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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL SYSTEMS

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Summary. In the paper we study the existence of nonoscillatory solutions of the system $x_i^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t) f_{ij}\Big(x_j\big(h_{ij}(t)\big)\Big)$, $n \geqslant 2$, $i = 1,2$, with the property $\lim_{t\to\infty} x_i(t)/t^{k_i} =$ const $\neq 0$ for some $k_i \in \{1, 2, \dots, n-1\}$, $i = 1,2$. Sufficient conditions for the oscillation of solutions of the system are also proved.

Keywords: Functional differential system, Schauder-Tychonov fixed point theorem, oscillatory solution, nonoscillatory solutions.

AMS classification: 34K25, 34K05

This paper is concered with the asymptotic properties of solutions of nonlinear functional differential systems in the form

$$\text{(S)}\qquad x_i^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t) f_{ij}\big(x_j\big(h_{ij}(t)\big)\big), \quad t \geqslant t_0 \geqslant 0, \quad i = 1,2, \quad n \geqslant 2,$$

under the following standing assumptions:

(1) $p_{ij}, h_{ij}\colon [t_0, \infty) \to \mathbf{R}$ $(i,j = 1,2)$ are continuous functions and $\lim h_{ij}(t) = \infty$ as $t \to \infty$ $(i,j = 1,2)$,

(2) $f_{ij}\colon \mathbf{R} \to \mathbf{R}$ $(i,j = 1,2)$ are continuous functions and $u f_{ij}(u) > 0$ for $u \neq 0$ $(i,j = 1,2)$,

(3) f_{ij} $(i,j = 1,2)$ are nondecreasing functions.

For any $t_1 \geqslant t_0$ denote

$$t_2 = \min\{\big(\inf h_{ij}(t); t \geqslant t_1\big),\ i,j = 1,2\}.$$

A function $X(t) = (x_1(t), x_2(t))$ is a solution of (S) if there exists a $t_1 \geqslant t_0$ such that $X(t)$ is continuous on $[t_2, \infty)$, n-times continuously differentiable on $[t_1, \infty)$ and satisfies the system (S) on $[t_1, \infty)$.

By a proper solution of the system (S) we mean a solution $X(t)$ of (S) such that $\sup\{|x_1(t)| + |x_2(t)| : t \geqslant T\} > 0$ for any $T \geqslant t_0$. Such a solution is called oscillatory if each of its component has arbitrarily large zeros. A proper solution of (S) is called nonoscillatory (weakly nonoscillatory), if each of its components (one component) is eventually of constant sign on $[T_x, \infty) \subset [t_0, \infty)$.

This paper has two parts. First we prove the existence of nonoscillatory solutions of the system (S) with the property $\lim_{t\to\infty} x_i(t)/t^{k_i} = \text{const} \neq 0$ for some $k_i \in \{0, 1, \ldots, n-1\}$, $i = 1, 2$. The assymptotic properties of solutions of this type of nonlinear differential equations of higher orders have been studied for example in the papers [1, 3–5].

Secondly, we establish criteria for oscillation of proper solutions of (S).

Denote

$$\gamma_{ij}(t) = \sup(s : h_{ij}(s) \leqslant t), \quad t \geqslant t_0,$$
$$\gamma(t) = \max(\gamma_{ij}(t); i, j = 1, 2), \quad t \geqslant t_0.$$

Theorem 1. *Let the conditions* (1)–(3) *hold and let* $k_i \in \{1, 2, \ldots, n-1\}$, $i = 1, 2$. *If*

$$\text{(4)} \qquad \int_{\gamma(t_0)}^{\infty} t^{n-k_i-1} \sum_{j=1}^{2} |p_{ij}(t)| f_{ij}\big(a_j (h_{ij}(t))^{k_j}\big)\, dt < \infty, \quad i = 1, 2$$

for some $a_j > 0$, $j = 1, 2$, *then for any couples* (k_1, k_2), $(k_i \in \{1, 2, \ldots, n-1\})$ *and* (c_1, c_2) $(c_i > 0,\ i = 1, 2)$ *there exists a nonoscillatory solution* $X(t) = (x_1(t), x_2(t))$ *of the system (S) such that*

$$\text{(5)} \qquad \lim_{t\to\infty} x_i(t)/t^{k_i} = c_i, \quad i = 1, 2,$$
$$\lim_{t\to\infty} x_i^{(m_i)}(t) = 0 \text{ for } m_i = k_i + 1, \ldots, n-1, \quad i = 1, 2.$$

P r o o f. Let a_i $(i = 1, 2)$ be positive numbers such that (4) holds and $k_i \in \{1, 2, \ldots, n-1\}$, $i = 1, 2$. We put $b_i = a_i/3$, $i = 1, 2$. In view of (2) there exists a $T \geqslant \gamma(t_0)$ such that

$$\text{(6)} \qquad \int_{T}^{\infty} t^{n-k_i-1} \sum_{j=1}^{2} |p_{ij}(t)| f_{ij}\big(a_j (h_{ij}(t))^{k_j}\big)\, dt < b_i, \quad i = 1, 2.$$

Let $T_0 = \min\{(\inf h_{ij}(t)\colon t \geqslant T),\ i,j = 1,2\} \geqslant t_0$. We denote by $C([T_0,\infty))$ the locally convex space of all vector continuous functions $X(t) = (x_1(t), x_2(t))$ defined on $[T_0,\infty)$ which are constant on $[T_0,T]$ with the topology of uniform convergence on any compact subinterval of $[T_0,\infty)$.

We consider a closed, convex subset Y of $C([T_0,\infty))$ defined by

$$
\begin{aligned}
(7)\qquad Y = \{X = (x_1, x_2) \in C([T_0,\infty));\ & x_i(t) = 2b_i\frac{T^{k_i}}{k_i!},\ t \in [T_0,T]\colon ,\\
& b_i\frac{t^{k_i}}{k_i!} \leqslant x_i \leqslant 3b_i\frac{t^{k_i}}{k_i!},\ t \geqslant T,\ i = 1,2\}.
\end{aligned}
$$

We define a mapping $F = (F_1, F_2)\colon Y \to C([T_0,\infty))$ by

$$
(8)\qquad (F_iX)(t) = \begin{cases} \dfrac{2b_iT^{k_i}}{k_i!}, & t \in [T_0,T],\\[2ex] \dfrac{2b_it^{k_i}}{k_i!} + (-1)^{n-k_i}\displaystyle\int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!}\int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} & \\ \qquad\times \displaystyle\sum_{j=1}^{2} p_{ij}(u) f_{ij}\big(x_j(h_{ij}(u))\big)\,du\,ds, & t \geqslant T, \quad i = 1,2. \end{cases}
$$

We shall show that F is a continuous operator which transforms Y into a compact of Y.

Ad 1. We prove that $F(Y) \subset Y$. From (8) in view of (3), (6), (7) we have

$$
\begin{aligned}
(9_i)\qquad (F_iX)(t) &\leqslant \frac{2b_it^{k_i}}{k_i!} + \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!}\int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!}\,.\\
&\qquad\times \sum_{j=1}^{2} |p_{ij}(u)| f_{ij}\big(a_j(h_{ij}(u))^{k_j}\big)\,du\,ds\\
&\leqslant \frac{2b_it^{k_i}}{k_i!} + b_i\int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!}\,ds\\
&\leqslant \frac{3b_it^{k_i}}{k_i!},\quad t \geqslant T,\quad i = 1,2,
\end{aligned}
$$

$$(10_i)\qquad (F_iX)(t) \geqslant \frac{2b_i t^{k_i}}{k_i!} - \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^{2} |p_{ij}(u)| f_{ij}\left(a_j\left(h_{ij}(u)\right)^{k_j}\right) \mathrm{d}u\,\mathrm{d}s$$

$$\geqslant \frac{2b_i t^{k_i}}{k_i!} - b_i \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!}\,\mathrm{d}s$$

$$\geqslant \frac{b_i t^{k_i}}{k_i!}, \quad t \geqslant T, \quad i = 1, 2.$$

Ad 2. We prove that F is continuous. Let $X_k = (x_{1k}, x_{2k}) \in Y$, $k = 1, 2, \ldots$, and $x_{ik} \to x_i$ $(i = 1, 2)$ for $k \to \infty$ in the space $C\big([T_0, \infty)\big)$. From (8) we than have

$$|(F_iX_k)(t) - (F_iX)(t)|$$

$$(11_i)\qquad \leqslant \int_T^t \frac{(t-s)^{k_i-1}}{(k_i-1)!} \int_s^\infty \frac{(u-s)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^{2} |p_{ij}(u)| \Big| f_{ij}\big(x_{jk}(h_{ij}(u))\big) - f_{ij}\big(x_j(h_{ij}(u))\big)\Big| \,\mathrm{d}u\,\mathrm{d}s$$

$$\leqslant t^{k_i} \int_T^\infty G_i^k(u)\,\mathrm{d}u,$$

where we set

$$G_i^k(u) = u^{n-k_i-1} \sum_{j=1}^{2} |p_{ij}(u)| \Big| f_{ij}\big(x_{jk}(h_{ij}(u))\big) - f_{ij}\big(x_j(h_{ij}(u))\big)\Big|.$$

It is easy to see that $\lim_{k\to\infty} G_i^k(u) = 0$ and $G_i^k(u) \leqslant M_i(u)$, where

$$M_i(u) = 2u^{n-k_i-1} \sum_{j=1}^{2} |p_{ij}(u)| f_{ij}\left(a_j\left(h_{ij}(u)\right)^{k_j}\right).$$

Using the fact that $\int_T^\infty M_i(u)\,\mathrm{d}u < \infty$ and the Lebesgue dominating convergence theorem, from (11_i) we get $(F_iX_k)(t) \to (F_iX)(t)$ for $k \to \infty$ $(i = 1, 2)$ in $C\big([T_0, \infty)\big)$. This implies the continuity of $F = (F_1, F_2)$.

Ad 3. We prove that $F(Y)$ has a compact closure. From (8), in view of (6), for any $X \in Y$ we have

$$|(F_iX)'(t)| \leqslant \frac{3b_i}{k_i - 1} t^{k_i-1}, \quad t \geqslant T, \quad i = 1, 2.$$

Hence $F(Y)$ is equicontinuous on any compact subinterval of $[T_0, \infty)$. Since $F(Y) \subset Y$, $F(Y)$ is uniformly bounded on such subintervals. Therefore by the Arzela-Ascoli theorem $F(Y)$ has a compact closure.

By the Schauder-Tychonov fixed point theorem there exists an $\bar{X} = (\bar{x}_1, \bar{x}_2)$ such that $F\bar{X} = (F_1\bar{X}, F_2\bar{X}) = \bar{X}$. The function $\bar{X}$ satisfies (8) in which $F_iX = x_i$ $(i = 1, 2)$.

Differentiating (8) in which $F_iX = x_i$ $(i = 1, 2)$ m_i-times, $m_i = k_i, \ldots, n-1$, for $X = (x_1, x_2) = \bar{X}$ we obtain

$$(12) \qquad x_i^{(k_i)}(t) = 2b_i + (-1)^{n-k_i} \int_t^\infty \frac{(u-t)^{n-k_i-1}}{(n-k_i-1)!} \times \sum_{j=1}^{2} p_{ij}(u) f_{ij}\big(x_j(h_{ij}(u))\big)\,du, \quad t \geqslant T, \quad i = 1, 2,$$

$$(13_{m_i}) \quad x_i^{(m_i)}(t) = (-1)^{n-m_i} \int_t^\infty \frac{(u-t)^{n-m_i-1}}{(n-m_i-1)!} \sum_{j=1}^{2} p_{ij}(u) f_{ij}\big(x_j(h_{ij}(u))\big)\,du,$$
$$t \geqslant T,\ m_i = k_i + 1, \ldots, n-1,\ (\text{if } k_i < 1),\ i = 1, 2,$$

Differentiating (13_{n-1}) we get the system (S). This implies that $X = (x_1, x_2) = \bar{X}$ is a nonoscillatory solution of (S). From (12), (13_{m_i}) in view of (4) we get $\lim_{t\to\infty} x_i^{(k_i)}(t) = 2b_i$, $\lim_{t\to\infty} x_i^{(m_i)}(t) = 0$ for $m_i = k_i + 1, \ldots, n-1$, $i = 1, 2$. This is equivalent to (5), where $c_i = 2b_i$ $(i = 1, 2)$. □

Theorem 2. *Let the conditions* (1)–(3) *hold and let*

$$(14) \qquad \int_{\gamma(t_0)}^{\infty} t^{n-1} \sum_{j=1}^{2} |p_{ij}(t)|\,dt < \infty, \quad i = 1, 2.$$

Then for any couple (c_1, c_2) $(c_i > 0,\ i = 1, 2)$ *there exists a nonoscillatory solution of the system (S) such that*

$$(15) \qquad \lim_{t\to\infty} |x_i(t)| = c_i, \quad \lim_{t\to\infty} x_i^{(k)}(t) = 0, \quad k = 1, 2, \ldots, n-1, \quad i = 1, 2.$$

Proof. Let $c_i > 0$ $(i = 1, 2)$ and $0 < \delta \leqslant \min(c_1, c_2)$. In view of (2) there exists a $K > 0$ such that for all (u_1, u_2): $|u_i - c_i| \leqslant \delta$ $(i = 1, 2)$ we have

$$(16) \qquad |f_{ij}(u_j)| \leqslant K, \quad i, j = 1, 2.$$

With regard to (14) there exists a $T \geqslant \gamma(t_0)$ such that

$$\int_T^\infty t^{n-1} \sum_{j=1}^2 |p_{ij}(t)|\, dt \leqslant \frac{\delta}{K}, \quad i = 1,2. \tag{17}$$

Let T_0 and $C([T_0,\infty))$ be the same as in the proof of Theorem 1. We consider a closed, convex subset Y_1 of $C([T_0,\infty))$ by

$$Y_1 = \{X = (x_1, x_2) \in C([T_0,\infty)) : |x_i(t) - c_i| \leqslant \delta,\ t \geqslant T,\ i = 1,2\}.$$

We define a mapping $F = (F_1, F_2) : Y_1 \to C([T_0,\infty))$ by

$$\begin{aligned}
(F_iX)(t) &= c_i + \frac{(-1)^n}{(n-1)!} \int_T^\infty (s-t)^{n-1} \sum_{j=1}^2 p_{ij}(t) f_{ij}\big(x_j(h_{ij}(s))\big)\, ds,\\
&\qquad t \in [T_0, T],\\
(F_iX)(t) &= c_i + \frac{(-1)^n}{(n-1)!} \int_T^\infty (s-t)^{n-1} \sum_{j=1}^2 p_{ij}(t) f_{ij}\big(x_j(h_{ij}(s))\big)\, ds,\\
&\qquad t \geqslant T, \quad i = 1,2.
\end{aligned} \tag{18}$$

If we proceed analogously as in the proof of Theorem 1 we can prove without difficulty that F maps Y_1 into itself, F is continuous and $F(Y_1)$ has a compact closure. Therefore there exists an $\bar{X} = (\bar{x}_1, \bar{x}_2) \in Y_1$ such that $F\bar{X} = (F_1\bar{X}, F_2\bar{X}) = \bar{X}$. The function $\bar{X}$ satisfies (18) in which $F_iX = x_i$ $(i = 1,2)$. We can easily verify that $X = (x_1, x_2) = \bar{X}$ is a nonoscillatory solution of (S) with the asymptotic behavior (15). □

Theorem 3. *Suppose that* (1)–(3) *hold and*

$$p_{ij}(t) = \sigma_i q_{ij}(t), \quad \sigma_i \in \{-1, 1\}, \quad q_{ij} : [t_0, \infty) \to (0,\infty), \quad i,j = 1,2. \tag{19}$$

Let (k_1, k_2) *be an arbitrary couple of integers* $k_i \in \{0, 1, \ldots, n-1\}$ $(i = 1,2)$. *Then there exists a nonoscillatory solution* (x_1, x_2) *of the system (S) such that*

$$\lim_{t\to\infty} \frac{x_i(t)}{t^{k_i}} = c_i > 0, \quad i = 1,2, \tag{20}$$

if and only if

$$\int_{\gamma(t_0)}^\infty t^{n-k_i-1} \sum_{j=1}^2 q_{ij}(t) f_{ij}\big(a_j (h_{ij}(t))^{k_j}\big)\, dt < \infty, \quad i = 1,2. \tag{21}$$

for some constants $a_j > 0$, $j = 1, 2$.

Proof. Let $X = (x_1, x_2)$ be a nonoscillatory solution of (S) which satisfies (20). Without loss of generality we suppose that $x_j(h_{ij}(t)) > 0$ for $t \geqslant T_1 \geqslant t_0$, $i, j = 1, 2$. Then in view of (2) $f_{ij}(x_j(h_{ij}(t))) > 0$ for $t \geqslant T_1$. From (20) we obtain

$$\lim_{t\to\infty} x_i^{(k_i)}(t) = c_i k_i! > 0, \quad i = 1, 2, \tag{22}$$
$$\lim_{t\to\infty} x_i^{(m_i)}(t) = 0, \quad m_i = k_i + 1, \ldots, n_1, \quad i = 1, 2.$$

Then integrating the system (S) $(n - k_i - 1)$-times (if $k_i < n - 1$), $i = 1, 2$, from t $(\geqslant T_1)$ to ∞ and using (22) we have

$$x_i^{(k_i+1)}(t) = (-1)^{n-k_i-1}\sigma_i \int_t^\infty \frac{(s-t)^{n-k_i-2}}{(n-k_i-2)!} \sum_{j=1}^{2} q_{ij}(s) f_{ij}\big(x_j(h_{ij}(s))\big)\, ds,$$
$$t \geqslant T, \quad i = 1, 2.$$

Integreating the last equation from T_1 to ∞ and using (20), after some modifications we obtain

$$\int_{T_1}^{\infty} s^{n-k_i-1} \sum_{j=1}^{2} q_{ij}(s) f_{ij}\big(x_j(h_{ij}(s))\big)\, ds < \infty, \quad i = 1, 2. \tag{23}$$

On the other hand, by virtue of (20) there exist constants $a_j > 0$ $(j = 1, 2)$ and $T_2 \geqslant T_1$ such that $x_j(h_{ij}(t)) \geqslant a_j\big(h_{ij}(t)\big)^{k_j}$ for $t \geqslant T_2$ $(i, j = 1, 2)$. Then the last inequality, (3) and (23) imply (21).

The "if" part follows from Theorem 1 a Theorem 2.

Oscillation criteria

Now we consider the system (S) in the form

(A) $x_i^{(n)}(t) = \sigma_i q_i(t) f_i\big(x_{3-i}(h_{3-i}(t))\big)$ $t \geqslant t_0$, $i = 1, 2$, where $\sigma_i \in \{-1, 1\}$.

(24) $q_i \colon [t_0, \infty) \to (0, \infty)$, $i = 1, 2$ are continuous functions,

(25) h_i and f_i, $i = 1, 2$ satisfy (1) and (2), respectively,

(26) for any $b > 0$ there exists $\delta > 0$ such that

$$\inf\{f_i(u)|; |u| \geqslant b\} \geqslant \delta, \quad i = 1, 2.$$

In the sequel we use Kiguradze's lemma. □

Lemma [2]. *Let $u \in C^n[t_0, \infty)$ be such that $(-1)^\nu u(t) u^{(n)}(t) < 0$ for $t \geqslant t_0$, $\nu \in \{1, 2\}$. Then there exist an integer $\ell \in \{0, 1, \ldots, n\}$, where $\ell + n + \nu$ is odd, and $T \geqslant t_0$ such that*

$$u(t)u^{(k)}(t) > 0 \text{ for } k = 0, 1, \ldots, \ell, \quad t \geqslant T,$$
$$(-1)^{\ell+k} u(t) u^{(k)}(t) > 0 \text{ for } k = \ell + 1, \ldots, n, \quad t \geqslant T.$$

Remark. Let $X = (x_1, x_2)$ be a weakly nonoscillatory solution of (A). Then in view of (24), (25) it follows, from (A) that X is a nonoscillatory solution.

Theorem 4. *Suppose that $\sigma_1\sigma_2 = -1$ and*

$$\int_{t_0}^{\infty} q_i(t)\,\mathrm{d}t = \infty, \quad i = 1,2. \tag{27}$$

Then every proper solution $(x_1(t), x_2(t))$ of (A) is oscillatory when n is odd, and for n even it is either oscillatory or $x_1(t)x_2(t) < 0$ and, moreover, for $\sigma_j = 1$, $\sigma_{3-j} = -1$ $(j = 1,2)$ $|x_j(t)|$ is increasing while $x_{3-j}^{(k)}(t)$, $(k = 0, 1, \ldots, n)$ tend monotonically to zero as $t \to \infty$.

Proof. Suppose that the system (A) has a wekly nonoscillatory solution $(x_1(t), x_2(t))$. Then in view of Remark it is a nonoscillatory solution. Without loss of generality we suppose that $\sigma_1 = 1$, $\sigma_2 = -1$.

I. Let n be odd. 1) Suppose that $x_1(t) > 0$, $x_2(t) > 0$ for $t \geqslant t_1$. (The proof in the case $x_1(t) < 0$, $x_2(t) < 0$ is similar.) Then from the system (A) with regard to (24), (25) we obtain $x_1^{(n)}(t) > 0$, $x_2^{(n)}(t) < 0$ for $t \geqslant t_2 \geqslant \gamma(t_1)$. Then by Lemma we get $x_1'(t) > 0$ and then $x_1(t) \geqslant b_1$ for $t \geqslant t_3 \geqslant t_2$ and some $b_1 > 0$. Therefore in view of (26) there exists $\delta_1 > 0$ such that $f_2(x_1(h_1(t))) \geqslant \delta_1$ for $t \geqslant t_4 \geqslant \gamma(t_3)$. Then from (A) we get $x_2^{(n)}(t) \leqslant -\delta_1 q_2(t)$, $t \geqslant t_4$. From the last inequality, in view of (27) we obtain $x_2^{(n-1)}(t) \to -\infty$ as $t \to \infty$. The inequalities $x_2^{(n)}(t) < 0$, $x_2^{(n-1)}(t) < 0$ for $t \geqslant t_5 \geqslant t_4$ imply that $x_2(t) < 0$ for all large t. This contradicts the assumption $x_2(t) > 0$ for $t \geqslant t_1$.

2) Let $x_1(t) > 0$, $x_2(t) < 0$ for $t \geqslant t_1$. (The proof in the case $x_1(t) < 0$, $x_2(t) > 0$ is similar). Then the system (A) in view of (24), (25) implies $x_i^{(n)}(t) < 0$, $i = 1,2$, $t \geqslant t_2 \geqslant \gamma(t_1)$. Because $x_2(t)x_2^{(n)}(t) > 0$ for $t \geqslant t_2$, by Lemma we get $x_2'(t) < 0$ and then $x_2(t) \leqslant -a_2$ for $t \geqslant \bar{t}_3 \geqslant t_2$ and some $a_2 > 0$. Therefore in view of (26) there exists $\delta_2 > 0$ such that $f_1(x_2(h_2(t))) \leqslant -\delta_2$ for $t \geqslant \bar{t}_4 \geqslant \delta(\bar{t}_3)$. Then from (A) with regard to (27) we get $x_1^{(n-1)}(t) < 0$ for $t \geqslant t_5 \geqslant \bar{t}_4$. From $x_1^{(n)}(t) < 0$, $x_1^{(n-1)}(t) < 0$ for $t \geqslant t_5$ we obtian $x_1(t) < 0$ for all large t. This contradicts the assumption $x_1(t) > 0$ for $t \geqslant t_1$.

II. Let n be even. 1) Suppose that $x_1(t) > 0$, $x_2(t) > 0$ for $t \geqslant t_1$. (The proof in the case $x_1(t) < 0$, $x_2(t) < 0$ is similar.) Then in view of (24), (25) the system (A) implies $x_1^{(n)}(t) > 0$, $x_2^{(n)}(t) < 0$ for $t \geqslant t_2 \geqslant \gamma(t_1)$ and by Lemma $x_2'(t) > 0$ and then $x_2(t) \geqslant b_3$ for $t \geqslant T_2 \geqslant t_2$ and some $b_3 > 0$. Therefore in view of (26) there exists $\delta_3 > 0$ such that $f_1(x_2(h_2(t))) \geqslant \delta_3$ for $t \geqslant T_3 \geqslant \gamma(T_2)$. Then from (A) with regard to (27) we get $x_1^{(n-1)}(t) \to \infty$ as $t \to \infty$. Therefore in view of (26) there exists $\delta_4 > 0$

such that $f_2(x_1(h_1(t))) \geqslant \delta_4$ for $t \geqslant T_4 \geqslant \gamma(T_3)$. Further we proceed analogously as in the case I-1) we obtaining $x_2(t) < 0$ for large t, which contradicts $x_2(t) > 0$ for $t \geqslant t_1$.

2) Suppose that $x_1(t) > 0$, $x_2(t) < 0$ for $t \geqslant t_1$. (The proof in the case $x_1(t) < 0$, $x_2(t) > 0$ is similar). Then in view of (24), (25) from (A) we get $x_i^{(n)}(t) < 0$, $i = 1, 2$, for $t \geqslant t_2 = \gamma(t_1)$. Using Lemma, we have $x_1'(t) > 0$ and either i) $x_2'(t) < 0$, $x_2''(t) < 0$, or ii) $x_2'(t) > 0$ for $t \geqslant t_3 \geqslant t_2$. In the case i) we proceed in the same way as in the case I-2), obtaining a contradiction to the assumption $x_1(t) > 0$ for $t \geqslant t_1$. Now we consider the case ii). The component $x_2(t)$ is increasing and $\lim_{t\to\infty} x_2(t) = -b \leqslant 0$. If we suppose that $b > 0$, we proceed in the same way as in the case i) arriving at a contradiction. Therefore $b = 0$, i.e. $\lim_{t\to\infty} x_2(t) = 0$. This in view of Lemma implies $\lim_{t\to\infty} x_2^{(k)}(t) = 0$ for $k = 0, 1, \ldots, n$.

The proof of Thoerem 4 is complete. □

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Súhrn

ASYMPTOTICKÉ VLASTNOSTI RIEŠENÍ FUNKCIONÁLNO-DIFERENCIÁLNYCH SYSTÉMOV

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V práci je študovaná existencia neoscilatorických riešení systému

$$x_i^{(n)}(t) = \sum_{j=1}^{2} p_{ij}(t) f_{ij}\Big(x_j\big(h_{ij}(t)\big)\Big), \quad n \geqslant 2, \ i = 1, 2,$$

s vlastnosťami $\lim\limits_{t\to\infty} x_i(t)/t^{k_i} = \text{const.} \neq 0$ pre nejaké $k_i \in \{1, 2, \ldots, n-1\}$, $i = 1, 2$. Ďalej sú dokázané postačujúce podmienky pre to, aby systém mal oscilatorické riešenie.

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