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Mathematica Bohemica, Vol. 117 (1992), No. 2, 133–136

Persistent URL: `http://dml.cz/dmlcz/125906`

AN ELEMENTARY PROOF OF THE ONE-DIMENSIONAL RADEMACHER THEOREM

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(Received July 9, 1990)

Summary. An elementary short proof of the one-dimensional Rademacher theorem on differentiability of Lipschitz functions is given.

Keywords: Rademacher theorem, derivative, Lipschitz function

AMS classification: 26A24

1. INTRODUCTION

The one-dimensional Lebesgue density theorem was proved in [5] without any covering theorem.

The aim of the present note is to show that also the one-dimensional Rademacher theorem can be easily proved by this (slightly modified) method without any covering theorem. For a very short proof we need only three obvious well-known lemmas and the classical Dini's theorem on Dini derivates which has a quite elementary proof.

Thus the present note completes in a sense the paper [2] which shows that the n-dimansional Rademacher theorem can be easily deduced from the one-dimensional Rademacher theorem and the Fubini theorem. Remind that the Rademacher theorem [3] asserts that each Lipschitz function on $\mathbf{R}^n$ is Frechet differentiable almost everywhere.

Finally, recall that the one-dimensional Rademacher theorem is an immediate consequence of the Lebesgue theorem on the differentability of monotone functions since the function $g(x) = f(x)+Kx$ is obviously monotone whenever f is K-Lipschitz. Another argument uses the fact that every Lipschitz function is locally absolutely continuous. Both these proofs use relatively deep theorems which are usually proved by the Vitali covering theorem or by similar means.

2. Preliminaries

In the sequel f is always a real function defined on a bounded interval (a,b). The one-sided Dini derivates of f at $x \in (a,b)$ will be denoted by the symbols $D_1 f(x) := D^+ f(x)$, $D_2 f(x) := D_+ f(x)$, $D_3 f(x) := D^- f(x)$ and $D_4 f(x) := D_- f(x)$. A function f is said to be K-Lipschitz on a set $M \subset \mathbf{R}$ if $|f(x) - f(y)| \leqslant K|x - y|$ for each $x, y \in M$. The symbol μ stands for the Lebesgue measure on $\mathbf{R}$.

We shall need three well-known lemmas. The following two are quite obvious.

Lemma 1.

$$\{x \in (a,b) \colon f'(x) \text{ does not exists}\} = \bigcup_{i,j=1}^{4} \{x \colon D_i f(x) < D_j f(x)\}.$$

Lemma 2. *Let $x \in (a,b)$ and $i \in \{1,2,3,4\}$. Then:*
(i) If $g'(x) = c \in \mathbf{R}$, then $D_i(f+g)(x) = D_i f(x) + c$.
(ii) If g is nondecreasing on (a,b), then $D_i(f+g)(x) \geqslant D_i f(x)$.

Lemma 3. *Let f be a Lipschitz function and $i \in \{1,2,3,4\}$. Then the function $D_i f(x)$ is Lebesgue measurable.*

Proof. We can suppose $i = 1$. If we define $f(x) = 0$ for $x \notin (a,b)$, then $D_1 f(x) = \lim_{n\to\infty} \sup\{(f(x+h) - f(x))/h \colon 0 < h < 1/n,\ h \text{ is rational}\}$. Since each function $g_h(x) := (f(x+h) - f(x))/h$ is obviously measurable, we obtain that $D_1 f(x)$ is measurable as well.

The only non-trivial fact we shall need is the classical Dini's theorem (cf. Theorem 88 of [1] or [4, p. 204]). It can be formulated in the following way. $\square$

Lemma 4. *Let g be a continuous function on $[c,d]$, $t \in \mathbf{R}$ and $i \in \{1,2,3,4\}$. If $D_i g(x) \geqslant t$ (or $D_i g(x) \leqslant t$) for each $x \in (c,d)$, then $g(d) - g(c) \geqslant t(d-c)$ ($g(d) - g(c) \leqslant t(d-c)$, respectively).*

3. Proof

Theorem. *Let f be a K-Lipschitz function on (a,b). Then f is differentiable almost everywhere.*

Proof. Suppose on the contrary that this is not the case. By Lemma 1 and Lemma 3 there exist $i,j\in\{1,2,3,4\}$ such that $\mu\{x\colon D_i f(x) < D_j(x)\} > 0$. Consequently, there exist rational numbers $r<s$ such that $\mu L > 0$, where $L := \{x\colon D_i f(x) < r < s < D_j f(x)\}$. Find $a<\alpha<\beta<b$ such that $\mu M > 0$, where $M := L\cap(\alpha,\beta)$. Further, find $\varepsilon>0$ such that

$$s > r + 4K\varepsilon \tag{1}$$

and an open set $M\subset G\subset(\alpha,\beta)$ such that $\mu M/\mu G > 1-\varepsilon$. It is easy to see that there exists a component (c,d) of G such that $\mu\big(M\cap(c,d)\big)/(d-c) > 1-\varepsilon$. Choose a closed set $F\subset M\cap(c,d)$ such that $\mu F/(d-c) > 1-\varepsilon$. Now put $H := (c,d) - F$ and $h(x) = \mu\big((c,x)\cap H\big)$, $u(x) = f(x) + 2Kh(x)$, $v(x) = f(x) - 2Kh(x)$ for $x\in(c,d)$. Since obviously h is nondecreasing and continuous, $h'(x) = 1$ for $x\in H$ and $\{D_i f(x), D_j f(x), r, s\}\subset[-K,K]$ for each $x\in(c,d)$, Lemma 2 easily implies that $D_j u(x)\geqslant s$ and $D_i v(x)\leqslant r$ for each $x\leqslant(c,d)$. Since $h(d)-h(c) = \mu\big((c,d)\cap H\big) < \varepsilon(d-c)$, Lemma 4 implies

$$s(d-c)\leqslant u(d)-u(c) = \big(f(d)-f(c)\big)+2K\big(h(d)-h(c)\big)\leqslant\big(f(d)-f(c)\big)+2K\varepsilon(d-c)$$

and

$$r(d-c)\geqslant v(d)-v(c) = \big(f(d)-f(c)\big)-2K\big(h(d)-h(c)\big)\geqslant\big(f(d)-f(c)\big)-2K\varepsilon(d-c).$$

These inequalities imply $s-r\leqslant 4K\varepsilon$, which contradicts (1). □

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Souhrn

ELEMENTÁRNÍ DŮKAZ JEDNOROZMĚRNÉ RADEMACHEROVY VĚTY

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V článku je podán jednoduchý elementární důkaz jednorozměrné Rademacherovy věty o derivování lipschitzovských funkcí.

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