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ON A THEOREM OF M. G. ARSOVE

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1. INTRODUCTION

In his paper [1] M. G. Arsove stated a condition for the continuity of the logarithmic potential

$$u(z)=\int \log|z-\zeta|^{-1}\,\mathrm{d}m_\zeta$$

in the plane. Here m is a non negative measure. This condition is both necessary and sufficient. But Arsove also explained that his theorem does not hold if m is a signed measure. In this case the theorem only gives a sufficient condition.

In the present paper the author investigates the case of signed merasures. Also more general kernels are considered and necessary and sufficient conditions for the continuity of such potentials are obtained.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE CONTINUITY OF POTENTIALS WITH A GENERAL KERNEL

We consider the potential

$$u(x)=\int_A K(x,\xi)\,\mathrm{d}\varphi.$$

Here A is a compact set of the n-dimensional Euclidean space $\mathbf{R}^n$ $(n=2,3,4,\ldots)$. φ is a signed measure defined on a σ-algebra $\mathfrak{A}$ of subsets of A. $\varphi(A)$ is finite and $\mathfrak{A}$ contains all Borel subsets of A. $K(x,\xi)$ is defined and continuous for (x,ξ) with $x\in\mathbf{R}^n$, $\xi\in A$ and $x\neq\xi$. For $x=\xi$ the equality $|K(x,\xi)|=+\infty$ holds.

Theorem 1.

$$u(x)=\int_A K(x,\xi)\,\mathrm{d}\varphi$$

is continuous in $x_0 \in A$ *iff*

$$\lim_{\varrho\to 0}\Big(\overline{\lim_{x\to x_0}}\Big|\int\limits_{A\cap S_\varrho(x_0)} K(x,\xi)\,\mathrm{d}\varphi\Big|\Big) = 0 \tag{1}$$

and

$$\lim_{\varrho\to 0}\int\limits_{A\cap S_\varrho(x_0)} |K(x,\xi)|\,\mathrm{d}|\varphi| = 0. \tag{2}$$

Proof. Let $x_0 \in A$, $x \in \mathbf{R}^n$ with $x \neq x_0$, $\varrho > 0$. We put

$$f_\varrho(x) = \int\limits_{A\setminus S_\varrho(x_0)} K(x,\xi)\,\mathrm{d}\varphi$$

and

$$g_\varrho(x) = \int\limits_{A\cap S_\varrho(x_0)} K(x,\xi)\,\mathrm{d}\varphi.$$

First we assume that u is continuous at x_0. This implies that u is finite at x_0. Hence (2) is fulfilled (comp. S. Dümmel [2], p. 355). Further, for a fixed ϱ the function f_ϱ is continuous at x_0 since x_0 is an exterior point of $A - S_\varrho(x_0)$. Thus also

$$g_\varrho(x) = u(x) - f_\varrho(x)$$

is continuous at x_0 and we obtain

$$\lim_{\varrho\to 0} g_\varrho(x) = g_\varrho(x_0). \tag{3}$$

From (3) and (2) it follows that

$$\begin{aligned} 0 &\leqslant \overline{\lim_{x\to x_0}}\Big(\overline{\lim_{x\to x_0}}|g_\varrho(x)|\Big) = \overline{\lim_{\varrho\to 0}}|g_\varrho(x_0)| \\ &\leqslant \lim_{\varrho\to 0}\int\limits_{A\cap S_\varrho(x_0)} |K(x,\xi)|\,\mathrm{d}|\varphi| = 0. \end{aligned}$$

Hence the condition (1) is also fulfilled.

Now we assume that the conditions (1) and (2) hold and we show that then u is continuous at x_0. From (1) we conclude: There exists a $\varrho_0 > 0$ such that

$$\overline{\lim_{x\to x_0}}|g_{\varrho_0}(x)|$$

is finite. This further implies that there exists a δ with $0 < \delta < \varrho_0$ such that $|g_{\varrho_0}(x)|$ is finite for all $x \in S_\delta(x_0) - \{x_0\}$. Then also $u(x)$ is finite for all these x. The finiteness of $u(x_0)$ follows from (2). Further, we obtain for arbitrary $\varrho > 0$ and $x \in S_\delta(x_0) - \{x_0\}$:

$$|u(x) - u(x_0)| \leqslant |f_\varrho(x) - f_\varrho(x_0)| + |g_\varrho(x)| + |g_\varrho(x_0)|,$$

and because of the continuity of f_ϱ at x_0:

$$\overline{\lim_{x \to x_0}} |u(x) - u(x_0)| \leqslant \overline{\lim_{x \to x_0}} |g_\varrho(x)| + |g_\varrho(x_0)|.$$

For $\varrho \to 0$ we obtain

$$\overline{\lim_{x \to x_0}} |u(x) - u(x_0)| \leqslant \lim_{\varrho \to 0} \Big(\overline{\lim_{x \to x_0}} |g_\varrho(x)| \Big) + \lim_{\varrho \to 0} |g_\varrho(x_0)|.$$

Using (1) and (2) we conclude

$$\lim_{x \to x_0} |u(x) - u(x_0)| = 0,$$

i.e. u is continuous at x_0. □

3. Necessary and sufficient conditions for the continuity of potentials with a special kernel

Now we consider the special kernel

$$K(x, \xi) = \Phi(r(x, \xi))$$

where $r(x, \xi)$ is the distance of $x, \xi \in \mathbf{R}^n$. We formulate the following supposition (α). (α): $\Phi \in C^1(0, +\infty)$, Φ is monotonously decreasing in $(0, +\infty)$,

$$\lim_{r \to +0} \Phi(r) = +\infty,$$

and there exists a $\varrho_1 > 0$ such that $\Phi(r) \geqslant 0$ for all r with $0 < r \leqslant \varrho_1$.

For instance, $\Phi(r) = r^{2-n}$ $(n \geqslant 3)$ satisfies the supposition (α). For the kernels $K(x, \xi) = \Phi(r(x, \xi))$ the condition (1) can be formulated in another way. We have

Lemma 1. *Let $x_0 \in A$, let $\Phi(r)$ satisfy the supposition (α) and let*

$$\lim_{r \to 0} \Phi(r) |\varphi| (A \cap \bar{S}_r(x_0)) = 0. \tag{4}$$

Then

$$\lim_{\varrho\to 0}\Big(\overline{\lim_{x\to x_0}}\Big(\int\limits_{A\cap S_\varrho(x_0)}\Phi(r(x,\xi))\,\mathrm{d}\varphi\Big)\Big)=0 \tag{5}$$

iff

$$\lim_{\varrho\to 0}\Big(\overline{\lim_{x\to x_0}}\int\limits_{A\cap S_\varrho(x)}\Phi(r(x,\xi))\,\mathrm{d}\varphi\Big)=0. \tag{6}$$

Proof. First we assume that (5) holds. Then

$$\begin{aligned}\Big|\int\limits_{A\cap S_\varrho(x)}\Phi(r(x,\xi))\,\mathrm{d}\varphi\Big| &\leqslant \Big|\int\limits_{A\cap S_\varrho(x)}\Phi(r(x,\xi))\,\mathrm{d}\varphi-\int\limits_{A\cap S_\varrho(x_0)}\Phi(r(x,\xi))\,\mathrm{d}\varphi\Big|\\ &\quad+\Big|\int\limits_{A\cap S_\varrho(x_0)}\Phi(r(x,\xi))\,\mathrm{d}\varphi\Big|.\end{aligned} \tag{7}$$

Further, for sufficiently small ϱ and $r(x,x_0)<\varrho$ we have

$$\begin{aligned}A(\varrho,x) &:= \Big|\int\limits_{A\cap S_\varrho(x)}\Phi(r(x,\xi))\,\mathrm{d}\varphi-\int\limits_{A\cap S_\varrho(x_0)}\Phi(r(x,\xi))\,\mathrm{d}\varphi\Big|\\ &\leqslant \int\limits_{A\cap S_\varrho(x)\setminus S_\varrho(x_0)}\Phi(r(x,\xi))\,\mathrm{d}|\varphi|+\int\limits_{A\cap S_\varrho(x_0)\setminus S_\varrho(x)}\Phi(r(x,\xi))\,\mathrm{d}|\varphi|\\ &\leqslant \Phi(\varrho-r(x,x_0))|\varphi|(A\cap S_\varrho(x))+\Phi(\varrho)|\varphi|(A\cap S_\varrho(x_0)).\end{aligned}$$

Hence

$$\begin{aligned}\overline{\lim_{x\to x_0}}A(\varrho,x) &\leqslant \Phi(\varrho)|\varphi|(A\cap\bar{S}_\varrho(x_0))+\Phi(\varrho)|\varphi|(A\cap S_\varrho(x_0))\\ &\leqslant 2\Phi(\varrho)|\varphi|(A\cap\bar{S}_\varrho(x_0)),\end{aligned}$$

and using (4) we obtain

$$\lim_{\varrho\to 0}\Big(\overline{\lim_{x\to x_0}}A(\varrho,x)\Big)=0. \tag{8}$$

The relations (5), (7) and (8) yield (6). Similarly one shows that (5) follows from (6). □

For $K(x,\xi) = \Phi(r(x,\xi))$ the condition (2) can also be written in another way.

Lemma 2. *Let $x_0 \in A$ and let Φ satisfy the supposition (α). Then*

$$\lim_{\varrho\to 0} \int\limits_{A\cap S_\varrho(x_0)} \Phi(r(x_0,\xi))\,\mathrm{d}|\varphi| = 0 \tag{9}$$

iff

$$\lim_{\varrho\to 0} \int\limits_0^\varrho \Phi'(r)|\varphi|(A\cap S_r(x_0))\,\mathrm{d}r = 0. \tag{10}$$

Proof. First we assume that (9) holds. We have

$$\begin{aligned}
0 &\leqslant \Phi(\varrho)|\varphi|(A\cap S_\varrho(x_0)) \leqslant \Phi(\varrho)|\varphi|(A\cap \bar{S}_\varrho(x_0))\\
&\leqslant \int\limits_{A\cap \bar{S}_\varrho(x_0)} \Phi(r(x_0,\xi))\,\mathrm{d}|\varphi| \leqslant \int\limits_{A\cap S_{2\varrho}(x_0)} \Phi(r(x_0,\xi))\,\mathrm{d}\varphi
\end{aligned}$$

for all sufficiently small ϱ. Thus we obtain from (9) the equalities

$$\lim_{\varrho\to 0} \Phi(\varrho)|\varphi|(A\cap S_\varrho(x_0)) = 0 \tag{11}$$

and

$$\lim_{\varrho\to 0} \Phi(\varrho)|\varphi|(A\cap \bar{S}_\varrho(x_0)) = 0. \tag{4}$$

By taking (11) into consideration we have

$$\begin{aligned}
\int\limits_{A\cap S_\varrho(x_0)} \Phi(r(x_0,\xi))\,\mathrm{d}|\varphi| &= \int\limits_0^\varrho \Phi(r)\,\mathrm{d}|\varphi|(A\cap S_r(x_0))\\
&= \lim_{\delta\to 0}\Big[\Phi(\varrho)|\varphi|(A\cap S_\varrho(x_0)) - \Phi(\delta)|\varphi|(A\cap S_\delta(x_0))\\
&\qquad - \int\limits_\delta^\varrho \Phi'(r)|\varphi|(A\cap S_r(x_0))\,\mathrm{d}r\Big]\\
&= \Phi(\varrho)|\varphi|(A\cap S_\varrho(x_0)) - \int\limits_0^\varrho \Phi'(r)|\varphi|(A\cap S_r(x_0))\,\mathrm{d}r. \qquad (12)
\end{aligned}$$

For $\varrho \to 0$ we obtain from (9), (11) and (12) the relation (10).

Now we assume that (10) holds. For sufficiently small ϱ and $0 < \delta < \varrho$ we have

$$\int_\delta^\varrho \Phi(r)\,\mathrm{d}|\varphi|(A \cap S_r(x_0)) \leqslant \Phi(\varrho)|\varphi|(A \cap S_\varrho(x_0)) - \int_\delta^\varrho \Phi'(r)|\varphi|(A \cap S_r(x_0))\,\mathrm{d}r.$$

For $\delta \to 0$ we obtain

$$\int_{A \cap S_\varrho(x_0)} \Phi(r(x_0,\xi))\,\mathrm{d}\varphi \leqslant \Phi(\varrho)|\varphi|(A \cap S_\varrho(x_0)) - \int_0^\varrho \Phi'(r)|\varphi|(A \cap S_r(x_0))\,\mathrm{d}r.$$

From (10) it follows that

$$\int_0^\varrho \Phi'(r)|\varphi|(A \cap S_r(x_0))\,\mathrm{d}r$$

is finite. Hence also

$$\int_{A \cap S_\varrho(x_0)} \Phi(r(x_0,\xi))\,\mathrm{d}|\varphi|$$

is finite and the relation (9) holds. □

Concerning the continuity of

$$u(x) = \int_A \Phi(r(x,\xi))\,\mathrm{d}\varphi$$

we prove

Theorem 2. *Let Φ satisfy the supposition (α). Then*

$$u(x) = \int_A \Phi(r(x,\xi))\,\mathrm{d}\varphi$$

is continuous at $x_0 \in A$ iff

(13) $$\lim_{\varrho\to 0}\Big(\overline{\lim_{x\to x_0}}\Big|\int_0^\varrho \Phi'(r)\varphi(A \cap S_r(x))\,\mathrm{d}r\Big|\Big) = 0,$$

(14) $$\lim_{\varrho\to 0}\int_0^\varrho \Phi'(r)|\varphi|(A \cap S_r(x_0))\,\mathrm{d}r = 0$$

and there exists a neighborhood $S_\delta(x_0)$ such that

$$(15) \qquad \lim_{\varrho\to 0} \Phi(\varrho)|\varphi|(A\cap S_\varrho(x)) = 0$$

for all $x \in S_\delta(x_0) - \{x_0\}$.

Proof. Let $u(x)$ be continuous at $x_0 \in A$. From Theorem 1 it follows that the relations (1) and (2) hold with $K(x,\xi) = \Phi(r(x,\xi))$. In the proof of Lemma 2 one can find that also the relation (4) is fulfilled. Thus from Lemma 1 and Lemma 2 we obtain the relations (6) and (10). But (10) is identical with (14). From the continuity of u in x_0 it follows that there exists a neighborhood $S_\delta(x_0)$ such that $u(x)$ is finite for all $x \in S_\delta(x_0)$. Hence for all these x we have

$$\lim_{\varrho\to 0} \int\limits_{A\cap S_\varrho(x)} \Phi(r(x,\xi))\,\mathrm{d}|\varphi| = 0.$$

From this relation we obtain (15) (comp. the begining of the proof of Lemma 2).

If we write $\varphi = \varphi^+ - \varphi^-$ where φ^+ and φ^- are the positive and negative variation respectivly of φ and if we consider the integrals with respect to φ^+ and φ^- separately we obtain in the same way as in the proof of Lemma 2 that

$$\int\limits_{A\cap S_\varrho(x)} \Phi(r(x,\xi))\,\mathrm{d}\varphi = \lim_{\varepsilon\to +0}\Big(\Phi(\varrho)\varphi(A\cap S_\varrho(x)) - \Phi(\varepsilon)\varphi(A\cap S_\varepsilon(x)) - \int\limits_\varepsilon^\varrho \Phi'(r)\varphi(A\cap S_r(x))\,\mathrm{d}r\Big)$$

for all $x \in S_\delta(x_0) - \{x_0\}$. Using the inequality

$$0 \leqslant \big|\Phi(\varepsilon)\varphi(A\cap S_\varepsilon(x))\big| \leqslant \Phi(\varepsilon)|\varphi|(A\cap S_\varepsilon(x))$$

and (15) we get

$$(16) \qquad \int\limits_{A\cap S_\varrho(x)} \Phi(r(x,\xi))\,\mathrm{d}\varphi = \Phi(\varrho)\varphi(A\cap S_\varrho(x)) - \int\limits_0^\varrho \Phi'(r)\varphi(A\cap S_r(x))\,\mathrm{d}r.$$

Thus for sufficiently small ϱ we have

$$(17) \qquad \Big|\int\limits_0^\varrho \Phi'(r)\varphi(A\cap S_r(x))\,\mathrm{d}r\Big| \leqslant \Phi(\varrho)|\varphi|(A\cap S_\varrho(x)) + \Big|\int\limits_{A\cap S_\varrho(x)} \Phi(r(x,\xi))\,\mathrm{d}\varphi\Big|.$$

Further, we have

$$\varlimsup_{x \to x_0} \Phi(\varrho)|\varphi|(A \cap S_\varrho(x)) \leqslant \Phi(\varrho)|\varphi|(A \cap \bar{S}_\varrho(x_0)).$$

Thus by (4) we obtain

$$\lim_{\varrho \to 0} \Big(\varlimsup_{x \to x_0} \Phi(\varrho)|\varphi|(A \cap S_\varrho(x)) \Big) = 0. \tag{18}$$

Then (6), (17), (18) imply that (13) holds.

Now we assume that (13), (14), (15) are valid. Then because of (14) and Lemma 2 the relation (9) is valid. Consequently, also (4) and (18) are fulfilled. From (15) we obtain that (16) holds for all $x \in S_\delta(x_0) - \{x_0\}$. Hence

$$\Big| \int\limits_{A \cap S_\varrho(x)} \Phi(r(x,\xi))\, \mathrm{d}\varphi \Big| \leqslant \Phi(\varrho)|\varphi|(A \cap S_\varrho(x)) + \Big| \int_0^\varrho \Phi'(r)\varphi(A \cap S_r(x))\, \mathrm{d}r \Big| \tag{19}$$

for sufficiently small ϱ. From (13), (18), (19) we obtain (6). By Lemma 1 the relations (4) and (6) imply (5). Thus (5) and (9) are valid and by Theorem 1 we obtain that u is continuous at x_0. □

4. Necessary and sufficient conditions for the continuity of potentials with respect to a non negative measure

In the special case φ is a non negative measure the conditions (14) and (15) can be ommited.

Theorem 3. *Let Φ satisfy the supposition (α) and let φ be a non negative measure. Then*

$$u(x) = \int_A \Phi(r(x,\xi))\, \mathrm{d}\varphi$$

is continuous at $x_0 \in A$ iff

$$\lim_{\varrho \to 0} \Big(\varlimsup_{x \to x_0} \int_0^\varrho |\Phi'(r)|\varphi(A \cap S_r(x))\, \mathrm{d}r \Big) = 0. \tag{20}$$

Proof. Since $\Phi' \leqslant 0$ and $\varphi \geqslant 0$, obviously the relation (20) is identical with (13). We show that the conditions (14) and (15) are consequences of (20). (20)

implies the existence of a neighborhood $S_\delta(x_0)$ and a sufficiently small $\varrho_0 > 0$ such that

$$\int_0^{\varrho_0} |\Phi'(r)| \varphi(A \cap S_r(x))\, \mathrm{d}r$$

is finite for all $x \in S_\delta(x_0) - \{x_0\}$. In the same way as in the proof of Lemma 2 we obtain for all such x that

$$\int_{A \cap S_{\varrho_0}(x)} \Phi(r(x,\xi))\, \mathrm{d}\varphi \leqslant \Phi(\varrho_0)\varphi(A \cap S_{\varrho_0}(x)) + \int_0^{\varrho_0} |\Phi'(r)| \varphi(A \cap S_r(x))\, \mathrm{d}r.$$

Thus

$$\int_{A \cap S_{\varrho_0}(x)} \Phi(r(x,\xi))\, \mathrm{d}\varphi$$

and $u(x)$ are finite and

$$\lim_{\varrho \to 0} \int_{A \cap S_\varrho(x)} \Phi(r(x,\xi))\, \mathrm{d}\varphi = 0.$$

From this equation we obtain (15) (comp. the proof of Lemma 2). Since $\varphi \geqslant 0$ we further have

$$\varphi(A \cap S_\varrho(x_0)) \leqslant \varliminf_{x \to x_0} \varphi(A \cap S_\varrho(x)),$$

and by the Fatou Theorem and (20) we obtain

$$0 \leqslant \lim_{\varrho \to 0} \int_0^{\varrho} |\Phi'(r)| \varphi(A \cap S_\varrho(x_0))\, \mathrm{d}r \leqslant \lim_{\varrho \to 0} \Big(\varliminf_{x \to x_0} \int_0^{\varrho} |\Phi'(r)| \varphi(A \cap S_r(x))\, \mathrm{d}r \Big)$$

$$\leqslant \lim_{\varrho \to 0} \Big(\varlimsup_{x \to x_0} \int_0^{\varrho} |\Phi'(r)| \varphi(A \cap S_r(x))\, \mathrm{d}r \Big) = 0.$$

Thus also (14) follows from (20). □

References

[1] *M. G. Arsove*: Continuous potentials and linear mass distributions, SIAM Review *2* (1960), 177–184.

[2] *S. Dümmel*: Verallgemeinerte Potentiale I, Math. Nachr. *23* (1961), 349–370.

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