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Anna Neubrunnová; Tibor Šalát

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ON ALMOST QUASICONTINUITY

A. NEUBRUNOVÁ, T. ŠALÁT, Bratislava

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Summary. The concept of almost quasicontinuity is investgated in this paper in several directions (e.g. the relation of this concept to other generalizations of continuity is described, various types of convergence of sequences of almost quasicontinuous function are studied, a.s.o.).

Keywords: quasicontinuity, almost quasicontinuity, almost continuity, cliquish function, residual set, transfinite sequence

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INTRODUCTION

The present paper was motivated by [4], where a characterization of continuity of a function was given. The autors of this paper tried to give an analogous characterization of quasicontinuity by introducing a new concept of generalized continuity (called weak quasicontinuity by the authors). However, during the preparation of the paper Dr. J. Doboš called the authors' attention to the fact that this concept had been already introduced (as almost quasicontinuity) in [1].[1] Using this concept the authors of [1] gave the following characterisation of quasicontinuity:

Let X be a topological space and Y a metric one. Then $f\colon X \to Y$ is quasicontinuous if and only if it is almost quasicontinuous and cliquish (see Theorem 2 in [1]).

Let us recall the notation and the concepts of almost quasicontinuity, quasicontinuity, almost continuity, and cliquishness.

Definition A. Let X, Y be two topological spaces. A function $f\colon X \to Y$ is called *quasicontinuous* (almost continuous, quasicontinuous) at a point $p \in X$,

[1] The authors are thankful to Dr. J. Doboš for this information and his comments on the original version of this paper.

if for any neighbouhood V of the point $f(p)$ in Y we have $p \in \mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}\, f^{-1}(V)$ ($p \in \mathrm{Int}\,\mathrm{Cl}\, f^{-1}(V), p \in \mathrm{Cl}\,\mathrm{Int}\, f^{-1}(V)$). Denote by B_f, (H_f, Q_f) the set of all points at which f is almost quasicontinuous (almost continuous, quasicontinuous). If $B_f = X$ $(H_f = X,\ Q_f = X)$ then f is said to be almost quasicontinuous (almost continuous,quasicontinuous) (see [1], [5], [6], [8]).

Definition B. Let X be a topological space and (Y, d) a metric one. A function $f\colon X \to Y$ is said to be *cliquish at a point* $p \in X$ if for every $\varepsilon > 0$ and each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $d\big(f(y), f(z)\big) < \varepsilon$ for each $y, z \in G$. Denote by A_f the set of all points at which f is cliquish. If $A_f = X$, then f is said to be cliquish (cf. [8], [15]).

In what follows the symbols $\mathbf{Q}$, $\mathbf{R}$ denote the sets of all rational and all real numbers, respectively.

This paper is devoted to the study of the notion of almost quasicontinuity from several standpoints. The results obtained are different from those in the quoted paper [1].

The paper is divided in four parts. In the firs part we discuss the relation of the almost continuity to various types of generalized continuity: quasicontinuity, cliquishness, almost continuity. In the second part we study the properties of the limit functions of almost quasicontinuous functions using different types of convergence. In the third part the structure of the set of points of almost quasicontinuity is examined. The fourth part is devoted to the study of the topological position of the class of bounded almost quasicontinuous functions in the space $M(X)$ (X is a topological space) of all bounded real functions on X.

1. Almost cotinuity and other types of generalized continuity

In this part of the paper we make some remarks concerning the relation of almost quasicontinuity to other types of generalized continuity.

Example 1.1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the Dirichlet function, hence $f = \chi_{\mathbf{Q}}$—the characteristic function of the set $\mathbf{Q}$. As one can easily verify, the function f is almost quasicontinuous, but it is not quasicontinuous at any point of $\mathbf{R}$.

Now a natural question arises concerning the relation between the almost quasicontinuity and cliquishness. If X is a topological space and Y a metric one, then the class of almost quasicontinuous functions intersects the class of cliquish functions, the intersection being the class of all quasicontinuous functions (see [1]). In general, none of the first mentioned classes is contained in the other one, as is shown in the following example.

Example 1.2. Let $X = \mathbf{Q}$ (with the usual topology). If $x \in \mathbf{Q}$, $x = \frac{p}{q}$ (in the standard form), put $f(x) = \frac{1}{q}$. It can be easily verified that the function $f: \mathbf{Q} \to \mathbf{R}$ is cliquish. However, it is not quasicontinuous at any $x \in \mathbf{Q}$. On the other hand, the Dirichlet function $\chi_{\mathbf{Q}}$ is an example of a function $f: \mathbf{R} \to \mathbf{R}$ which is almost quasicontinuous but not cliquish at any $p \in \mathbf{R}$.

It is useful to mention the relation of almost quasicontinuity to the almost continuity. Evidently, the almost continuity implies the almost quasicontinuity. The converse is not true in general, as the following example shows.

Example 1.3. Choose $X = \langle -1, 1 \rangle$, $Y = \mathbf{R}$ (both spaces with the usual topology). Put $f(x) = 0$ if $x \in \langle -1, 0 \rangle$, $f(x) = 1$ otherwise. Then f is almost quasicontinuous but not almost continuous at the point 0.

Let X, Y be two topological spaces and $f: X \to Y$. Denote by C_f the set of all points at which the function f is continuous.

The structure of sets C_f, Q_f, A_f has been studied in [8], [9], [11], [15]. Results included in the following Theorem A are known.

Theorem A. a) *Let X be a topological space and Y a metric one. Let $f: X \to Y$. Then the set A_f is closed in X (see [8]), and the set $A_f \setminus C_f$ is of the type G_δ of the first category in X (see [11]).*

b) Let X be a Baire topological space and Y a metric space. Then following conditions are equivalent (see [3], [10]):

(i) *the function f is cliquish,*

(ii) *the set C_f is dense in X,*

(iii) *the set C_f is residual in X,*

(iv) *the set Q_f is dense in X,*

(v) *the set Q_f is residual in X.*

From a) we obtain the following result showing that one part of b) holds without the assumption on X to be a Baire space.

Theorem 1.1. *Let X be a topological space and Y a metric one. Let $f: X \to Y$ be such that C_f (or Q_f) is dense in X. Then f is cliquish.*

Proof. Since $C_f \subset Q_f \subset A_f$, the set A_f is dense in X. Hence $A_f = \mathrm{Cl}\, A_f = X$. $\square$

The following example shows that the converse is not true.

Example 1.4. We construct a space X which is of second category in itself, and a cliquish function $f\colon X \to \mathbf{R}$ such that C_f is not dense in X. Set $X = [(-\infty, 0) \cap \mathbf{Q}] \cup (0, \infty)$ with the Euclidean topology. Define $f\colon X \to \mathbf{R}$ such that $f(x) = x$ if $x \in (0, \infty)$ and $f(x) = \frac{1}{q}$ if $x \in (-\infty, 0) \cap \mathbf{Q}$, $x = \frac{p}{q}$ (in the standard form). Evidently, f is discontinuous at every point of the set $(-\infty, 0) \cap \mathbf{Q}$, but it is cliquish on X.

2. Sequences of almost quasicontinuous functions

The quasiuniform convergence is not sufficient for preserving the almost quasicontinuity.

Example 2.1. Let $f_n : \mathbf{R} \to \mathbf{R}$ ($n = 1, 2, \ldots$) be defined by

$$f_n(x) = \chi_{\left(0, \frac{1}{n}\right)}\left((-1)^n x\right).$$

Evidently, f_n are almost quasicontinuous and the sequence $\{f_n\}$ quasiuniformly converges to $f = \chi_{\{0\}}$. The function f is not almost quasicontinuous. Note that in [2] the above example was used to prove that quasicontinuity is not preserved under quasiuniform convergence.

From this example it follows that the class of almost quasicontinuous functions $f\colon X \to Y$ is not closed with respect piontwise convergence even X, Y are spaces with nice properties. However, the class of almost quasicontinuous functions is closed with respect to uniform convergence.

Theorem 2.1. *Let X be a topological space and (Y, d) a metric one. Let $f_n : \mathbf{R} \to \mathbf{R}$ ($n = 1, 2, \ldots$) be almost quasicontinuous and $f_n \rightrightarrows f$ on X. Then f is almost quasicontinuous.*

Proof. Let x_0 be an arbitrary point in X. Let $\varepsilon > 0$. Due to the uniform convergence of $\{f_n\}$ there exists $m \in \mathbf{N}$ such that

$$\forall x \in X : d\big(f_m(x), f(x)\big) < \frac{\varepsilon}{4}$$

Since the function f_m is almost quasicontinuous at x_0, for an arbitrary neighbourhood $U(x_0)$ of the point x_0 there exists a nonempty open set $G \subset U(x_0)$ such that the set

$$H = \left\{x \in X : d(f_m(x), f_m(x_0) < \frac{\varepsilon}{4}\right\}$$

is dense in G. Let $x \in H$. Then the triangle inequality gives

$$d\big(f(x), f(x_0)\big) \leqslant d\big(f(x), f_m(x)\big) + d\big(f_m(x), f_m(x_0)\big) + d\big(f_m(x_0), f(x_0)\big) < \frac{3}{4}\varepsilon.$$

So $f(H) \subset \{y \in Y : d(f(x_0), y) < \varepsilon\}$. The inclusion implies that f is almost quasicontinuous at x_0. □

A question arises whether the transfinite convergence preserves the almost quasi-continuity.

Recall the notion of convergence of a transfinite sequence. Let (M, d) be a metric space. Let Ω be the first uncountable ordinal number. The transfinite sequence $\{x_\xi\}_{\xi<\Omega}$ of points $x_\xi \in M$ is said to be convergent to a point $a \in M$, if for every $\varepsilon > 0$ there exists a ordinal number $\eta < \Omega$ such that for every for every ξ, $\eta < \xi < \Omega$ we have

$$d(x_\xi, a) < \varepsilon$$

Let X be a set and Y a metric space. If $f_\xi\colon X \to Y$ ($\xi < \Omega$), then we say that the transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions f_ξ conveges (pointwise) to a function $f\colon X \to Y$ if at an arbitrary $x \in X$ the (transfinite) sequence $\{f_\xi(x)\}_{\xi<\Omega}$ converges to $f(x)$ (see [12]).

The following assertion yields the negative answer to the above mentioned question.

Theorem 2.2. *Let X be a second countable topological space in which every nonempty open set is uncountable. Let Y be a metric space containing at least two different points. Let $a \in X$ be an arbitrary point. Then there exists a transfinite sequence of almost continuous functions $f_\xi\colon X \to Y$ ($\xi < \Omega$) converging to a function $f\colon X \to Y$, where f is not almost quasicontinuous at the point a.*

Proof. Let S be a countable dense set in X and let $a \notin S$. First we construct a transfinite sequence $\{x_\xi\}_{\xi<\Omega}$ of elements belonging to $X \setminus (S \cup \{a\})$, such that the set $\{x_\xi : \xi > \eta\}$ is dense in X for every $\eta < \Omega$. The construction will proceed by transfinite induction in the following way:

Let $\{U_n\}_{n=1}^{\infty}$ be a countable base of nonempty open sets in X. Choose $x_1 \in U_1 \setminus (S \cup \{a\})$ arbitrary. Let $\xi < \Omega$. Suppose that the sequence $\{x_\eta\}_{\eta<\xi}$ is constructed such that for $\eta_1 \neq \eta_2$ we have $x_{\eta_1} \neq x_{\eta_2}, x_{\eta_1}, x_{\eta_2} \in X \setminus (S \cup \{a\})$, and such that for a limit ordinal number $\eta < \xi$ the corresponding $x_\eta \in X \setminus (S \cup \{a\})$ is chosen arbitrarily. For a non limit number of the form $\eta = \eta_0 + n$, where η_0 is a limit number and n a positive integer (such representation of η is unique [7], p. 230), $x_\eta \in X \setminus (S \cup \{a\})$, $x_\eta \in U_n \setminus (S \cup \{a\})$, $x_\eta \neq x_\alpha$ for $\alpha < \eta$. Now, if ξ is a limit number we choose $x_\xi \in X \setminus (S \cup \{a\})$ arbitrarily. If ξ is not a limit number and its representation is of the form $\xi = \xi_0 + n$, where ξ_0 is a limit number, then we choose $x_\xi \in U_n, x_\xi \in X \setminus (S \cup \{a\})$ such that $x_\eta \neq x_\xi$ for $\eta < \xi$. By the construction the set $M_\eta = \{x_\xi : \xi > \eta\}$ is dense in X for each $\eta < \Omega$, because in every U_n is at least one point of M_η. Let $b \neq c$ be two points belonging to Y. For every $\xi < \Omega$ define

$$f_\xi(x) = \begin{cases} b & \text{if } x = x_\eta,\ \eta \geqslant \xi \text{ or } x = a, \\ c & \text{otherwise.} \end{cases}$$

The sequence $\{f_\xi\}_{\xi<\Omega}$ converges to the function f defined by

$$f(x) = \begin{cases} b & \text{if } x = a, \\ c & \text{if } x \neq a\,. \end{cases}$$

It is not difficult to verify that each f_ξ is almost continuous while f is not almost continuous at a. □

3. Points of almost quasicontinuity

If nothing else is said, in this part X is a topological space and (Y, d) a metric one. If $p \in Y$ and $\delta > 0$, then the set $K(p, \delta) = \{y \in Y : d(p, y) < \delta\}$ is called a ball with the center p and radius δ.

We associate with a given function f a function $\omega_f : X \to \langle 0, +\infty)$ resembling the oscilation of the function f. We characterize the points of the set B_f by means of ω_f. Let $f : X \to Y$, let $U(x)$ be a neighbourhood of a point $x \in X$. Put

(1)
$$\omega_f\big(x, U(x)\big) = \inf\big\{\sup\big\{d\big(f(x), f(z)\big) : z \in M\big\} : \emptyset \neq G \subset U(x), M \subset G \subset \operatorname{Cl} M\big\}$$

(hence the infimum on the right-hand side is taken over all nonempty open subsets G of the set $U(x)$ and over all sets M dense in G).

Further, we put

$$\omega_f(x) = \sup_{U(x)} \omega_f\big(x, U(x)\big). \tag{2}$$

It will be shown that the above defined function ω_f makes the characterization of points of the set B_f possible.

Theorem 3.1. *A function $f : X \to Y$ is almost quasicontinuous at a point $x \in X$ if and only if $\omega_f(x) = 0$.*

Proof. 1. Let $\omega_f(x) = 0$ and $\varepsilon > 0$. Then on the basis (2), for an arbitrary neighbourhood $U(x)$ of the point x we have $\omega_f\big(x, U(x)\big) < \varepsilon$. But then, due to (1), there exists a nonempty open set $G \subset U(x)$ and a set $M \subset G$ dense in G such that $f(M) \subset K(f(x), \varepsilon)$. So it is evident that the function f is almost quasicontinuous at x.

2. Let f be almost quasicontinuous at x and let $\eta > 0$. Since

$$x \in \operatorname{Cl}\operatorname{Int}\operatorname{Cl} f^{-1}\left(K\left(f(x), \frac{\eta}{2}\right)\right),$$

an arbitrary neighbourhood $U(x)$ of the point x contains a point

$$y_0 \in \operatorname{Int} \operatorname{Cl} f^{-1}\left(K\left(f(x), \frac{\eta}{2}\right)\right).$$

This implies that there exists a neighbourhood G of the point y_0 such that

$$G \subset \operatorname{Cl} f^{-1}\left(K\left(f(x), \frac{\eta}{2}\right)\right). \tag{3}$$

We may suppose that $G \subset U(x)$. It follows immediately from (3) that the set M of all those $y \in G$ which belong to $f^{-1}(K(f(x), \frac{\eta}{2})$ is dense in G. If $y \in M$, then $d(f(y), f(x)) \leqslant \frac{\eta}{2}$. Hence $\sup\limits_{y \in M} d(f(y), f(x)) \leqslant \frac{\eta}{2}$. Thus we get (see (1)) $\omega_f(x, U(x)) \leqslant \frac{\eta}{2}$, which holds for an arbitrary neighbourhood $U(x)$ of the point x. Hence (see (2)) $\omega_f(x) = \sup\limits_{U(x)} \omega_f(x, U(x)) \leqslant \frac{\eta}{2}$. So $\omega_f(x) < \eta$. Since η has been arbitrary we get $\omega_f(x) = 0$. □

We will show that the uniform convergence of a sequence of functions f_n is transferred to the sequence of functions ω_{f_n}.

Theorem 3.2. *Let $f: X \to Y$, $f_n: X \to Y$ $(n = 1, 2, \ldots)$. If $f_n \rightrightarrows f$ on X, then $\omega_{f_n} \rightrightarrows \omega_f$ on X.*

Proof. Let $\varepsilon > 0$. By the assumption there exists m_0 such that for every $m \geqslant m_0$ we have

$$\forall x \in X : d(f_m(x), f(x)) < \varepsilon$$

Let $x \in X$ and let $U(x)$ be a neighbourhood of the point x. Let $G \subset U(x)$ be a nonempty open set and $M \subset G$ a dense set in G. Then by triangle inequality we have for $y \in M$

$$\begin{aligned} d(f(y), f(x)) &\leqslant d(f(y), f_m(y)) + d(f_m(y), f_m(x)) + d(f_m(x), f(x)) \\ &< 2\varepsilon + d(f_m(y), f_m(x)). \end{aligned}$$

In a similar way the inequality $d(f_m(y), f_m(x)) < 2\varepsilon + d(f(y), f(x))$ can be obtained. The last two inequalities immediately imply

$$|\omega_{f_m}(x, U(x)) - \omega_f(x, U(x))| \leqslant 2\varepsilon$$

So $|\omega_{f_m}(x) - \omega_f(x)| \leqslant 2\varepsilon$ for every $m \geqslant m_0$ and every $x \in X$. □

It is well known that the topological structure of the continuity points of an arbitrary function can be described by means of the oscilation of the function ([13], pp. 120–122). This set is of the first Borel class. In the case of almost quasicontinuity points of a function f, one cannot obtain a similar result by means of the function ω_f. Even in the case of simple function f defined on $\mathbf{R}$ the set B_f is not necessarily a Borel set and, moreover, it need not be L-measurable. The following example illustrates such a situation.

E x a m p l e 3.1. Let $M \subset \mathbf{R}$ be nowhere dense, L-nonmeasurable set. Let $f = \chi_M$ be the charasteristic function of the set M. It is easy to verify that $B_f = \mathbf{R} \setminus M$. Thus B_f is not L-measurable.

In the paper [16] it is proved that if $f\colon X \to Y$, where Y is a second countable topological space, then the set $X \setminus H_f$ is of the first category in X. This yields the following theorem.

Theorem 3.3. *Let $f\colon X \to Y$, where X is a Baire space and Y a second countable topological space. Then the set B_f is residual in X.*

4. Almost quasicontinuous functions in the space $M(X)$

Let $X \neq \emptyset$ be a topological space. Denote by $M(X)$ the linear normed space of all bounded functions $f\colon X \to \mathbf{R}$ with the norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

In the paper [14] the topological position of quasicontinuous functions in the space $M(X)$ is described. We present now a similar result for the class $W(X)$ of all $f \in M(X)$ which are quasicontinuous on X.

Theorem 4.1. *The set $W(X)$ is a perfect and nowhere dense set in the space $M(X)$.*

P r o o f. Since the convergence in the space $M(X)$ coincides with the uniform convergence, we have from Theorem 2.1 that $W(X)$ is a closed set in $M(X)$. From the simple fact that for a function $g \in W(X)$ also $g + c$ (where $c \in \mathbf{R}$) belongs to $W(X)$, we obtain that $W(X)$ is dense in itself. Thus $W(X)$ is a perfect set in $M(X)$. To complete the proof, it suffices to prove that $W(X)$ is a boundary set in $M(X)$. To prove it it is sufficient to show that if $K(f,\delta)$ is an arbitrary ball in $M(X)$, then

$$K(f,\delta) \cap \big(M(X) \setminus W(X)\big) \neq \emptyset \tag{4}$$

Since f is bounded on X, there exists $A = \sup_{x \in X} f(x) \in \mathbf{R}$. On the basis of the definition of the supremum there exists a point $x_1 \in X$ such that $f(x_1) > A - \frac{\delta}{4}$. Define a function $h\colon X \to \mathbf{R}$ by $h(x) = f(x)$ if $x \neq x_1$, $h(x_1) = A + \frac{\delta}{4}$.

Then evidently $h \in M(X)$ and

$$\|f - h\| = h(x_1) - f(x_1) < A + \frac{\delta}{4} - \left(A - \frac{\delta}{4}\right) = \frac{\delta}{2} < \delta.$$

Thus $h \in K(f,\delta)$. It is sufficient to prove that $h \notin W(X)$. For every $x \neq x_1$ we have $h(x) = f(x) \leqslant A < A + \frac{\delta}{4} = h(x_1)$, $h(x_1) - h(x) \geqslant \frac{\delta}{4}$. So it is evident that h is not almost quasicontinuous at the point x_1. Thus (4) is true. □

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Authors' address: MFF UK, Mlynská dolina, 842 15 Bratislava, Czechoslovakia.