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*Mathematica Bohemica*, Vol. 118 (1993), No. 3, 297–312

Persistent URL: http://dml.cz/dmlcz/125924

# EXTENSIONS OF THE REPRESENTATION THEOREMS OF RIESZ AND FRÉCHET

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(Received June 8, 1992)

*Summary.* We present two types of representation theorems: one for linear continuous operators on spaces of Banach valued regulated functions of several real variables and the other for bilinear continuous operators on cartesian products of spaces of regulated functions of a real variable taking values on Banach spaces. We use generalizations of the notions of functions of bounded variation in the sense of Vitali and Fréchet and the Riemann-Stieltjes-Dushnik or interior integral. A few applications using geometry of Banach spaces are given.

*Keywords*: Riesz type representation theorem, Fréchet type representation theorem, regulated functions

*AMS classification*: 46B99, 46E15, 46E40

## 1. INTRODUCTION

We extend the Riesz type representation theorem for linear continuous functionals on the space of continuous real valued functions of several variables to the case Banach space valued regulated functions of several variables. The second topic concerns the extension of the representation result of Fréchet for bilinear continuous functionals on the cartesian product of spaces of real valued continuous functions of a real variable for bilinear continuous operators on the cartesian product of spaces of Banach space valued regulated functions of a real variable.

We use generalizations of the notions of functions of bounded variation in the sense of Vitali and Fréchet and the notion of the Riemann-Stieltjes-Dushnik or the interior integral.

This work has been done under the partial support of C.N.Pq.

In the second section we define and present the most basic properties of the regulated functions of several real variables, in the third the notion of functions of bounded variation and of bounded semi-variation in the sense of Vitali and the representation theorem, in the fourth the notion of functions of bounded semi-variation in the sense of Fréchet and the corresponding representation theorems and in the final section a few applications.

To keep the notation clear we will only give proofs for the two dimensional case, one sees easily how to cope with the general case.

## 2. Regulated functions of several real variables

**Definition 2.1.** A *partition* $d$ of an interval $a, b \subseteq \mathbf{R}$ is a finite sequence $d$: $t_0 = a < \cdots < t_m = b$. Set $|d| = m$, $\Delta d = \sup\{t_j - t_{j-1} \colon 1 \leqslant j \leqslant |d|\}$ (the mesh of the partition).

We call $\{t_j\}$, $0 \leqslant j \leqslant |d|$ and $]t_{j-1}, t_j[$, $1 \leqslant j \leqslant |d|$ the *basic intervals* of $d$.

The set of all partitions of $a$, $b$ is $\mathbf{D}_{[a,b]}$ or $\mathbf{D}$. We order $\mathbf{D}$ by the inclusion relation, that is, $\forall d, d' \in \mathbf{D}$ $d \geqslant d' \Longleftrightarrow d \supseteq d'$, in this way $\mathbf{D}$ becomes a net.

Similarly a *partition* of $[c,d] \times [a,b] \subseteq \mathbf{R}^2$ is a product $d = d'' \times d'$ where $d'' \in \mathbf{D}_{[c,d]}$ and $d' \in \mathbf{D}_{[a,b]}$. Now we set $|d| = |d''| \cdot |d'|$, $\Delta d = \sup\{\Delta d'', \Delta d'\}$.

A *basic interval* of $d$ is now a cartesian product of a *basic interval* of $d''$ by one of $d'$. We say that $d = d'' \times d'$ is finer than $\bar{d} = \overline{d''} \times \overline{d'}$ and write $d \geqslant \bar{d}$ if and only if $d'' \geqslant \overline{d''}$ and $d' \geqslant \overline{d'}$. We write $\mathbf{D}_{[c,d]\times[a,b]}$ for the set of all partitions of $[c,d] \times [a,b]$ or more simply $\mathbf{D}$ when there is no danger of confusion.

$S([c,d] \times [a,b], X)$ will be the set of all *step functions* from $[c,d] \times [a,b]$ to the Banach space $X$ where by a step function we mean a function for which there exists a partition $d \in \mathbf{D}_{[c,d]\times[a,b]}$ such that the function is constant on the basic intervals of $d$. $S([a,b], X)$ is defined analogously. $G([c,d] \times [a,b], X)$ will be the closure of $S([c,d] \times [a,b], X)$ in the sup norm inside the Banach space of the bounded functions. $G([a,b], X)$ has an analogous meaning.

It is easy to show that the following result holds.

**Lemma 2.1.** *If $E$ is a normed space and $\overline{E}$ its completion then*

$$G([a,b], \overline{E}) = \overline{S([a,b], E)}.$$

It is also easy to prove the following lemma:

**Lemma 2.2.** *The mapping*

$$f \in S([c,d] \times [a,b], X) \mapsto f^{\square} \in S([c,d], S([a,b], X))$$

*is an isometry where $f^{\square}(t)(s) = f(t,s)$.*

From these two results one gets the following theorem that relates the spaces of regulated functions of two variables with those of a single variable.

**Theorem 2.1.** *The mapping*

$$f \in G([c,d] \times [a,b], X) \mapsto f^{\square} \in G([c,d], G([a,b], X))$$

*is an isometry from the first Banach space onto the other.*

Now we present a result that shows that regulated functions of several variables are indeed those that possess limits from "all sides".

**Theorem 2.2.** *For a function $f\colon [c,d] \times [a,b] \longrightarrow X$ the following properties are equivalent:*

*1. $\forall \varepsilon > 0\ \exists d_\varepsilon = d''_\varepsilon \times d'_\varepsilon \in \mathbf{D}_{[c,d]\times[a,b]}$ such that $\omega_{\dot{d}_\varepsilon}(f) < \varepsilon$ where $\omega_{\dot{d}}(f) = \sup\{\omega_I(f)\colon I$ is a basic interval of $d \in \mathbf{D}_{[c,d]\times[a,b]}\}$ and by $\omega_A(f)$ for a subset $A$ of a set $S$ and a function $f$ from $S$ into a metric space $M$ we mean the oscillation of the function $f\colon S \longrightarrow M$ on $A$, that is, $\omega_A(f) = \sup\{\operatorname{dist}(f(x), f(y))\colon x, y \in A\}$, dist being the distance in $M$.*

*2. $f \in G([c,d] \times [a,b], X)$.*

*3. For all $(t,s) \in [c,d] \times [a,b]$ the limits*

$$\lim_{\tau\downarrow 0,\sigma\downarrow 0} f(t+\tau, s+\sigma) = f(t+,s+),\quad \lim_{\tau\downarrow 0,\sigma\downarrow 0} f(t+\tau, s-\sigma) = f(t+,s-),$$
$$\lim_{\tau\downarrow 0,\sigma\downarrow 0} f(t-\tau, s+\sigma) = f(t-,s+),\quad \lim_{\tau\downarrow 0,\sigma\downarrow 0} f(t-\tau, s-\sigma) = f(t-,s-),$$
$$\lim_{\tau\downarrow 0} f(t+\tau, s) = f(t+,s),\quad \lim_{\tau\downarrow 0} f(t-\tau, s) = f(t-,s),$$
$$\lim_{\sigma\downarrow 0} f(t, s+\sigma) = f(t,s+),\quad \lim_{\sigma\downarrow 0} f(t, s-\sigma) = f(t,s-)$$

*exist whenever they are meaningful.*

Proof. (1. $\Rightarrow$ 2.) Given $\varepsilon > 0$ take $d_\varepsilon$ as in 1. Choose one point $p_I \in I$ where $I$ is a basic interval of $d_\varepsilon$ and consider the step function $g = \sum_I 1_I f(p_I)$, $1_I$ being the characteristic or indicator function of $I$. Then $\|f - g\| \leqslant \varepsilon$ where $\|f\|$ is the sup-norm of $f$. Consequently $f \in G([c,d] \times [a,b], X)$.

(2. $\Rightarrow$ 3.) It follows from Theorem 2.1, we also have $f(t\pm, s\pm) = f(t\pm)(s\pm)$ and the limits $f(t\pm)(s) = f(t\pm, s)$ are uniform in $s$.

(3. $\Rightarrow$ 1.) Consider $\varepsilon > 0$, then there exists a $\delta > 0$ such that if $V^1 = ]t, t+\delta[\times]s, s+\delta[$, $V^2 = ]t, t-\delta[\times]s, s+\delta[$, $\cdots$, $V^8 = \{t\}\times]s, s-\delta[$ then $\omega_{V^i}(f) < \varepsilon$,

$1 \leqslant i \leqslant 8$. At the boundary points we choose the same type of neighborhoods. Considering $V = \bigcup_{i=1}^{\infty} V^i \cup \{(t,s)\}$ we get an open covering of $[c,d] \times [a,b]$. Extracting a finite subcover and projecting the vertices and centers of their elements we obtain divisions $d''_\varepsilon \in \mathbf{D}_{[c,d]}$ and $d'_\varepsilon \in \mathbf{D}_{[a,b]}$. When $I$ is one of the basic intervals of $d_\varepsilon = d''_\varepsilon \times d'_\varepsilon$ then it will be a subset of some of the $Vi$, hence $\varepsilon > \omega_{Vi}(f) \geqslant \omega_I(f)$. □

A regulated function of one variable can have at most a denumerable set of discontinuities. An analogous result is valid for regulated functions of several variables:

**Theorem 2.3.** *The discontinuities of $f \in G(c, d \times a, b, X)$ lie on a denumerable set of lines parallel to the axis.*

**Proof.** Let $\mathscr{D}(f)$ be the set of discontinuities of $f$. Then $\mathscr{D}(f) = \bigcup_{n=1}^{\infty} \mathscr{D}^n(f)$ where $\mathscr{D}^n(f) = \{(t,s) \in [c,d] \times [a,b] : \omega_{(t,s)}(f) > 1/n\}$. Now if $g$ is a step function such that $\|f - g\| < \frac{1}{3n}$ then $\mathscr{D}^n(f) \subseteq \mathscr{D}^{3n}(g)$ and $\forall m \geqslant 1$, $\mathscr{D}^m(g)$ lies in a finite set parallel to the axis. □

We will define a Riemann-Stieltjes type of integral for regulated functions of several variables with respect to functions of bounded semi-variation, and it will be shown later that the resulting integral is not changed if we make our regulated function "left (or right) continuous". With this aim we introduce

**Definition 2.2.** Let $f \in G([c,d] \times [a,b], X)$. We say that 1. $f \in G_{--}([c,d] \times [a,b], X)$ if $f(c,s) = f(t,a) = 0$ and $f(t-,s) = f(t,s-) = f(t-,s-) = f(t,s)$ for $t \neq c$ and $s \neq a$.

2. $f \in \Omega_0([c,d] \times [a,b], X)$ if $\forall \varepsilon > 0$ $\exists d_\varepsilon \in \mathbf{D}_{[c,d]\times[a,b]}$ such that the set $\Omega_\varepsilon = \{(t,s) \in [c,d] \times [a,b] : \|f(t,s)\| \geqslant \varepsilon\}$ is contained in the set of lines parallel to the axis defined by the points of $d_\varepsilon$.

A similar definition is given for $G_{-}([c,d], X)$, for details see [1]. We also let $\Pi f$ be the function $\Pi f(c,s) = \Pi f(t,a) = 0$ and $\Pi f(t,s) = f(t-,s-)$ if $t \neq c$ and $s \neq a$. To simplify we will write $f_{--}$ for $\Pi f$. For the one dimensional case see [1].

The next result shows that our $\Pi$ is a projection.

**Theorem 2.4.** *$\Pi$ is a continuous projection from $G([c,d] \times [a,b], X)$ onto $G_{--}([c,d] \times [a,b], X)$. Its kernel is $\Omega_0([c,d] \times [a,b], X)$, hence $G([c,d] \times [a,b], X) = G_{--}([c,d] \times [a,b], X) \oplus \Omega_0([c,d] \times [a,b], X)$.*

**Proof.** To save space we will only prove that the kernel of the projection $\Pi$ is $\Omega_0([c,d] \times [a,b], X)$. Let $f \in \Omega_0([c,d] \times [a,b], X)$ and $\varepsilon > 0$ be given, take $d_\varepsilon$ as in Definition 2.2. Then $\forall (t,s) \in [c,d] \times [a,b]$ we have $\|f_{--}(t,s)\| < \varepsilon$. □

The next theorem is important in the sense that it provides a dense subset of $G_{--}([c,d]\times[a,b],X)$ that is suitable for giving the proofs of the representation theorems we will deal with later on.

**Theorem 2.5.** *The closure of the linear span of the set* $\{1_{]c,\tau]\times]a,\sigma]}(t,s)\cdot x\colon \tau\in[c,d],\sigma\in[a,b],x\in X\}$ *is* $G_{--}([c,d]\times[a,b],X)$.

Proof. The functions $1_{]c,\tau]}(t)\cdot x$ form a set the closure of the linear span of which is $G_{-}([c,d],X)$ ([1]). Since the function $1_{]c,\tau]\times]a,\sigma]}(t,s)\cdot x$ corresponds to $1_{]c,\tau]}(t)\cdot 1_{]a,\sigma]}(s)\cdot x$ we get the result taking into account an easy extension of Theorem 2.1. $\square$

Remark. Generally speaking if $|c,\tau|$ is an interval of the form $]c,\tau]$, $]c,\tau[$, $[c,\tau]$, $[c,\tau[$ or $\{c=\tau\}$ the closure of the linear span of the set of functions $1_{|c,\tau|}\cdot x$ is $G([c,d],X)$ while that of the functions $1_{|c,\tau|\times|a,\sigma|}\cdot x$ is $G([c,d]\times[a,b],X)$.

## 3. Functions of bounded Vitali semi-variation and representation theorems

We begin this section with the usual notion of a function of bounded Vitali variation and show some of its properties, later we generalize it to the notion of Vitali semi-variation and obtain the representation theorems.

**Definition 3.1.** For a $K\colon[c,d]\times[a,b]\longrightarrow X$ and $d=d''\times d'\in\mathbf{D}$ we set $K_{ji}=K(t_j,s_i)-K(t_j,s_{i-1})+K(t_{j-1},s_{i-1})-K(t_{j-1},s_i)$, $1\leqslant j\leqslant|d''|$, $1\leqslant i\leqslant|d'|$. Set also $V_d[K]=\sum\limits_{j,i}^{|d|}\|K_{ji}\|$ where $\sum\limits_{j,i}^{|d|}\|K_{ji}\|=\sum\limits_{j=1}^{|d''|}\sum\limits_{i=1}^{|d'|}\|K_{ji}\|$, $V[K]=V_{[c,d]\times[a,b]}K=\sup\{V_dK\colon d\in\mathbf{D}\}$. $V[K]$ is the *Vitali variation* of $K$ and the functions of finite Vitali variation are collected in the set $BV([c,d]\times[a,b],X)$, this is a linear space and $V[K]$ is a semi norm.

The trouble one incurs with the definition above is that the sections of $K$, that is $Kt(s)=K(t,s)$ and $K_s(t)=K(t,s)$ need not be of bounded variation, for example, if $K(t,s)=1_{\mathbf{Q}\cap[c,d]}(t)$ then $K_s$ is not of bounded variation though $K$ is. To overcome this nuisance we require $K(t,a)=K(c,s)=0$. With this normalization we also get a norm with $V[K]$, the corresponding space of functions with bounded Vitali variation that satisfy the requirements above will be denoted by $BV_{ca}([c,d]\times[a,b],X)$. Note that if $K\in BV([c,d]\times[a,b],X)$ then $\tilde{K}(t,s)=K(t,s)-K(t,a)+K(c,a)-K(c,s)$ is in $BV_{ca}([c,d]\times[a,b],X)$ and $\tilde{K}_{ji}=K_{ji}$, hence, as we will soon see, they will define the same higher dimensional Riemann-Stieltjes-Dushnik integral.

The usual arguments of limits get us the following theorem whose proof is omitted.

**Theorem 3.1.** *The mapping*

$$K \in BV_{ca}([c,d] \times [a,b], X) \mapsto K^{\square} \in BV_c([c,d], BV_a([a,b], X))$$

*is an isometry from the first Banach space onto the other.*

Next, for the sake of completeness, we state some results that though interesting in themselves are a bit away from the direction this paper points to. For this reason they are presented without proofs.

**Theorem 3.2.** *For $K \in C([c,d]\times[a,b], X)$ (here $C$ stands for the set of continuous functions) we have $V_{t_0,t\times[a,b]}K = V_{t_0,t\times[a,b]}K$ where $V_{t_0,t\times[a,b]}K = \sup\{V_{t_0+\delta,t\times[a,b]}K\colon \delta > 0\}$.*

**Theorem 3.3.** *The mapping*

$$K \in BVC_{ca}([c,d] \times [a,b], X) \mapsto K^{\square} \in BVC_c([c,d], BVC_a([a,b], X))$$

*is an isometry from the first Banach space onto the other, where $BVC = BV \cap C$.*

Now we proceed to the generalization of the functions of bounded Vitali variation and the first type of representation theorem.

**Definition 3.2.** Let $K\colon [c,d] \times [a,b] \longrightarrow L(X,Y)$ where $L(X,Y)$ is the Banach space of the continuous linear operators from $X$ into $Y$, let $d \in \mathbf{D}$. We set $SV_d[K] = \sup\left\{\left\|\sum_{ji}^{|d|} K_{ji}\cdot x_{ji}\right\|\colon x_{ji} \in X, \|x_{ji}\| \leqslant 1\right\}$, $SV[K] = \sup\{SV_dK\colon d \in \mathbf{D}\}$ and call $SVK$ the *Vitali semi-variation* of $K$ and let $SV([c,d] \times [a,b], X)$ be the set of all $K$ with finite Vitali semi variation. As it was done with the functions of bounded Vitali variation we normalize them by requiring that $K(t,a) = K(c,s) = 0$ and the resulting space is written $SV_{ca}([c,d] \times [a,b], L(X,Y))$.

When $Y = \mathbf{C}$ one has $SV([c,d] \times [a,b], X') = BV([c,d] \times [a,b], X')$ so we have in fact a true extension of the old concept.

Next we define the higher dimensional Riemann-Stieltjes-Dushnik integral that will represent our operators.

**Definition 3.3.** For a function $K\colon [c,d] \times [a,b] \longrightarrow L(X,Y)$ and $f\colon [c,d] \times [a,b] \longrightarrow X$ the *Riemann-Stieltjes-Dushnik integral of $f$ with respect to $K$* or the *interior integral of $f$ with respect to $K$* is the

$$\lim_{d\in\mathbf{D}} \sum_{ji}^{|d|} K_{ji} \cdot f(\eta_j, \xi_i) = \int_c^d \int_a^b d_{ts} K(t,s) \cdot f(t,s),$$

where $\eta_j \in ]t_{j-1}, t_j[$, $1 \leqslant j \leqslant |d''|$, $\xi_i \in ]s_{i-1}, s_i[$, $1 \leqslant i \leqslant |d'|$ $(d = d'' \times d')$ and the limit is to be understood in the sense of the net $\mathbf{D}$.

As a first result linking the notions of regulated functions of two variables and the the functions of bounded Vitali variation we have

**Theorem 3.4.** *Given the function $f \in G([c,d] \times [a,b], X)$ and the function $K \in SV([c,d] \times [a,b], L(X,Y))$ we have: 1. It exists $F_K(f) = \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot f(t,s)$.*

*2. $\|F_K(f)\| \leqslant SV[K]\, \|f\|$.*

*3. If $f \in \Omega_0([c,d] \times [a,b], X)$ then $F_K(f) = 0$.*

*4. $F_K(f) = F_K(f_{--})$ and $\|F_K(f)\| \leqslant SV[K]\, \|f_{--}\|$.*

*5. $\|F_K(f) - \sum_{ji}^{|d|} K_{ji} \cdot f(\eta_j, \xi_i)\| \leqslant SV[K]\, \omega_d(f), : \forall d \in \mathbf{D}$.*

Proof. (1. and 5.): We show that the Cauchy criterion holds. Given $\varepsilon > 0$, choose $d_\varepsilon \in \mathbf{D}$ such that $\omega_{d_\varepsilon} < \frac{\varepsilon}{2SV[K]}$ (see 1. of Theorem 2.2). Now let $d \geqslant d_\varepsilon$ and consider (if any) $t* \in d''_\varepsilon$, $s* \in d - d'_\varepsilon$, say, $t* \in ]t_{j-1}, t_j[$, $s* \in ]s_{i-1}, s_i[$. For the division $\{t_{j-1}, t*, t_j\} \times \{s_{i-1}, s*, s_i\}$ of $[t_{j-1}, t_j] \times [s_{i-1}, s_i]$ we have $K_{ji} = K_{11} + K_{12} + K_{21} + K_{22}$. Hence $\left\| \sum_{m,l}^{|d|} K_{ml} \cdot f(\eta_m, \xi_l) - \sum_{j,i}^{|d_\varepsilon|} K_{ji} \cdot f(\eta_j, \xi_i) \right\| = \left\| \sum_{m,l}^{|d|} K_{ml} \cdot f(\eta_m, \xi_l) - f(\eta_{m_j}, \xi_{l_i}) \right\| \leqslant SVK \frac{\varepsilon}{2SVK} = \frac{1}{2}\varepsilon$, where $\eta_{m_j}$ and $\xi_{l_i}$ are points in the $d$ partition. Clearly if $\bar{d} \geqslant d$ we also have

$$\left\| \sum_{m,l}^{|\bar{d}|} K_{ml} \cdot f(\eta_m, \xi_l) - \sum_{j,i}^{|d|} K_{ji} \cdot f(\eta_j, \xi_i) \right\| \leqslant SV[K]\, \omega_{d_\varepsilon},$$

hence taking the limit in $\bar{d}$ we get 5.

(2.): Let $\varepsilon > 0$ be given and take $d_\varepsilon \in \mathbf{D}$ such that $\left\| F_K(f) - \sum_{j,i}^{|d_\varepsilon|} K_{ji} \cdot f(\eta_j, \xi_i) \right\| \leqslant \varepsilon$, then $\|F_K(f)\| \leqslant \left\| \sum_{j,i}^{|d_\varepsilon|} K_{ji} \cdot f(\eta_j, \xi_i) \right\| + \varepsilon \leqslant SV[K]\, \|f\| + \varepsilon$.

(3.): If $f \in \Omega_0([c,d] \times [a,b], X)$ and $\frac{\varepsilon}{SV[K]} > 0$ then $\exists d_\varepsilon \in \mathbf{D}$ such that $\{(t,s) \in [c,d] \times [a,b] : \|f(t,s)\| \geqslant \frac{\varepsilon}{SV[K]}\}$ lies in the set of lines parallel to the axis defined by $d_\varepsilon$. Hence $\|F_K(f)\| \leqslant \frac{\varepsilon}{SV[K]} SV[K] = \varepsilon$.

(4.): Obvious. □

Observe that we actually have the following equality:

$$\int_c^d \int_a^b d_{ts}\, K(t,s) \cdot f(t,s) = \int_c^d \int_a^b d_{ts}\, \tilde{K}(t,s) \cdot f_{--}(t,s).$$

The representation theorem follows.

**Theorem 3.5.** *The mapping*

$$K \in SV_{ca}([c,d] \times [a,b], L(X,Y)) \mapsto F_K \in L(G_{--}([c,d] \times [a,b], X), Y)$$

*where* $F_K(f) = \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot f(t,s)$ *is an isometry from the first Banach space onto the other.*

Proof. By 2. of Theorem 3.4 $F_K \in L(G_{--}([c,d] \times [a,b], X), Y)$ and $\|F_K\| \leqslant SVK$. To show that the mapping is injective just note that if $K \in SV_{ca}([c,d] \times [a,b], L(X,Y))$, $K \neq 0$ then by taking $(\tau,\sigma) \in [c,d] \times [a,b]$ and $x \in X$ such that $K(\tau,\sigma) \cdot x \neq 0$ we get $F_K(1_{]c,\tau] \times ]a,\sigma]}(t,s) \cdot x) = K(\tau,\sigma) \cdot x$. To see why the mapping is onto take any $F \in L(G_{--}([c,d] \times [a,b], X), Y)$ and define $K$ by $K(t,s) \cdot x = F(1_{]c,\tau] \times ]a,\sigma]}(t,s) \cdot x)$. Then $K(t,s) \in L(X,Y)$ and for $d \in \mathbf{D}$ we have $\left\| \sum_{j,i}^{|d|} K_{ji} \cdot x_{ji} \right\| = \left\| \sum_{j,i}^{|d|} F(1_{]t_{j-1},t_j] \times ]s_{i-1},s_i]}(t,s) \cdot x_{ji}) \right\| = \left\| F(\sum_{j,i}^{|d|} 1_{]t_{j-1},t_j] \times ]s_{i-1},s_i]}(t,s) \cdot x_{ji})\right\| \leqslant \|F\|$. Consequently $SV[K] \leqslant \|F\|$ and since $F = F_K$ at $1_{]c,\tau] \times ]a,\sigma]}(t,s) \cdot x$ they are the same by Theorem 2.5. □

Next we present our form of the classical convergence theorem of Helly.

**Theorem 3.6.** *Let $K_n$ be a sequence in $SV([c,d] \times [a,b], L(X,Y))$ and $K\colon [c,d] \times [a,b] \longrightarrow L(X,Y)$ such that $\forall x \in X$, $\forall (t,s) \in [c,d] \times [a,b]$ we have $K_n(t,s) \cdot x \longrightarrow K(t,s) \cdot x$ and $\exists M > 0$ such that $SV[K_n] \leqslant M$. Then*

*1. $K \in SV([c,d] \times [a,b], L(X,Y))$ and $SV[K] \leqslant M$.*

*2. $\forall f \in G([c,d] \times [a,b], X)$ we have*

$$\int_c^d \int_a^b d_{ts}\, K_n(t,s) \cdot f(t,s) \longrightarrow \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot f(t,s).$$

Proof. (1.): $\forall d \in \mathbf{D}$, $\forall x_{ji} \in X$ with $\|x_{ji}\| \leqslant 1$ and $\forall \varepsilon > 0$ we have

$$\begin{aligned}\left\| \sum_{j,i}^{|d|} K_{ji} \cdot x_{ji} \right\| &= \left\| \sum_{j,i}^{|d|} (K_{ji} - K_{ji}n + K_{ji}n) \cdot x_{ji} \right\| \\ &\leqslant \left\| \sum_{j,i}^{|d|} (K_{ji} - K_{ji}n) \cdot x_{ji} \right\| + \left\| \sum_{j,i}^{|d|} K_{ji}n \cdot x_{ji} \right\| \leqslant \varepsilon + M\end{aligned}$$

for a suitable $n \in \mathbb{N}$.

(2.): From

$$\begin{aligned}
\left\| \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot f(t,s) - \int_c^d \int_a^b d_{ts}\, K_n(t,s) \cdot f(t,s) \right\| \\
\leqslant \left\| \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot f(t,s) - \sum_{j,i}^{|d|} K_{ji} \cdot f(\eta_j, \xi_i) \right\| \\
+ \left\| \sum_{j,i}^{|d|} K_{ji} \cdot f(\eta_j, \xi_i) - \sum_{j,i}^{|d|} K_{ji}n \cdot f(\eta_j, \xi_i) \right\| \\
+ \left\| \sum_{j,i}^{|d|} K_{ji}n \cdot f(\eta_j, \xi_i) - \int_c^d \int_a^b d_{ts}\, K_n(t,s) \cdot f(t,s) \right\| \\
\leqslant 2M\omega_{\dot{d}}(f) + \left\| \sum_{j,i}^{|d|} (K_{ji} - K_{ji}n) \cdot f(\eta_j, \xi_i) \right\|
\end{aligned}$$

one sees that by choosing $d \in \mathbf{D}$ such that $\omega_{\dot{d}}(f) < \frac{\epsilon}{2M}$ and $n \in \mathbf{N}$ such that $\left\| \sum_{j,i}^{|d|} (K_{ji} - K_{ji}n) \cdot f(\eta_j, \xi_i) \right\| \leqslant \frac{1}{2}\epsilon$ the result follows. □

The results that follow explain how one does the composition of operators in the context of spaces of regulated functions. They are usually known under the heading of "Bray theorems".

**Theorem 3.7.** *Take $J = [c,d] \times [a,b]$, $I = [\gamma,\delta] \times [\alpha,\beta]$ and $K \in SV(J, L(Y,Z))$, $g \in G(I,X)$ and $L \colon J \times I \longrightarrow L(X,Y)$ such that $\forall q \in I$ we have $L_q \in G(J, L(X,Y))$ and $\forall p \in J$ we have $L^p \in SV(I, L(X,Y))$ with $\sup\{SV[L^p] \colon p \in J\} < \infty$. Then*

1.
$$\int_\gamma^\delta \int_\alpha^\beta d_{vu} \left[ \int_c^d \int_a^b d_{ts}\, K(t,s) \circ L(t,s,v,u) \right] \cdot g(v,u) = \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot \left[ \int_\gamma \delta \int_\alpha^\beta d_{vu}\, L(t,s,v,u) \cdot g(v,u) \right].$$

2.
$$\left\| \int_\gamma^\delta \int_\alpha^\beta d_{vu} \left[ \int_c^d \int_a^b d_{ts}\, K(t,s) \circ L(t,s,v,u) \right] \cdot g(v,u) \right\| \leqslant SV[K] \cdot \sup\{SV[L^p] \colon p \in J\} \cdot \|g\|.$$

If we wish to compose operators on spaces of regulated functions of distinct number of variables we have similar results. For example, the one below.

**Theorem 3.8.** *Take intervals $J = [c,d] \times [a,b]$, $I = [\alpha,\beta]$, and functions $K \in SV(J, L(Y,Z))$, $g \in G(I,X)$ and $h\colon J \times I \longrightarrow L(X,Y)$ such that $\forall u \in I$ we have $h_u \in G(J, L(X,Y))$ and $\forall p \in J$ we have $h^p \in SV(I, L(X,Y))$ with $\sup\{SV[h^p]\colon p \in J\} < \infty$. Then*

1.
$$\int_\alpha^\beta d_u \left[\int_c^d \int_a^b d_{ts}\, K(t,s) \circ h(t,s,u)\right] \cdot g(u) = \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot \left[\int_\alpha^\beta d_u\, h(t,s,u)\right] \cdot g(u).$$

2.
$$\left\| \int_\alpha^\beta d_u \left[\int_c^d \int_a^b d_{ts}\, K(t,s) \circ h(t,s,u)\right] \cdot g(u) \right\| \leqslant SV[K] \sup\{SV[h^p]\colon p \in J\} \cdot \|g\|.$$

We will only prove Theorem 3.7, the proof of Theorem 3.8 being totally analogous.

**Proof.** We begin by showing that the function

$$\Phi\colon (v,u) \in I \mapsto \int_c^d \int_a^b d_{ts}\, K(t,s) \circ L(t,s,v,u) \in L(X,Y)$$

is of bounded semi-variation and $SV[\Phi] \leqslant SV[K] \cdot \sup\{SV[L^p]\colon p \in J\}$. $\Phi$ is well defined because we are in a setting like that of Theorem 3.4. Since $\forall x \in X$, $\left[\int_c^d \int_a^b d_{ts}\, K(t,s) \circ L(t,s,v,u)\right] \cdot x = \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot [L(t,s,v,u) \cdot x]$, if $d \in \mathbf{D}_I$ and $x_{ji} \in X$ with $\|x_{ji}\| \leqslant 1$, we have $\left\| \sum_{j,i}^{|d|} \Phi_{ji} x_{ji} \right\| = \left\| \sum_{j,i}^{|d|} \int_c^d \int_a^b d_{ts}\, K(t,s)[L_{ji} \cdot x_{ji}] \right\| = \left\| \int_c^d \int_a^b d_{ts}\, K(t,s) \sum_{j,i}^{|d|} L_{ji} \cdot x_{ji} \right\| \leqslant SV[K] \cdot \sup\{SV[L^p]\colon p \in J\}$.

Now we show that $\Psi\colon (t,s) \in J \mapsto \int_\gamma^\delta \int_\alpha^\beta d_{vu}\, L(t,s,v,u) \cdot g(v,u)$ is regulated. Again $\Psi$ is well defined by Theorem 3.4 and if $(t,s) \in J$, $t_n \uparrow t$, $s_n \uparrow s$ then $L(t_n, s_n, v, u) \longrightarrow L(t-, s-, v, u)$. Resorting now to Theorem 3.6 we get $\int_\gamma^\delta \int_\alpha^\beta d_{vu}\, L(t_n, s_n, v, u) \cdot g(v,u) \longrightarrow \int_\gamma^\delta \int_\alpha^\beta d_{vu}\, L(t-, s-, v, u) \cdot g(v,u)$.

So let $S, T\colon G(I,X) \longrightarrow Z$ be $S(g) = \int_\gamma^\delta \int_\alpha^\beta d_{vu}\, \Phi(v,u) \cdot g(v,u)$ and $T(g) = \int_c^d \int_a^b d_{ts}\, K(t,s) \cdot \Psi(t,s)$. Because of the inequalities

$$\|S(g)\| \leqslant SV[K] \cdot \sup\{SV[L^p]\colon p \in J\} \cdot \|g\|$$
$$\|T(g)\| \leqslant SV[K] \cdot \sup\{SV[L^p]\colon p \in J\} \cdot \|g\|$$

they are both in $L(G_{--}(I,X), Z)$. Also, if $g(v,u) = 1_{]\gamma,\nu]\times]\alpha,\mu]}(v,u) \cdot x$, $x \in X$ then $S(g) = \Phi(\nu,\mu) \cdot x = \int_c^d \int_a^b d_{ts}\, K(t,s) L(t,s,\nu,\mu) \cdot x = T(g)$, hence $S = T$ according to Theorems 2.5, 2.4 and 3.4. □

## 4. Representation of bilinear operators

In this section we represent bilinear operators on cartesian products of spaces of Banach valued regulated functions by means of a Riemann-Stieltjes-Dushnik integral with respect to functions of bounded Fréchet semi-variation. $B(Y \times X, Z)$ is the Banach space of all continuous bilinear operators from $Y \times X$ into $Z$. If $K \in B(Y \times X, Z)$ we write $K(1)$ for the linear operator $K(1) \colon Y \longrightarrow L(X,Z)$ obtained from the canonical isometry $B(Y \times X, Z) \equiv L(Y, L(X,Z))$ and $K(2)$ for the linear operator $K(2) \colon X \longrightarrow L(Y,Z)$. Now we introduce the definition of a function of bounded semi-variation in the sense of Fréchet and of the interior integral.

**Definition 4.1.** Let $K \colon [c,d] \times [a,b] \longrightarrow B(Y \times X, Z)$ and $d \in \mathbf{D}$ be given. We set $SF_d[K] = \sup\left\{\left\|\sum_{j,i}^{|d|} K_{ji}(y_j, x_i)\right\| : \|y_j\| \leqslant 1, \|x_i\| \leqslant 1\right\}$ and $SF[K] = \sup\left\{SF_d[K] : d \in \mathbf{D}\right\}$.

We call $SF[K]$ the *Fréchet semi-variation of* $K$ and write $SF([c,d] \times [a,b], B(Y \times X, Z))$ for the linear space of all functions with finite Fréchet semi-variation with the semi norm $SF[K]$.

As before we write $SF_{ca}([c,d] \times [a,b], B(Y \times X, Z))$ for the subspace of $SF([c,d] \times [a,b], B(Y \times X, Z))$ whose elements $K$ satisfy $K(t,a) = K(c,s) = 0$, in $SF_{ca}([c,d] \times [a,b], B(Y \times X, Z))$ $SF[K]$ is a norm.

When $X = Y = Z = \mathbf{C}$ we write simply $SF([c,d] \times [a,b])$ and $SF_{ca}([c,d] \times [a,b])$. In this case one has $SF_d[K] = \sum_{j,i}^{|d|} K_{ji}\beta_j\alpha_i$ for suitable $\beta_j$, $\alpha_i \in \mathbf{C}$.

Given $g \colon [c,d] \longrightarrow Y$ and $f \colon [a,b] \longrightarrow X$ we say that *the pair* $(g,f)$ *is* $K$*-integrable* if the limit $\lim_{d \in \mathbf{D}} \sum_{j,i}^{|d|} K_{ji}(g(\eta_j), f(\xi_i)) = \int_c^d \int_a^b d_{ts}\, K(t,s)(g(t), f(s))$ exists where $\eta_j \in ]t_{j-1}, t_j[, : \xi_i \in ]s_{i-1}, s_i[$ in the net $\mathbf{D}$.

R e m a r k . Note that for $K \in SF_{ca}([c,d] \times [a,b], B(Y \times X, Z))$, $t \in [c,d]$, $s \in [a,b]$ we have $K_s^{(1)} \in SV_c([c,d], L(Y, L(X,Z))$ and $K^{t(2)} \in SV_a([a,b], L(X, L(Y,Z))$.

A result analogous to Theorem 3.4 follows.

**Theorem 4.1.** *Let* $f \in SV([c,d] \times [a,b], L(X,Y))$, $g \in G([c,d], Y)$ *and* $f \in G([a,b], X)$. *Then we have*

1. *There exists* $B_K(g,f) = \int_c^d \int_a^b d_{ts}\, K(t,s)(g(t), f(s))$.
2. $\|B_K(g,f)\| \leqslant SF[K] \cdot \|g\| \cdot \|f\|$.
3. *If* $g \in \Omega_0([c,d], Y)$ *or* $f \in \Omega_0([a,b], X)$ *(the one dimensional analogues of* $\Omega_0([c,d] \times [a,b], X)$ *(see Definition 2.2) then* $B_K(g,f) = 0$.

4.

$$\left\| \int_c^d \int_a^b d_{ts}\, K(t,s)(g(t), f(s)) - \sum_{j,i}^{|d|} K_{ji}(g(\eta_j), f(\xi_i)) \right\| \leqslant SF[K](\|g\| \cdot \omega_{d''}(f) + \|f\| \cdot \omega_{d'}(g)).$$

**Proof.** (1. and 4.): Suppose $g \neq 0$, $f \neq 0$ and $K \neq 0$, otherwise the proof is evident. Let us prove that the Cauchy criterion holds. So take $\varepsilon > 0$ and choose $d''_\varepsilon \in \mathbf{D}_{[c,d]}$ and $d'_\varepsilon \in \mathbf{D}_{[a,b]}$ such that $\omega_{d''_\varepsilon} < \frac{\varepsilon}{2SF[K]\|g\|}$ and $\omega_{d'_\varepsilon} < \frac{\varepsilon}{2SF[K]\|f\|}$. Now if $d \geqslant d_\varepsilon = d''_\varepsilon \times d'_\varepsilon$ we have

$$\begin{aligned}
&\left\| \sum_{j,i}^{|d|} K_{ji}(g(\eta_j), f(\xi_i)) - \sum_{n,m}^{|d_\varepsilon|} K_{ji}(g(\eta_n), f(\xi_m)) \right\| \\
&= \left\| \sum_{j,i}^{|d|} K_{ji}(g(\eta_j) - g(\eta_{n_j}), f(\xi_i)) + \sum_{j,i}^{|d|} K_{ji}(g(\eta_n), f(\xi_i) - f(\xi_{m_i})) \right\| \\
&\leqslant \left\| \sum_{j,i}^{|d|} K_{ji}\Big(\frac{2\|f\|SF[K]}{\varepsilon}(g(\eta_j) - g(\eta_{n_j})), \frac{f(\xi_i)}{\|f\|}\Big) \right\| \frac{\varepsilon}{2SF[K]} \\
&\quad + \left\| \sum_{j,i}^{|d|} K_{ji}\Big(\frac{g(\eta_j)}{\|g\|}, \frac{2\|g\|SF[K]}{\varepsilon}(f(\xi_i) - f(\xi_{m_i}))\Big) \right\| \frac{\varepsilon}{2SF[K]}
\end{aligned}$$

where $\eta_{n_j}$ and $\xi_{m_i}$ have the same meaning as in Theorem 3.4. Now 1. is evident and 2. follows by taking limit in $d$.

(2.): $\forall d \in \mathbf{D}$ we have

$$\begin{aligned}
\left\| \sum_{j,i}^{|d|} K_{ji}(g(\eta_j), f(\xi_i)) \right\| &= \left\| \sum_{j,i}^{|d|} K_{ji}\Big(\frac{g(\eta_j)}{\|g\|}, \frac{f(\xi)}{\|f\|}\Big) \right\| \cdot \|g\| \cdot \|f\| \\
&\leqslant SF[K] \cdot \|g\| \cdot \|f\|.
\end{aligned}$$

(3.): Let $g \in \Omega_0([c,d], Y)$. Given $\varepsilon > 0$ there exists $d_\varepsilon \in \mathbf{D}$ such that $\{t \in [c,d] : \|g(t)\| \geqslant \frac{\varepsilon}{SF[K]\|f\|}\} \subset d''_\varepsilon$. If $d'' \geqslant d''_\varepsilon$ then for $d = d'' \times d'$ we have

$$\begin{aligned}
&\left\| \sum_{j,i}^{|d|} K_{ji}(g(\eta_j), f(\xi_i)) \right\| \\
&= \left\| \sum_{j,i}^{|d|} K_{ji}\Big(\frac{SF[K]\|f\|g(\eta_j)}{\varepsilon}, \frac{f(\xi_i)}{\|f\|}\Big) \right\| \frac{\varepsilon}{SF[K]} \leqslant \varepsilon.
\end{aligned}$$

□

Next we establish the representation theorem.

**Theorem 4.2.** *The mapping*

$$K \in SF_{ca}([c,d] \times [a,b], B(Y \times X, Z)) \\ \mapsto B_K \in B(G_-([c,d],Y) \times G_-([a,b],X), Z)$$

*where* $B_K(g,f) = \int_c^d \int_a^b d_{ts}\, K(t,s)(g(t), f(s))$ *is an isometry from the first Banach space onto the other.*

**Proof.** From the previous result one immediately sees that the mapping is well defined and that $\|B_K\| \leqslant SF[K]$. If $K \neq 0$ there are $\tau \in ]c,d]$, $\sigma \in ]a,b]$, $y \in Y$ and $x \in X$ such that $B_K(1_{]c,\tau]}(t)\cdot y, 1_{]a,\sigma]}(s)\cdot x) = K(]\tau,\sigma])(y,x) \neq 0$, hence the mapping is injective. Let us now show that it is onto as well. So take $B \in B(G_-([c,d],Y) \times G_-([a,b],X), Z)$ and define $K$ by $K(\tau,\sigma)(y,x) = B_K(1_{c,\tau}(t)\cdot y, 1_{a,\sigma}(s)\cdot x)$. Let us show now that such $K$ is in $SF_{ca}([c,d] \times [a,b], B(Y \times X, Z))$. So take $d = d'' \times d' \in \mathbf{D}$, $y_j \in Y$ and $x_i \in X$ with $\|y_j\| \leqslant 1$, $\|x_i\| \leqslant 1$. We then have

$$\begin{aligned}
\Big\| \sum_{j,i}^{|d|} K_{ji}(y_j, x_i) \Big\| &= \Big\| \sum_{j,i}^{|d|} B(1_{]t_{j-1},t_j]}(t)\cdot y_j, 1_{]s_{i-1},s_i]}(s)\cdot x_i) \Big\| \\
&= \Big\| B\Big( \sum_{j=1}^{|d''|} 1_{]t_{j-1},t_j]}(t)\cdot y_j, \sum_{i=1}^{|d'|} 1_{]s_{i-1},s_i]}(s)\cdot x_i \Big) \Big\| \\
&\leqslant \|B\|.
\end{aligned}$$

By the usual density argument, since $B$ and $B_K$ agree at $(1_{]c,\tau]}(t)\cdot y, 1_{]a,\sigma]}(s)\cdot x)$ they are really the same operator. □

We end this section by a theorem of Fubini type.

**Theorem 4.3.** *The following identity holds for any* $g \in G([c,d],Y)$, $f \in G([a,b], X)$ *and* $K \in SF([c,d] \times [a,b], B(Y \times X, Z))$.

$$\begin{aligned}
\int_c^d \int_a^b d_{ts}\, K(t,s)\cdot(g(t), f(s)) &= \int_a^b d_s \left[ \int_c^d d_t\, K(t,s)(1) \right] \cdot g(t) \cdot f(s) \\
&= \int_c^d d_t \left[ \int_a^b d_s\, K(t,s)(2) \cdot f(s) \right] \cdot g(t)
\end{aligned}$$

**Proof.** To begin with consider the function $\Phi\colon s \in [a,b] \longrightarrow \int_c^d d_t K(t,s)(1) \cdot g(t) \in L(X,Z)$. It is well defined due to the remark following Definition 4.1. We now show it is of bounded semi-variation. So choose a $d' \in \mathbf{D}_{[a,b]}$, $x_i \in X$ with $\|x_i\| \leqslant 1$. We have $\left\| \sum_{i=1}^{|d'|} \Phi(s_i) - \Phi(s_{i-1}) \cdot x_i \right\| = \left\| \sum_{i=1}^{|d'|} \int_c^d d_t K(t, s_i)(1) - K(t, s_{i-1})(1) \cdot g(t) \cdot x_i \right\| \leqslant \left\| \sum_{i=1}^{|d'|} \int_c^d d_t K(t, s_i)(1) - K(t, s_{i-1})(1) \cdot g(t) \cdot x_i - \sum_{i=1}^{|d'|} \sum_{j=1}^{|d''|} K_{ji}(1) \cdot g(\eta_j) \cdot x_i \right\| + \left\| \sum_{i=1}^{|d'|} \sum_{j=1}^{|d''|} K_{ji}(1) \cdot g(\eta_j) \cdot x_i \right\|$. Now choose $d'' \in \mathbf{D}_{[c,d]}$ such that $\left\| \int_c^d d_t K(t, s_i)(1) - K(t, s_{i-1})(1) \cdot g(t) - \sum_{j=1}^{|d''|} K_{ji}(1) \cdot g(\eta_j) \right\| < \frac{\varepsilon}{\|d'\|}$. With this choice of $d''$ we get $\left\| \sum_{i=1}^{|d'|} \Phi(s_i) - \Phi(s_{i-1}) \cdot x_i \right\| \leqslant \varepsilon + SF[K]\|g\|$, therefore $SV\Phi \leqslant SF[K]\|g\|$ and the integral $\int_a^b d_s \Phi(s) \cdot f(s)$ is meanigful.

To prove the identity we resort to the usual density argument. Consider $S, T$: $G_-([c,d], Y) \times G_-([a,b], X) \longrightarrow Z$ defined by $S(g,f) = \int_c^d \int_a^b d_{ts}\, K(t,s)(g(t), f(s))$ and $T(g,f) = \int_a^b d_s \int_c^d d_t\, K(t,s)(1) \cdot g(t) \cdot f(s)$. If $g(t) = 1_{c,\tau}(t) \cdot y$ and $f(s) = 1_{a,\sigma}(s) \cdot x$ we have $S(g,f) = K(\tau,\sigma)(y,x)$. On the other hand $T(g,f) = \int_a^b d_s \int_c^d d_t\, K(t,s)(1) \cdot g(t) \cdot f(s) = \int_a^b d_s\, K(\tau, s)(1) \cdot 1_{a,\sigma}(s) \cdot x = K(\tau,\sigma)(y,x)$. Consequently $S = T$. □

## 5. Some applications

We present here a few applications of the results obtained so far. We begin with a proof of the well known result that $BV_{ca}([c,d] \times [a,b]) \not\subseteq SF_{ca}([c,d] \times [a,b])$. When $X = \mathbf{C}$ we write $BW([a,b], Y)$ instead of $SV([a,b], L(X,Y))$.

Let us first state a result which is nothing but an easy reformulation of the representation theorems.

**Theorem 5.1.** *We have the following isometries:*

$$SF_{ca}([c,d] \times [a,b], B(Y \times X, Z)) \equiv SV_c([c,d], L(Y, SV_a([a,b], L(X,Z))))$$
$$SF_{ca}([c,d] \times [a,b], B(Y \times X, Z)) \equiv SV_c([c,d], L(Y, SV_a([a,b], L(X,Z))))$$
$$SF_{ca}([c,d] \times [a,b], B(\mathbf{C} \times X, Z)) \equiv BW_c([c,d], SV_a([a,b], L(X',Z)))$$
$$SF_{ca}([c,d] \times [a,b]) \equiv BW_c([c,d], BV_a([a,b]))$$
$$SF_{ca}([c,d] \times [a,b], B(Y \times X, Z)) \equiv L(G_-([c,d], Y), SV_a([a,b], L(X,Z)))$$
$$SF_{ca}([c,d] \times [a,b], B(Y \times X, \mathbf{C})) \equiv L(G_-([c,d], Y), BV_a([a,b], X'))$$
$$SF_{ca}([c,d] \times [a,b], B(Y \times \mathbf{C}, Z)) \equiv L(G_-([c,d], Y), BW_a([a,b], Z))$$
$$SF_{ca}([c,d] \times [a,b]) \equiv L(G_-([c,d], Y), BV_a([a,b]))$$

Proof. By Theorem 4.2 we have

$$\begin{aligned} SF_{ca}([c,d]\times[a,b], B(Y\times X, Z)) \\ &\equiv B(G_{-}([c,d],Y)\times G_{-}([a,b],X),Z) \\ &\equiv L(G_{-}([c,d],Y), L(G_{-}([a,b],X),Z)) \\ &\equiv L(G_{-}([c,d],Y), SV_a([a,b],L(X,Z))) \\ &\equiv SV_c([c,d], L(Y, SV_a([a,b],L(X,Z)))). \end{aligned}$$

For the third isometry see [1], all the other cases being simply particularizations of what was written above. □

Now we give a "simple" proof of $BV \not\subseteq SF$. Of course the burden must lie somewhere, in this case it is in the following result due to Rocha, [4], the proof of which relies on the theorem of Dvoretzky-Rogers.

**Lemma 5.1.** *For a Banach space $X$, $BV([a,b],X) = BW([a,b],X)$ if and only if $X$ is finite dimensional.*

Now we prove

**Theorem 5.2.** $BV_{ca}([c,d]\times[a,b]) \not\subseteq SF_{ca}([c,d]\times[a,b])$.

Proof. By Theorem 3.1, $BV_{ca}([c,d]\times[a,b]) \equiv BV_c([c,d], BV_a([a,b]))$. Now use the fourth isometry of Theorem 5.1 and Lemma 5.1. □

Next we prove that $SF \subseteq G$. To obtain it we will use the following two results the first of which can be found in [4].

**Lemma 5.2.** *If $X$ is a normed space and $Y$ a reflexive Banach space (or more generally a weakly sequentially complete Banach space) then $SV([a,b],L(X,Y)) \subseteq G([a,b],L(X,Y))$.*

The other can be found in [7].

**Lemma 5.3.** *The Banach space $BV_a([a,b])$ is weakly sequentially complete.*

Now from the two previous results and the fourth isometry of Theorem 5.1 we easily get

**Theorem 5.3.** $SF_{ca}([c,d]\times[a,b]) \subseteq G([c,d]\times[a,b])$.

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