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A NOTE ON ONE OF THE BERNSTEIN THEOREMS

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Summary. One of the Bernstein theorems asserts that the class of bounded functions of the exponential type is dense in the space of bounded and uniformly continuous functions. This theorem follows from a convergence theorem for some interpolating operators on the real axis.

Keywords: Bernstein's inequality, function of exponential type, interpolating operator, uniform norm, space of uniformly continuous functions.

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1. Preliminaries and Notation

An entire function is said to be of the exponential type if there are $A, B \in \mathbf{R}$ such that

$$|f(z)| \leqslant A e^{B|z|}$$

for all $z \in \mathbf{C}$.

The type of the function f is the number

$$\sigma = \limsup_{|z|\to\infty} \frac{\ln|f(z)|}{|z|}.$$

We denote by C_{UB} the normed space of all bounded and uniformly continuous real (or complex) functions on the real axis with topology induced by the uniform norm

$$\|f\| = \sup\{|f(x)|;\, x \in \mathbf{R}\},$$

and by B_σ the class of all entire functions of the exponential type less than or equal to σ which are bounded on the real axis $\mathbf{R}$. We put

$$B_\infty = \bigcup_{\sigma \geqslant 0} B_\sigma .$$

For every function $f \in B_\sigma$ the so called Bernstein's inequality

$$\|f'\| \leqslant \sigma \|f\|$$

can be proved (cf. e.g. [1]). It follows from this inequality that every function from B_∞ is uniformly continuous. Thus $B_\infty \subset C_{UB}$.

Using Cauchy's method of decomposition meromorphic functions into simple fractions we can deduce the well known fact that for all $x \in \mathbf{R}$

$$\sum_{k=-\infty}^{+\infty} \frac{\sin^2(\sigma x - k\pi)}{(\sigma x - k\pi)^2} = 1 \tag{1}$$

and the series on the left hand side of (1) is absolutely and locally uniformly convergent in $\mathbf{C}$. (We define $\frac{\sin z}{z} = 1$ for $z = 0$.) The function $f_\sigma \in C_{UB}$:

$$f_\sigma(x) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin^2(\sigma x - k\pi)}{(\sigma x - k\pi)^2}$$

is a function of exponential type since the following estimate holds (where $x = re^{i\varphi}$, $m\pi < \sigma r \leqslant (m+1)\pi$, $m \geqslant 0$ integer):

$$\begin{aligned}
|f_\sigma(x)| &\leqslant \|f\| \sum_{k=-\infty}^{+\infty} \left|\frac{\sin \sigma x}{\sigma x - k\pi}\right|^2 \\
&= \|f\| \Bigg(\sum_{|k| \leqslant m-1} \left|\frac{\sin \sigma x}{\sigma x - k\pi}\right|^2 + \sum_{|k|=m}^{m+2} \left|\frac{\sin \sigma x}{\sigma x - k\pi}\right|^2 + \sum_{|k| \geqslant m+3} \left|\frac{\sin \sigma x}{\sigma x - k\pi}\right|^2 \Bigg) \\
&= 2|\sin \sigma x|^2 O\Bigg(\sum_{k=0}^{m-1} \frac{1}{(\sigma r - k\pi)^2} \Bigg) + O(e^{2\sigma r}) \\
&\quad + 2|\sin \sigma x|^2 O\Bigg(\sum_{k=m+3}^{+\infty} \frac{1}{(\sigma r - k\pi)^2} \Bigg) \\
&= O(e^{2\sigma r}).
\end{aligned}$$

So we can introduce correctly an interpolating operator $\mathbf{T}_\sigma : C_{UB} \to B_\infty$ for every $\sigma > 0$ by the formula

$$(2) \qquad (\mathbf{T}_\sigma f)(x) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin^2(\sigma x - k\pi)}{(\sigma x - k\pi)^2}.$$

We shall prove a convergence theorem for the operators (2).

2. Convergence Theorem

Theorem. *For each function $f \in C_{UB}$*

$$\mathbf{T}_\sigma f \to f \quad \textit{for} \quad \sigma \to \infty$$

in the strong topology of the space C_{UB}.

Proof. Let us choose any $\varepsilon > 0$. Because f is uniformly continuous there exists $\delta > 0$ such that for $x_1, x_2 \in \mathbf{R}$, $|x_1 - x_2| < \delta$, we have $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2}$. Let us choose any real number x and divide all integers into disjoint sets A_n, $n = 0, 1, \ldots$:

$$A_n = \left\{k \in \mathbf{Z};\, n\delta \leqslant \left|x - \frac{k\pi}{\sigma}\right| < (n+1)\delta\right\}.$$

It is easy to see that card $A_n \leqslant \frac{2\delta\sigma}{\pi}$. Using (2) we then have

$$\begin{aligned}
&\left|f(x) - \sum_{k=-\infty}^{+\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin^2(\sigma x - k\pi)}{(\sigma x - k\pi)^2}\right| \\
&\quad = \left|\sum_{k=-\infty}^{+\infty} \left[f(x) - f\left(\frac{k\pi}{\sigma}\right)\right] \frac{\sin^2(\sigma x - k\pi)}{(\sigma x - k\pi)^2}\right| \\
&\quad = \left|\sum_{n=0}^{\infty}\sum_{k\in A_n} \left[f(x) - f\left(\frac{k\pi}{\sigma}\right)\right] \frac{\sin^2(\sigma x - k\pi)}{(\sigma x - k\pi)^2}\right| \\
&\quad < \frac{\varepsilon}{2} + \sum_{n=1}^{\infty}\left|\sum_{k\in A_n} \left[f(x) - f\left(\frac{k\pi}{\sigma}\right)\right] \frac{\sin^2(\sigma x - k\pi)}{(\sigma x - k\pi)^2}\right| \\
&\quad \leqslant \frac{\varepsilon}{2} + \frac{2\|f\|}{\sigma^2}\sum_{n=1}^{\infty}\sum_{k\in A_n}\frac{1}{n^2\delta^2} \leqslant \frac{\varepsilon}{2} + \frac{2\|f\|}{\sigma^2}\sum_{n=1}^{\infty}\frac{2\sigma}{\pi n^2\delta} \\
&\quad = \frac{\varepsilon}{2} + \frac{2\pi\|f\|}{3\delta\sigma} < \varepsilon \quad \textit{for} \quad \sigma > \sigma_1 = \frac{4\pi\|f\|}{3\delta\varepsilon}.
\end{aligned}$$

Hence $\|\mathbf{T}_\sigma f - f\| \to 0$ for $\sigma \to \infty$ and the proof is complete. □

3. Remark

The theorem that has just been proved yields $\overline{B}_\infty = C_{UB}$. This result is well-known as the Bernstein theorem (see e.g. [2]).The reader can compare our proof with that of [3].

References

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Souhrn

POZNÁMKA K JEDNÉ BERNŠTEJNOVĚ VĚTĚ

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V článku je definován soubor operátorů $(\mathbf{T}_\sigma)_{\sigma>0}$ vztahem (1) a dokázáno, že $\mathbf{T}_\sigma f \rightarrow f$ $(\sigma \rightarrow \infty)$ v prostoru omezených a stejnoměrně spojitých funkcí s topologií indukovanou stejnoměrnou normou.

Z této věty plyne, že v tomto prostoru je hustá třída celých funkcí exponenciálního typu omezených na reálné ose (což je jedna z Bernštejnových vět).

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