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EQUIVALENCES BETWEEN ISOMORPHISM CLASSES ON INFINITE GRAPHS

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Summary. The paper studies some equivalence relations between isomorphism classes of countable graphs which correspond in a certain sense to various distances between isomorphism classes of finite graphs.

Keywords: distance between graphs, isomorphism class graphs, tree, edge rotation, equivalence relation

AMS classification: 05C99, 05C05

Various distances between isomorphism classes of graphs were defined. An isomorphism class of graphs is the class of all graphs which are isomorphic to a given graph. For the sake of brevity we may speak about distances between graphs instead of distances between isomorphism classes of graphs; in this case we must have in mind that two graphs whose distance is zero need not be identical, but they are isomorphic.

Let G_1, G_2 be two finite graphs with the same number n of vertices. The distance $\delta(G_1, G_2)$ is equal to n minus the maximum number of vertices of a graph which is isomorphic simultaneously to an induced subgraph of G_1 and to an induced subgraph of G_2 [4].

Now let G_1, G_2 be arbitrary two finite graphs. The edge distance $\delta_E(G_1, G_2)$, introduced by V. Baláž, J. Koča, M. Kvasnička and M. Sekanina [1], is defined so that

$$\delta_E(G_1, G_2) = |E_1| + |E_2| - |E_{12}| + \big||v_1| - |v_2|\big|,$$

where V_1, V_2 are the vertex sets of G_1, G_2 and E_1, E_2 are their edge sets, the symbol E_{12} denotes the edge set of a graph which is isomorphic simultaneously to a subgraph of G_1 and to a subgraph of G_2 and has the maximum number of edges among all graphs with this property.

Let T_1, T_2 be two finite trees with the same number n of vertices. The tree distance $\delta_T(T_1, T_2)$ is equal to n minus the maximum number of vertices of a tree which is isomorphic simultaneously to a subtree of T_1 and to a subtree of T_2 [5].

Now we shall define the edge rotation and the edge shift. Let x, y, z be three vertices of an undirected graph G such that x is adjacent to y and not to z. To perform an edge rotation of the edge xy to the position xz means to delete the edge xy from G and to add the edge xz to it. If, moreover, y and z are adjacent in G, such an edge rotation is called an edge shift of the edge xy to the position xz along the edge yz.

Let G_1, G_2 be two finite graphs with the same number of vertices and the same number of edges. The edge-rotation distance $\delta_R(G_1, G_2)$ is the minimum number of edge rotations which are necessary for transforming the graph G_1 and G_2 into a graph isomorphic to G_2. If, moreover, both G_1 are connected, then the edge-shift distance $\delta_S(G_1, G_2)$ is the minimum number of edge shifts which are necessary for transforming the graph G_1 into a graph isomorphic to G_2. The edge-rotation distance was introduced by G. Chartrand, F. Saba and H.-B. Zou in [2], the edge-shift distance by M. Johnson in [3].

All these distances were defined and studied for finite graphs. Here we will introduce some equivalence relations between isomorphism classes of infinite graphs which correspond in a certain sense to the above mentioned distances. We will limit our considerations to countable graphs, i. e. graphs in which the cardinality of the vertex set is $\aleph_0$. For the sake of simplicity again we will speak about equivalences between graphs instead of equivalences between isomorphism classes of graphs. Obviously, if two graphs belong to the same isomorphism class, then they are equivalent in each of the described equivalence relations.

Thus we consider countable undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ denotes its vertex set and $E(G)$ its edge set.

First we shall define the relation ε. For two countable graphs G_1, G_2 we have $(G_1, G_2) \in \varepsilon$ if and only if there exists an induced subgraph G_1' of G_1 and an induced subgraph G_2' of G_2 such that $G_1' \cong G_2'$ and the sets $V(G_1) - V(G_1')$, $V(G_2) - V(G_2')$ are finite.

Now let us define the relation ε_E. For two countable graphs G_1, G_2 we have $(G_1, G_2) \in \varepsilon_E$ if and only if there exists a subgraph G_1' of G_1 and a subgraph G_2' of G_2 such that $G_1' \cong G_2'$ and the set $E(G_1) - E(G_1')$, $E(G_2) - E(G_2')$ are finite.

The next relation is ε_R. If G_1, G_2 are countable graphs, then $(G_1, G_2) \in \varepsilon_R$ if and only if G_1 can be transformed by a finite number of edge rotations into a graph isomorphic to G_2.

Finally, let T_1, T_2 be two countable trees. We have $(T_1, T_2) \in \varepsilon_T$ if and only if there exists a subtree T_1' of T_1 and a subtree T_2' of T_2 such that $T_1' \cong T_2'$ and the sets $V(T_1) - V(T_1')$, $V(T_2) - V(t_2')$ are finite.

Evidently, the following assertion holds.

Theorem 1. *The relation ε, ε_E, ε_R are equivalences on the set of all isomorphism classes of countable graphs, the relation ε_T is an equivalence on the set of all isomorphism classes of countable trees.*

We can call ε the subgraph equivalence, ε_E the edge equivalence, ε_R the edge-rotation equivalence, ε_T the tree equivalence.

Theorem 2. *Let G_1, G_2 be two countable graphs without infinite sets of isolated vertices, and let $(G_1, G_2) \in \varepsilon_E$. Then $(G_1, G_2) \in \varepsilon$.*

Proof. Let G_1', G_2' be the graphs described in the definition of ε_E. Let V_1 be the set of all end vertices of edges from the set $E(G_1) - E(G_1')$ which belong to $V(G_1')$. Similarly, let V_2 be the set of all end vertices of edges from $E(G_2) - E(G_2')$ which belong to $V(G_2')$. The sets V_1, V_2 are evidently finite. Let φ be an isomorphic mapping of G_1' onto G_2'. Let G_1'' be the subgraph of G_1' induced by the set $V(G_1') - (V_1 \cup \varphi^{-1}(V_2))$, let G_2'' be the subgraph of G_2' induced by the set $V(G_2') - (V_2 \cup \varphi(V_1))$. Evidently, the restriction of φ onto $V(G_1'')$ maps G_1'' isomorphically onto G_2'' and therefore $G_1'' \cong G_2''$. The subgraph of G_1 induced by the set $V(G_1) - V(G_1')$ has a finite edge set (a subset of $E(G_1) - E(G_1')$), and therefore a finite number of non-isolated vertices. As we have assumed that G_1 does not contain an infinite set of isolated vertices, the set $V(G_1) - V(G_1')$ is finite. The set $V(G_1) - V(G_1'')$ is the union of three finite sets $V(G_1) - V(G_1')$, V_1, $\varphi^{-1}(V_2)$ and thus it is finite. Analogously, $V(G_2) - V(G_2'')$ is finite. Hence $(G_1, G_2) \in \varepsilon$. □

Remark 1. There exist countable graphs G_1, G_2 without infinite sets of isolated vertices such that $(G_1, G_2) \in \varepsilon$ and $(G_1, G_2) \notin \varepsilon_E$.

For G_2 we may take an arbitrary locally finite countable graph without an infinite set of isolated vertices; the graph obtained from G_2 by adding a new vertex v and joining it by edges with all vertices of G_2 will be G_1. Evidently $(G_1, G_2) \in \varepsilon$. Now let G_1' be a subgraph of G_1 such that $E(G_1) - E(G_1')$ is finite. The vertex v is incident in G_1 with infinitely many edges; as only a finite number of them is in $E(G_1) - E(G_1')$, the vertex v has the infinite degree also in G_1'. The graph G_1' is not locally finite and cannot be isomorphic to any subgraph of the locally finite graph G_2. This implies $(G_1, G_2) \notin \varepsilon_E$.

The following assertion is evident.

Theorem 3. *Let G_1, G_2 be two locally finite countable graphs such that $(G_1, G_2) \in \varepsilon$. Then $(G_1, G_2) \in \varepsilon_E$.*

Now we turn to trees.

Theorem 4. *Let T_1, T_2 be two countable trees such that $(T_1, T_2) \in \varepsilon_T$. Then $(T_1, T_2) \in \varepsilon_E$.*

Proof. Let T_1', T_2' be the trees used in the definition of ε_T. The set $V(T_1) - V(T_1')$ is finite. Consider the set $E(T_1) - E(T_1')$. No edge of this set may join two vertices of T_1'; otherwise there would be a circuit in T_1 and T_1 would not be a tree. For the same reason each vertex of $V(T_1) - V(T_1')$ may be adjacent at most to one vertex of T_1'. The set $E(T_1) - E(T_1')$ consists of the edge set of the subgraph of G_1 induced by $V(T_1) - V(T_1')$ and of all edges joining a vertex of $V(T_1) - V(T_1')$ with a vertex of T_1'. The former set is the edge of a graph with a finite vertex set, therefore it is finite. The latter set has a cardinality not exceeding the cardinality of $V(T_1) - V(T_1')$, therefore it is also finite. Hence $E(T_1) - E(T_1')$ is finite and $(T_1, T_2) \in \varepsilon_E$. □

Remark 2. There exist countable trees T_1, T_2 such that $(T_1, T_2) \in \varepsilon_E$ and $(T_1, T_2) \notin \varepsilon_T$.

Let C_1, C_2 be two vertex-disjoint two-way infinite paths. Choose a vertex u_1 of C_1 and a vertex u_2 of C_2. Add a new vertex v and join it by edges with u_1 and u_2; the tree thus obtained will be denoted by T_1. If instead of v we take two new vertices w_1, w_2 and new edges u_1w_1, w_1w_2, u_2w_2, we obtain a tree T_2. Evidently $(T_1, T_2) \in \varepsilon_E$. Evidently, every proper subtree T_1' of T_1 has the property that there exists a one-way infinite path in T_1 which is disjoint with T_1', and thus T_1 is in the relation ε_T only with itself. The same holds for T_2. Hence $(T_1, T_2) \notin \varepsilon_T$.

Now we shall treat edge rotations. The following assertion is evident.

Theorem 5. *Let G_1, G_2 be two countable graphs, let $(G_1, G_2) \in \varepsilon_R$. Then $(G_1, G_2) \in \varepsilon_E$.*

Note that at each edge rotation the number of vertices of odd (finite) degrees in the graph either increases by two, or decreases by two, or remains the same. The same holds also for the number of vertices of even degrees. This implies the following assertion.

Theorem 6. *Let G_1, G_2 be two locally finite countable graphs such that $(G_1, G_2) \in \varepsilon_R$. Then the numbers of vertices of odd degrees in G_1 and G_2 are* *either both infinite, or both finite and congruent modulo 2, and the same assertion holds for the numbers of vertices of even degrees.*

Remark 3. There exist countable graphs G_1, G_2 such that $(G_1, G_2) \in \varepsilon_T$ and $(G_1, G_2) \notin \varepsilon_R$.

The vertex set of the graph G_1 consists of the vertices u_i for all non-negative integers i and of the vertices v_i for all positive integers i. The edges of G_1 are $u_i u_{i+1}$ for all non-negative integers i and $u_i v_i$ for all positive integers i. The graph G_2 is obtained from G_1 by deleting the vertex v_1 and the edge $u_1 v_1$. Both G_1, G_2 are trees and evidently $(G_1, G_2) \in \varepsilon_T$. All vertices of G_1 have odd degrees, while G_2 has exactly one vertex, namely u_1, of an even degree. According to Theorem 6 we have $(G_1, G_2) \notin \varepsilon_R$.

At the end we state an evident assertion on edge shifts.

Theorem 7. *Let G_1, G_2 be two connected countable graphs and let $(G_1, G_2) \in \varepsilon_R$. Then G_1 can be transformed into a graph isomorphic to G_2 by a finite number of edge shifts.*

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Souhrn

EKVIVALENCE MEZI TŘÍDAMI ISOMORFISMŮ U NEKONEČNÝCH GRAFŮ

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Článek se zabývá třídami ekvivalence mezi třídami isomorfismu spočetných grafů, které v jistém smyslu odpovídají různým vzdálenostem mezi třídami isomorfismu konečných grafů.

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