pi}{b-a}\right)^n$ such that the function $f(x) - \beta x$ of the variable $x$ is nonincreasing in $\mathbb{R}$.*

(c) *If $n = 4m-2$, then the function $f(.) + h(t,.)$ is nondecreasing in $\mathbb{R}$ for every $t \in [a,b]$ and there exists a $\beta$, $0 \leqslant \beta < \left(\frac{2\pi}{b-a}\right)^n$ such that the function $f(x) + \beta x$ of the variable $x$ is nondecreasing in $\mathbb{R}$. Then there exists a closed set $R_b \subset C^0$ such that for each $g \in C^0 \setminus R_b$ the BVP (37), (35) has a unique solution, for each $g \in R_b$ the set $S_g$ of all solutions of that BVP is convex and $R_b = \overline{R}_2$, where $R_2 \subset R_b$ is the set of all $g \in C^0$ for which the BVP (37), (35) has infinitely many solutions.*

Remark 7. We see that under the assumptions of Corollary 3 the following alternative holds: Either the BVP (37), (35) has a unique solution or it has infinitely many solutions, more precisely a nontrivial convex compact set of solutions.

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