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ON AN ESTIMATE OF A FUNCTIONAL IN THE CLASS OF HOLOMORPHIC UNIVALENT FUNCTIONS

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Summary. Let S denote the class of functions $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ univalent and holomorphic in the unit disc $\Delta = \{z\colon |z| < 1\}$. In the paper we obtain an estimate of the functional $|a_3 - ca_2^2| + c|a_2|^n$ in the class S for arbitrarily fixed $c \in \mathbf{R}$ and $n = 1, 2, 3, \ldots$. Hence, for some special values of the parameters, we obtain estimates of several interesting functionals and numerous applications. A few open problems of a similar type are also formulated.

Keywords: univalent function, coefficient problem

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1. Introduction

Let S stand for the well-known class of functions

$$f(z) = z + a_2z^2 + a_3z^3 + \ldots \tag{1.1}$$

holomorphic and univalent in the unit disc $\Delta = \{z\colon |z| < 1\}$. As known, in many papers the functional $|a_3 - ca_2^2|$ was studied for different classes of univalent functions of the form (1.1). The first results are due to Fekete and Szegö [4] and Goluzin [5]. For this reason, we shall call $|a_3 - ca_2^2|$ the Fekete-Szegö-Goluzin functional (abbr. FSG functional). In turn, the interest in studying this functional arises from the possibilities of applying it in many other extremal problems. The studies were also taken up (e.g. [6], [7]) for the purpose of establishing the influence of the coefficient a_2 on the form of extremal functions with respect to the FSG functional, for instance, by considering the functional $Re[a_2^2(a_3 - ca_2^2)]$, $c \in \mathbf{R}$, $f \in S$ [6] or the functional $a_2^m(a_3 - ca_2^2)$, $c \in \mathbf{R}$, $m = 0, 1, 2, \ldots$, $f \in S_R \subset S$ [7].

It is well known that, for each function $f \in S$,

$$(1.2) \qquad |a_2| \leqslant 2$$

with equality occurring only for the Koebe function

$$(1.3) \qquad f_0(z) = \frac{z}{(1-\varepsilon z)^2}, \qquad z \in \Delta, \ |\varepsilon| = 1.$$

However, the maximum of the FSG functional for $c \in (0,1)$ ([5]) is not attained for function (1.3).

During the 10^{th} Conference on Analytic Functions at Szczyrk, Zyskowska presented the main result from [10] indirectly connected with the particular case of the FSG functional.

During the discussion, S. Ruscheweyh raised the question concerning the estimation of the functional $|a_3 - ca_2^2| + c|a_2|^2$, $c \in \mathbf{R}$, $f \in S$. Hence the idea arises to consider the functional

$$(1.4) \qquad \mathcal{F}(f) = |a_3 - ca_2^2| + c|a_2|^n, \qquad c \in \mathbf{R}, \ \ n \in \mathbf{N} \equiv \{1,2,3,\ldots\},$$

in the class S. The aim of our paper is to determine the maximum of functional (1.4) for all n and c belonging to the respective intervals. For the remaining values of c, we shall give the bounds from above. The set of pairs (c,n) such that the Koebe function is extremal seems to be rather interesting. In the paper we also show some applications and related open problems.

Obviously, from (1.2) and the well-known result of Jenkins [8] we immediately get

$$|a_3 - ca_2^2| + c|a_2|^n \leqslant 4c - 3 + 2^n c, \qquad c \geqslant 1, \ \ n \in \mathbf{N},$$

with equality occurring only for Koebe function (1.3). Consequently, it would be sufficient to limit our further considerations to the case $c < 1$. However, on account of the method applied, our reasoning is carried out for all $c \in \mathbf{R}$.

One also knows that if $f \in S$, then, for all $\theta \in \langle 0, 2\pi)$, the function $f_\theta(z) = e^{i\theta} f(e^{-i\theta} z)$, $z \in \Delta$, belongs to S, too. Thus, the determination of the maximum of functional (1.4) is equivalent to the determination of the maximum of the functional

$$(1.5) \qquad G(f) = Re(a_3 - ca_2^2) + c|a_2|^n, \qquad f \in S, \ \ c \in \mathbf{R}, \ \ n \in \mathbf{N}.$$

2. An application of the Valiron-Landau lemma

From the Löwner theorem we have the following representations:

$$a_2 = -2\int_0^{\infty} e^{-t}k(t)\,dt,$$

$$a_3 = -2\int_0^{\infty} e^{-2t}k^2(t)\,dt + 4\Big(\int_0^{\infty} e^{-t}k(t)\,dt\Big)^2,$$

where $k(t)$ is a piecewise continuous complex-valued function with $|k(t)| = 1$ for all t. Setting $k(t) = e^{i\theta(t)}$, we obtain

$$\begin{aligned} G(f) = {} & 4(1-c)\Big[\Big(\int_0^{\infty} e^{-t}\cos\theta(t)\,dt\Big)^2 - \Big(\int_0^{\infty} e^{-t}\sin\theta(t)\,dt\Big)^2\Big] \\ & - 4\int_0^{\infty} e^{-2t}\cos^2\theta(t)\,dt + 1 \\ & + 2^n c\Big[\Big(\int_0^{\infty} e^{-t}\cos\theta(t)\,dt\Big)^2 + \Big(\int_0^{\infty} e^{-t}\sin\theta(t)\,dt\Big)^2\Big]^{n/2}. \end{aligned} \tag{2.1}$$

Put

$$u = \int_0^{\infty} e^{-t}\cos\theta(t)\,dt, \qquad v = \int_0^{\infty} e^{-t}\sin\theta(t)\,dt. \tag{2.2}$$

Let x be a nonnegative root of the equation

$$\int_0^{\infty} e^{-2t}\cos^2\theta(t)\,dt = w(x) \tag{2.3}$$

where

$$w(x) = \Big(x + \frac{1}{2}\Big)e^{-2x}, \qquad x \geqslant 0. \tag{2.4}$$

Then, by the Valiron-Landau lemma [3], we get

$$\Big|\int_0^{\infty} e^{-t}\cos\theta(t)\,dt\Big| \leqslant \omega(x), \tag{2.5}$$

with

$$\omega(x) = (x+1)e^{-x}, \qquad x \geqslant 0,$$

where the form of the functions $\theta(t)$ for which the equality sign in (2.5) holds is known.

Since

$$(2.6) \qquad w(y) = \int_0^\infty e^{-2t} \sin^2 \theta(t)\, dt = \frac{1}{2} - \int_0^\infty e^{-2t} \cos^2 \theta(t)\, dt = \frac{1}{2} - w(x),$$

we also have

$$(2.7) \qquad \left| \int_0^\infty e^{-t} \sin \theta(t)\, dt \right| \leqslant \omega(y)$$

where $y = w^{-1}(\frac{1}{2} - w(x))$, $y \geqslant 0$.

From (2.2) and (2.1) we obtain

$$(2.8) \qquad \begin{aligned} G(f) = {} & 4(1-c)(u^2 - v^2) \\ & - 4\int_0^\infty e^{-2t} \cos^2 \theta(t)\, dt + 1 + 2^n c(u^2 + v^2)^{n/2}. \end{aligned}$$

Estimate (1.2) and formula (2.2) imply

$$(2.9) \qquad u^2 + v^2 \leqslant 1.$$

By using the Valiron-Landau lemma and taking account of inequality (2.9), the problem of determining the maximum of $G(f)$ will be reduced to the investigation of the maxima of some functions of the variable x (or y) where x is defined by (2.3).

Denote by $G(u^2, v^2; x)$ the right-hand side of (2.8), i.e.

$$(2.8') \qquad \begin{aligned} G(u^2, v^2; x) \equiv {} & 4(1-c)(u^2 - v^2) \\ & - 4w(x) + 1 + 2^n c(u^2 + v^2)^{n/2}. \end{aligned}$$

Note that, for a fixed admissible x (and, in consuequence, for a certain subclass of S), the form of function (2.8′) and (2.5), (2.7), (2.9) imply that the maximum of $G(u^2, v^2; x)$ can be attained only in the cases when

1^0 $u^2 = 0$ and $v^2 = 0$,
2^0 $u^2 = \omega^2(x)$ and $v^2 = 0$,
3^0 $u^2 = \omega^2(x)$ and $v^2 = 1 - u^2$,
4^0 $u^2 = 1 - v^2$ and $v^2 = \omega^2(y)$,
5^0 $u^2 = 0$ and $v^2 = \omega^2(y)$,

with $x \geqslant 0$ and $y \geqslant 0$.

In this way, we shall obtain the above-mentioned functions of x or y, whose maxima can determine the sought-for maximum of the functional $G(f)$.

From (2.8′) and (2.3)–(2.7) we have successively for $c \in \mathbf{R}$ and $n \in \mathbf{N}$:

by case 1^0, $G(u^2, v^2; x) = H_1(x)$ where

$$H_1(x) = -4\left(x + \frac{1}{2}\right) e^{-2x} + 1, \qquad x \geqslant 0; \tag{2.10}$$

by case 2^0, $G(u^2, v^2; x) = H_2(x)$ where

$$\begin{aligned} H_2(x) = 4&\left[(1-c)\,x^2 + (1-2c)\,x + \frac{1}{2} - c\right] e^{-2x} \\ &+ 1 + 2^n c\left[(x+1)\,e^{-x}\right]^n, \quad x \geqslant 0; \end{aligned} \tag{2.11}$$

by case 3^0, $G(u^2, v^2; x) = H_3(x)$ where

$$\begin{aligned} H_3(x) = 4&\left[2(1-c)\,x^2 + (3-4c)\,x + \frac{3}{2} - 2c\right] e^{-2x} \\ &+ 4c - 3 + 2^n c, \qquad x \geqslant 0; \end{aligned} \tag{2.12}$$

by case 4^0, $G(u^2, v^2; y) = H_4(y)$ where

$$\begin{aligned} H_4(y) = -4&\left[2(1-c)\,y^2 + (3-4c)\,y + \frac{3}{2} - 2c\right] e^{-2y} \\ &- 4c + 3 + 2^n c, \qquad y \geqslant 0; \end{aligned} \tag{2.13}$$

by case 5^0, $G(u^2, v^2; y) = H_5(y)$ where

$$\begin{aligned} H_5(y) = -4&\left[(1-c)\,y^2 + (1-2c)\,y + \frac{1}{2} - c\right] e^{-2y} \\ &- 1 + 2^n c[(y+1)\,e^{-y}]^n, \qquad y \geqslant 0. \end{aligned} \tag{2.14}$$

Now, we shall determine the maxima of the functions H_k, $k = 1,2,3,4,5$, for any fixed $c \in \mathbf{R}$ and $n \in \mathbf{N}$. It is easily seen that

$$H_1(x) \leqslant H_1(\infty) = 1, \qquad c \in \mathbf{R}, \ \ n \in \mathbf{N}, \tag{2.15}$$

where H_1 is defined by formula (2.10).

Examining the function $H_2(x)$ given by (2.11), we conclude:

if $n = 1$, then

$$H_2(x) \leqslant \begin{cases} H_2(0) = 3 - 2c & \text{when } \ c \leqslant 0, \\ H_2(x_0) & \text{when } \ c > 0, \end{cases} \tag{2.16}$$

where x_0 is the positive root of the equation

$$4e^{-x}[(1-c)\,x - c] + c = 0; \tag{2.17}$$

if $n = 2$, then

$$H_2(x) \leqslant H_2(0) = 3, \qquad c \in \mathbf{R}; \tag{2.18}$$

if $n \geqslant 3$, then

$$H_2(x) \leqslant \begin{cases} H_2(x_1) & \text{when } c < 0, \\ H_2(0) = 3 - 4c + 2^n c & \text{when } c \geqslant 0, \end{cases} \tag{2.19}$$

where x_1 is the positive root of the equation

$$8e^{-x}[(1-c)x - c] + 2^n n c[(x+1)e^{-x}]^{n-1} = 0. \tag{2.20}$$

In turn, examining the function $H_3(x)$ given by (2.12), we obtain, for $n \in \mathbf{N}$,

$$H_3(x) \leqslant \begin{cases} H_3(0) = 3 - 4c + 2^n c & \text{when } c \leqslant \frac{1}{2}, \\ H_3(x_2) & \text{when } \frac{1}{2} < c < 1, \\ H_3(\infty) = -3 + 4c + 2^n c & \text{when } c \geqslant 1, \end{cases} \tag{2.21}$$

where $x_2 = (2c-1)/2(1-c)$.

From the examination of the function $H_4(y)$ given by (2.13) we have, for $n \in \mathbf{N}$,

$$H_4(y) \leqslant \begin{cases} H_4(\infty) = 3 - 4c + 2^n c & \text{when } c < \frac{3}{4}, \\ H_4(0) = -3 + 4c + 2^n c & \text{when } c \geqslant \frac{3}{4}. \end{cases} \tag{2.22}$$

Finally, from the investigation of the function $H_5(y)$ given by formula (2.14) we get:

if $n = 1$, then

$$H_5(y) \leqslant \begin{cases} H_5(\infty) = -1 & \text{when } c \leqslant 0, \\ H_5(y_0) & \text{when } 0 < c < \frac{1}{3}, \\ H_5(0) = 3(2c-1) & \text{when } c \geqslant \frac{1}{3}, \end{cases} \tag{2.23}$$

where y_0 is the greater positive root of the equation

$$4e^{-y}[(1-c)y - c] - c = 0; \tag{2.24}$$

if $n \geqslant 2$, then

$$H_5(y) \leqslant \begin{cases} H_5(\infty) = -1 & \text{when } c < 2/(4+2^n), \\ H_5(0) = -3 + 4c + 2^n c & \text{when } c \geqslant 2/(4+2^n). \end{cases} \tag{2.25}$$

Next, we shall carry out a comparison of the estimates of functions H_k, $k = 1,2,3,4,5$, obtained for suitable values of c and n.

The estimates of the functions $H_2(x)$ and $H_5(y)$ imply that it is necessary to consider separately the cases $n = 1$, $n = 2$ and $n \geqslant 3$, for suitable values of $c \in \mathbf{R}$.

Let $n = 1$. From (2.15), (2.16), (2.21), (2.22) and (2.23) it follows that, for any function $f \in S$,

$$G(f) \leqslant \begin{cases} \max\{1,\ 3-2c,\ -1\} & \text{when } c \leqslant 0, \\ \max\{1,\ H_2(x_0),\ 3-2c,\ H_5(y_0)\} & \text{when } 0 < c < \frac{1}{3}, \\ \max\{1,\ H_2(x_0),\ 3-2c,\ 3(2c-1)\} & \text{when } \frac{1}{3} \leqslant c \leqslant \frac{1}{2}, \\ \max\{1,\ H_2(x_0),\ H_3(x_2),\ 3-2c,\ 3(2c-1)\} & \text{when } \frac{1}{2} < c < \frac{3}{4}, \\ \max\{1,\ H_2(x_0),\ H_3(x_2),\ 3(2c-1)\} & \text{when } \frac{3}{4} \leqslant c < 1, \\ \max\{1,\ H_2(x_0),\ 3(2c-1)\} & \text{when } c \geqslant 1, \end{cases}$$

where H_2, H_3, H_5 are given by (2.11), (2.12), (2.14), while x_0 and y_0 are the positive roots of equations (2.17) and (2.24), respectively and $x_2 = (2c-1)/2(1-c)$.

From the examination of the function $H_2(x)$ defined by (2.11) it follows that $H_2(x_0) > 3-2c = H_2(0)$ and $H_2(x_0) > 1 = H_2(\infty)$ where x_0 is the root of equation (2.17). It is easily verified that $H_5(y_0) < 1$ where H_5 is given by (2.14) and y_0 is the root of equation (2.24). It can be shown that if $c \leqslant \frac{1}{2}$, then the function $H_3(x)$ defined by (2.12) is decreasing; hence $H_3(0) = 3-2c > 3(2c-1) = H_3(\infty)$. If $c > \frac{1}{2}$, it suffices to examine the function $H_3(x)$ and the difference $H_3(x) - H_2(x)$. Then we obtain that $H_2(x_0) > H_3(x_2)$ if $\frac{1}{2} < c < c^*$, whereas $H_3(x_2) > H_2(x_0)$ if $c^* \leqslant c < 1$, with c^* being the only root of the equation

$$e^{(2c-1)/2(c-1)} - 3c + 2 = 0, \tag{3.1}$$

x_0 being the positive root of equation (2.17), and $x_2 = (2c-1)/2(1-c)$. Moreover, $H_3(\infty) = 3(2c-1) > H_2(x_0)$ if $c \geqslant 1$.

Consequently, if $n = 1$, then, for each function $f \in S$, the following estimate of functional (1.5) holds:

$$G(f) \leqslant \begin{cases} H_2(0) = 3-2c & \text{when } c \leqslant 0, \\ H_2(x_0) & \text{when } 0 < c < c^*, \\ H_3(x_2) & \text{when } c^* \leqslant c < 1, \\ H_3(\infty) = 3(2c-1) & \text{when } c \geqslant 1, \end{cases} \tag{3.2}$$

where H_2, H_3 are defined by formulae (2.11), (2.12), x_0 is the only root of equation (2.17), $x_2 = (2c-1)/2(1-c)$ and c^* is the only root of equation (3.1).

Next, let $n = 2$. From (2.15), (2.18), (2.21), (2.22) and (2.25) it follows that, for any function $f \in S$,

$$G(f) \leqslant \begin{cases} \max\{1,\, 3,\, -1\} & \text{when } c < \frac{1}{4}, \\ \max\{1,\, 3,\, 8c-3\} & \text{when } \frac{1}{4} \leqslant c \leqslant \frac{1}{2}, \\ \max\{1,\, 3,\, H_3(x_2),\, 8c-3\} & \text{when } \frac{1}{2} < c < 1, \\ \max\{1,\, 3,\, 8c-3\} & \text{when } c \geqslant 1. \end{cases}$$

In this case, the examination of the function H_3 yields $H_3(x_2) > 3 = H_3(0)$ when $c \in (\frac{1}{2}, \frac{3}{4})$, and $H_3(x_2) > 8c-3 = H_3(\infty)$ when $c \in \langle \frac{3}{4}, 1)$, $x_2 = (2c-1)/2(1-c)$.

So, in this case, the following estimate of functional (1.5) holds:

$$\text{(3.3)} \qquad G(f) \leqslant \begin{cases} H_3(0) = 3 & \text{when } c \leqslant \frac{1}{2}, \\ H_3(x_2) & \text{when } \frac{1}{2} < c < 1, \\ H_3(\infty) = 8c-3 & \text{when } c \geqslant 1, \end{cases}$$

where H_3 is defined by formula (2.12), and $x_2 = (2c-1)/2(1-c)$.

Finally, consider the case when $n \geqslant 3$. From (2.15), (2.19), (2.21), (2.22) and (2.25) it follows that, for any function $f \in S$,

$$G(f) \leqslant \begin{cases} \max\{1,\, H_2(x_1),\, 3-4c+2^n c,\, -1\} & \text{when } c < 0, \\ \max\{1,\, 3-4c+2^n c,\, -1\} & \text{when } 0 \leqslant c < \frac{2}{4+2^n}, \\ \max\{1,\, 3-4c+2^n c,\, -3+4c+2^n c\} & \text{when } \frac{2}{4+2^n} \leqslant c \leqslant \frac{1}{2}, \\ \max\{1,\, 3-4c+2^n c,\, H_3(x_2),\, -3+4c+2^n c\} & \text{when } \frac{1}{2} < c < 1, \\ \max\{1,\, 3-4c+2^n c,\, -3+4c+2^n c\} & \text{when } c \geqslant 1, \end{cases}$$

where H_2, H_3 are given by (2.11), (2.12), while x_1 is the only root of equation (2.20) and $x_2 = (2c-1)/2(1-c)$.

From the examination of the function H_2 in this case it follows that $H_2(x_1) > 3-4c+2^n c = H_2(0)$ and $H_2(x_1) > 1$. Whereas, examining the function H_3, we obtain in this case that $H_3(x_2) > 3-4c+2^n c = H_3(0) > H_3(\infty)$ when $c \in (\frac{1}{2}, \frac{3}{4})$, while $H_3(x_2) > -3+4c+2^n c = H_3(\infty) > H_3(0)$ when $c \in \langle \frac{3}{4}, 1)$.

Consequently, if $n \geqslant 3$, then, for each function $f \in S$, the following estimate of functional (1.5) holds:

$$\text{(3.4)} \qquad G(f) \leqslant \begin{cases} H_2(x_1) & \text{when } c < 0, \\ H_3(0) = 3-4c+2^n c & \text{when } 0 \leqslant c \leqslant \frac{1}{2}, \\ H_3(x_2) & \text{when } \frac{1}{2} < c < 1, \\ H_3(\infty) = -3+4c+2^n c & \text{when } c \geqslant 1, \end{cases}$$

where H_2, H_3 are defined by formulae (2.11), (2.12), x_1 is the root of equation (2.20) and $x_2 = (2c-1)/2(1-c)$.

We shall next examine whether, and for which $c \in \mathbf{R}$, estimates (3.2), (3.3) and (3.4) are sharp.

Let us first notice that, in those estimates from (3.2), (3.3) and (3.4) in which the maximum of the function H_k is attained at the points 0 or ∞, the extremal function is Koebe function (1.3).

In the case $n = 1$, equality in (3.2) takes place for Koebe function (1.3) with $\varepsilon = \pm 1$ when $c \leqslant 0$ and with $\varepsilon = \pm \mathrm{i}$ when $c \geqslant 1$.

If $n = 2$, equality in (3.3) takes place for Koebe function (1.3) with $\varepsilon = \pm 1$ when $c \leqslant \frac{1}{2}$ and with $\varepsilon = \pm \mathrm{i}$ when $c \geqslant 1$.

Similarly, in the case $n \geqslant 3$, equality in (3.4) holds for Koebe function (1.3) with $\varepsilon = \pm 1$ when $0 \leqslant c \leqslant \frac{1}{2}$ and with $\varepsilon = \pm \mathrm{i}$ when $c \geqslant 1$.

Next, we shall prove that estimates (3.2) for $0 < c < c^*$ and (3.4) for $c < 0$ are sharp.

It is sufficient to prove the latter case, the proof of the former is analogous.

In order to show that, for $c < 0$, estimate (3.4) in the class S is sharp, it is enough to prove, in view of case 2^0 from Section 2 and on account of the Valiron-Landau lemma, that there exists a function $\theta_*(t)$ for which $v^2 = 0$, i.e. (compare [3], pp. 104–107)

$$\int_0^\infty \mathrm{e}^{-t} \sin \theta_*(t)\, \mathrm{d}t = 0 \tag{3.5}$$

and $|\mathrm{e}^{-t} \cos \theta_*(t)| = \varphi(t)$ where

$$\varphi(t) = \begin{cases} \mathrm{e}^{-x} & \text{for} \quad 0 \leqslant t \leqslant x, \\ \mathrm{e}^{-t} & \text{for} \quad x < t < \infty. \end{cases}$$

Let x_1, $x_1 > 0$, be a solution of equation (2.20), whereas $\theta_*(t)$ a function defined by the formulae

$$\cos \theta_*(t) = \begin{cases} \mathrm{e}^{t-x_1} & \text{for} \quad 0 \leqslant t \leqslant x_1, \\ 1 & \text{for} \quad x_1 < t < \infty. \end{cases}$$

Then

$$\sin \theta_*(t) = \begin{cases} \pm\sqrt{1 - \mathrm{e}^{2(t-x_1)}} & \text{for} \quad 0 \leqslant t \leqslant x_1, \\ 0 & \text{for} \quad x_1 < t < \infty, \end{cases}$$

whence one can easily obtain the formulae for the function $k_*(t) = \mathrm{e}^{i\theta_*(t)}$.

Choosing different signs in portions of the interval $\langle 0, x_1\rangle$, one can satisfy condition (3.5). Indeed, let us consider, for instance, the function

$$\psi(\tau) = \int_0^\tau e^{-t}\sqrt{1-e^{2(t-x_1)}}\,dt - \int_\tau^{x_1} e^{-t}\sqrt{1-e^{2(t-x_1)}}\,dt, \qquad \tau \in \langle 0, x_1\rangle.$$

It is continuous in the interval $\langle 0, x_1\rangle$, $\psi(0) < 0$, $\psi(x_1) > 0$, thus there exists a point $\tau_0 \in (0, x_1)$ such that $\psi(\tau_0) = 0$. Putting then

$$\sin\theta_*(t) = \begin{cases} \sqrt{1-e^{2(t-x_1)}} & \text{for } 0 \leqslant t \leqslant \tau_0, \\ -\sqrt{1-e^{2(t-x_1)}} & \text{for } \tau_0 \leqslant t \leqslant x_1, \\ 0 & \text{for } x_1 < t < \infty, \end{cases}$$

we eventually obtain condition (3.5).

Let us observe that the functions $H_3(x)$ and $H_4(y)$ given by formulae (2.12) and (2.13), respectively, have been obtained in the case when $u^2 + v^2 = 1$ (compare 3^0, 4^0). It is known from estimate (1.2) that this equality is possible only for Koebe function (1.3). From (1.5) and (1.3), putting $\varepsilon = e^{i\varphi}$, $0 \leqslant \varphi < 2\pi$, we get

$$G(f_0) = (3-4c)\cos 2\varphi + 2^n c, \qquad c \in \mathbf{R}, \quad n \in \mathbf{N}.$$

It is easily verified that

$$G(f_0) \leqslant \max_{0\leqslant\varphi\leqslant 2\pi}[(3-4c)\cos 2\varphi + 2^n c] = \begin{cases} 3-4c+2^n c & \text{when } c < \frac{3}{4}, \\ -3+4c+2^n c & \text{when } c \geqslant \frac{3}{4}, \end{cases}$$

$n \in \mathbf{N}$.

Thus, this implies that, in cases 3^0 and 4^0, we can find a sharp estimate of the functional $G(f)$ at the points $(0,1)$, $(1,0)$ only. In the remaining cases, we shall only get an estimate from above. Such cases take place for those n and c for which, in (3.2), (3.3) or (3.4), the value of the function H_3 at the point $x_2 = (2c-1)/2(1-c)$ occurs. For, then, we should have $u^2 + v^2 = 1$, $|u| = \omega(x_2)$ and, respectively, $|v| = |\int_0^\infty e^{-t}\sin\theta(t)\,dt|$. Consequently, for those c for which the value of the function H_3 occurs and for a fixed x, the polygon with vertices given by the conditions 1^0–5^0 seems too large.

Thereby, (3.2), (3.3), (3.4) and (1.5) and (1.4) imply the following

Theorem 1. *If $f \in S$ is any function of form* (1.1), *then we have*

$$|a_3 - c\,a_2^2| + c|a_2| \leqslant \begin{cases} 3-2c & \text{when } c \leqslant 0, \\ c(x_0+1)\,\mathrm{e}^{-x_0} + 2\mathrm{e}^{-2x_0} + 1 & \text{when } 0 < c < c^*, \\ H_3(x_2) & \text{when } c^* \leqslant c < 1, \\ 3(2c-1) & \text{when } c \geqslant 1 \end{cases}$$

where x_0 is the only root of the equation

$$4\mathrm{e}^{-x}[(1-c)\,x - c] + c = 0,$$

H_3 is defined by formula (2.12), $x_2 = (2c-1)/2(1-c)$, while c^* is the only root of equation (3.1). For any $c < c^*$ and $c \geqslant 1$, there exist functions of the class S for which the equality sign in the above estimates takes place.

Theorem 2. *If $f \in S$ is any function of form* (1.1), *then we have*

$$|a_3 - c\,a_2^2| + c|a_2|^2 \leqslant \begin{cases} 3 & \text{when } c \leqslant \frac{1}{2}, \\ H_3(x_2) & \text{when } \frac{1}{2} < c < 1, \\ 8c-3 & \text{when } c \geqslant 1, \end{cases}$$

where H_3 is given by (2.12) and $x_2 = (2c-1)/2(1-c)$. For each $c \leqslant \frac{1}{2}$ and each $c \geqslant 1$, there exist functions of the class S for which the equality sign in the above estimates takes place.

Theorem 3. *If $f \in S$ is any function of form* (1.1), *n—any integer number, $n \geqslant 3$, then we have*

$$|a_3 - c\,a_2^2| + c|a_2|^n \leqslant \begin{cases} -c(n-2)\,2^{n-1}[(x_1+1)\,\mathrm{e}^{-x_1}]^n & \\ +2\mathrm{e}^{-2x_1} + 1 & \text{when } c < 0, \\ 3 - 4c + 2^n c & \text{when } 0 \leqslant c \leqslant \frac{1}{2}, \\ H_3(x_2) & \text{when } \frac{1}{2} < c < 1, \\ -3 + 4c + 2^n c & \text{when } c \geqslant 1, \end{cases}$$

where x_1 is the positive root of the equation

$$8\mathrm{e}^{-x}[(1-c)\,x - c] + c\,n\,2^n[(x+1)\,\mathrm{e}^{-x}]^{n-1} = 0,$$

H_3 is given by (2.12), and $x_2 = (2c-1)/2(1-c)$. For any $c \leqslant \frac{1}{2}$ and $c \geqslant 1$ there exist functions of the class S for which the equality sign in the above estimates takes place.

4. Applications

We shall give some simple applications of the results obtained, admitting that some of them are known.

Let us put, for example,

$$
\begin{aligned}
&(4.1) \qquad g_1(z) = [f(1/z)]^{-1} = z + a_0^{(1)} + a_1^{(1)}/z + \dots, && 1 < |z| < \infty,\\
&(4.2) \qquad g_2(z) = z\sqrt{f(z^2)/z^2} = z + a_2^{(2)} z^3 + a_3^{(2)} z^5 + \dots, && z \in \Delta,\\
&(4.3) \qquad g_3(z) = \log[f(z)/z] = 2(a_1^{(3)} z + a_2^{(3)} z^2 + \dots), && z \in \Delta,\\
&(4.4) \qquad g_4(z) = 1/f'(z) = 1 + a_1^{(4)} z + a_2^{(4)} z^2 + \dots, && z \in \Delta,\\
&(4.5) \qquad g_5(w) = f^{-1}(w) = w + a_2^{(5)} w^2 + a_3^{(5)} w^3 + \dots, && |w| < \frac{1}{4},\\
&(4.6) \qquad g_6(z) = [z/f(z)]^k = 1 + a_1^{(6)} z + a_2^{(6)} z^2 + \dots, && z \in \Delta,\\
& && k = 2, 3 \dots
\end{aligned}
$$

(g_5 denotes the inverse function of f).

Making use of Theorems 1, 2 and the relations between the coefficients a_2, a_3 in expansion (1.1) and suitable coefficients $a_l^{(p)}$ in (4.1)–(4.6), we obtain, for instance, the following eestimates:

$$|a_1^{(1)}| + |a_0^{(1)}| \leqslant 3;$$

$$|a_3^{(2)}| + \frac{1}{2}|a_2^{(2)}|^2 \leqslant \frac{3}{2};$$

$$|a_2^{(3)}| + |a_1^{(3)}|^2 \leqslant \frac{3}{2};$$

$$2|a_2^{(3)}| + |a_1^{(3)}| \leqslant 1 + 2e^{-2x_0} + \frac{1}{2}(x_0 + 1)\,e^{-x_0},$$

where x_0 is the root of the equation $4e^{-x}(x-1) + 1 = 0$;

$$|a_2^{(4)}| + |a_1^{(4)}|^2 \leqslant 23;$$

$$|a_3^{(5)}| + 2|a_2^{(5)}|^2 \leqslant 13;$$

$$|a_2^{(6)}| + \frac{k+1}{2}|a_1^{(6)}| \leqslant 3k^2, \qquad k = 2, 3, \dots.$$

5. On the class $S(|a_2|)$

Let $S(|a_2|)$ denote a subclass of the family S of functions (1.1) with a fixed $|a_2| \in \langle 0, 2\rangle$. Note that modifying the procedure presented before, we can examine the maximum of functional (1.5) in the class $S(|a_2|)$. The consideration will be carried out in the case $n = 2$.

From (2.1), (2.2) and (2.8′) we then have

$$G(f) = (2c-1)|a_2|^2 + 8(1-c)\,u^2 - 4\int_0^\infty e^{-2t}\cos^2\theta(t)\,dt + 1. \tag{5.1}$$

With no loss of generality we may assume that

$$u = \int_0^\infty e^{-t}\cos\theta(t)\,dt \in \langle 0, 1\rangle.$$

In virtue of the variational lemma from I.E. Bazylevič's paper [1], it is known that, among all admissible functions $\theta(t)$ taking values such that $u = A_0 \in \langle 0, 1\rangle$, the minimum of the integral $\int_0^\infty y^2(t)\,dt$, where $y(t) = e^{-t}\cos\theta(t)$, is attained for the function $y_0(t) = \min[e^{-t}, C]$. The constant C satisfies the equality

$$\int_0^\infty y_0(t)\,dt = A_0.$$

Let $y_0(t) = e^{-x}$ for $t \in \langle 0, x\rangle$ and $y_0(t) = e^{-t}$ for $t \geqslant x$. We shall then get $A_0 = \omega(x)$. As $A_0 \in \langle 0, 1\rangle$ is arbitrarily fixed, there exists exactly one $x \in \langle 0, \infty)$ satisfying the above equation. Moreover,

$$\int_0^\infty (e^{-t}\cos\theta(t))^2\,dt \geqslant \int_0^\infty y_0^2(t)\,dt = w(x).$$

In the problem considered, we have $0 \leqslant u = A_0 \leqslant \frac{1}{2}|a_2| \leqslant 1$, the function ω being decreasing, thus the subsequent considerations should be confined to those x for which $A_0 = \omega(x) \leqslant \frac{1}{2}|a_2|$.

Let $\tilde{x} = \tilde{x}(|a_2|)$ stand for a root of the equation

$$\omega(x) = \frac{1}{2}|a_2|. \tag{5.2}$$

So, in view of (5.1), we are to estimate the function

$$\tilde{H}_3(x) = 8(1-c)\omega^2(x) - 4w(x) + 1, \qquad x \geqslant \tilde{x}, \tag{5.3}$$

where $\omega(x) = (x+1)e^{-x}$, $w(x) = (x+\frac{1}{2})e^{-2x}$ (cf. Section 2). Since $\tilde{H}_3'(x) = -8x e^{-2x}[2(1-c)x + 1 - 2c]$, $x \geqslant \tilde{x}$, therefore, for $c \geqslant 1$, we have $\max \tilde{H}_3(x) = \tilde{H}_3(\infty) = 1$, whereas for $c \leqslant \frac{1}{2}$, $\max \tilde{H}_3(x) = \tilde{H}_3(\tilde{x})$.

Let $\frac{1}{2} < c < 1$. Then $\tilde{H}_3'(x) > 0$ for $0 < x < x_2$ and $\tilde{H}_3'(x) < 0$ for $x > x_2$ where $x_2 = (2c-1)/2(1-c) > 0$. Hence $\max \tilde{H}_3(x) = \tilde{H}_3(x_2)$ when $\tilde{x} \leqslant x_2$ and $\max \tilde{H}_3(x) = \tilde{H}_3(\tilde{x})$ when $\tilde{x} > x_2$. The formula for x_2 and (5.2) imply that

$$\max \tilde{H}_3(x) = \begin{cases} \tilde{H}_3(x_2) & \text{when } |a_2| \geqslant 2\omega(x_2), \\ \tilde{H}_3(\tilde{x}) & \text{when } |a_2| \leqslant 2\omega(x_2). \end{cases}$$

From (5.2), (5.3) and the analysis we have carried out we obtain

Theorem 4. *If a function f of form (1.1) belongs to the class $S(|a_2|)$, then*

$$(5.4)\quad Re(a_3 - c a_2^2) + c|a_2|^2 \leqslant 4e^{-2\tilde{x}}(\tilde{x}^2 + \tilde{x} + \frac{1}{2}) + 1 \qquad \textit{when } c \leqslant \frac{1}{2},\ |a_2| \leqslant 2,$$

$$(5.5)\quad Re(a_3 - c a_2^2) + c|a_2|^2 \leqslant 4e^{-2\tilde{x}}(\tilde{x}^2 + \tilde{x} + \frac{1}{2}) + 1 \qquad \textit{when } \frac{1}{2} < c < 1,\ |a_2| \leqslant 2\omega(x_2),$$

$$(5.6)\quad Re(a_3 - c a_2^2) + c|a_2|^2 \leqslant (2c-1)|a_2|^2 + 2e^{-2x_2} + 1 \qquad \textit{when } \frac{1}{2} < c < 1,\ 2\omega(x_2) \leqslant |a_2| \leqslant 2,$$

$$(5.7)\quad Re(a_3 - c a_2^2) + c|a_2|^2 \leqslant (2c-1)|a_2|^2 + 1 \qquad \textit{when } c \geqslant 1,\ |a_2| \leqslant 2,$$

where $\omega(x) = (x+1)e^{-x}$, $x_2 = (2c-1)/2(1-c)$, $\omega(\tilde{x}) = \frac{1}{2}|a_2|$, $\tilde{x} \geqslant 0$.

The equality in (5.4) and (5.5) can hold only if $u = \frac{1}{2}|a_2|$, $v = 0$, $\omega(x) = u$, $w(x) = \int_0^\infty y^2(t)dt$ with the corresponding x. So, in cases (5.4) and (5.5), we have the conditions

$$(5.8)\qquad \int_0^\infty e^{-t}\cos\theta(t)\,dt = \omega(\tilde{x}),$$

$$(5.9)\qquad \int_0^\infty e^{-t}\sin\theta(t)\,dt = 0,$$

$$(5.10)\qquad \int_0^\infty e^{-2t}\cos^2\theta(t)\,dt = w(\tilde{x}).$$

If, however, the equality in (5.10) holds, then $e^{-t}\cos\theta(t) = y_0(t)$ with $x = \tilde{x}$. Consequently, $\cos\theta(t) = e^{-\tilde{x}+t}$ for $t \leqslant \tilde{x}$ and $\cos\theta(t) = 1$ for $t \geqslant \tilde{x}$. Then, of course, $u = \omega(\tilde{x})$ and one may choose $\sin\theta(t)$ (cf. the proof of the estimate in Theorem 3) so that (5.9) takes place. Consequently, estimates (5.4) and (5.5) are sharp.

In the case of the equality in (5.6), the relations $u = \omega(x_2)$, $\int_0^\infty e^{-2t}\cos^2\theta(t)\,dt = w(x_2)$ and $\omega(\tilde{x}) = \frac{1}{2}|a_2|$, $\tilde{x} < x_2$ should hold. In the last case, the equalities $x = \infty$, $\int_0^\infty y^2(t)dt = w(\infty) = 0$ necessarily hold, so $u = 0$, $|\sin\theta(t)| = 1$ and $|a_2| = 2|v|$. Let

$$v = v(\tau) = \int_0^\tau e^{-t}dt - \int_\tau^\infty e^{-t}dt, \quad \tau \geqslant 0.$$

Thus there exists, in this case, a function of the class $S(|a_2|)$, $0 \leqslant |a_2| \leqslant 2$, realizing in (5.7) the equality sign.

From (5.4), (5.6) and (5.7) we immediately obtain the above Theorem 2.

Analogous investigations for $n \neq 2$ have not been the aim of the present paper.

6. Open problems

To conclude with, let us notice that functional (1.4) can be modified a little. In particular, it seems interesting to investigate, for example, the following functionals defined on the class S:

$$|a_3 - \alpha a_2^2| + |\alpha a_2^n|, \qquad \alpha \in \mathbf{C}, \quad n \in \mathbf{N},$$

$$|a_3 - \alpha a_2^2| + \beta|a_2|^n, \quad \alpha \in \mathbf{C}, \quad \beta \in \mathbf{R}, \quad n \in \mathbf{N}, \tag{6.1}$$

$$|a_3 - \alpha a_2^2| + \alpha|a_2|^n, \qquad \alpha < 0, \quad n \in \mathbf{N}. \tag{6.2}$$

In the case of (6.2), the problem concerns the determination of the minimum of this functional. Case (6.1) when $n = 1$ will provide, in particular, an estimate of the sum of the moduli of coefficients in various classes associated with the class S (see 4. Applications).

Of course, it remains open to determine in Theorems 1, 2 and 3 sharp estimates in the intervals $c^* \leqslant c < 1$ and $\frac{1}{2} < c < 1$. In spite of several different attempts, these cases have resisted so far.

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