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POLYNOMIAL APPROACH TO POLE PLACEMENT IN MIMO n-D SYSTEMS[1]

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Matrix polynomial equations in n-D polynomials are employed to assign desired invariant polynomials to general n-D multi-input multi-output systems.

1. INTRODUCTION

A lot of dynamical properties of a linear system can be naturally expressed in terms of the positions of its poles. That is why the problem of pole placement has become so popular in control. In scalar linear systems, it is sufficient to assign just the characteristic polynomial. In multi-input multi-output systems, however, one must assign separately all the invariant polynomials (and not merely their multiple — the characteristic polynomial). The reason is that the characteristic polynomial itself does not say enough about the internal dynamics of a multi-input multi-output system.

For a scalar 2-D system, the problem of pole placement has been solved by several authors recently. We shall follow here the approach of [7] which is based on 2-D polynomial equations. The progress described in the present paper is twofold: at first, the multi-input multi-output systems are considered which call for matrix (instead of scalar) polynomial equations. At second, n-D systems are involved so that general n-D equations ($n \geqq 2$) apply.

2. BASIC DEFINITIONS

Throughout the paper, the n-D multi-input multi-output systems are described by their transfer matrices which are expressed as matrix polynomial fractions of the type

$$D_L^{-1} N_L \tag{1}$$

[1] The original version of the paper was presented at the IMACS World Congress, Paris, July 1988.

or

$$N_R D_R^{-1} \tag{2}$$

where the matrices D_L, N_L and D_R, N_R have their entries in $\mathscr{R}[z_1, z_2, \ldots, z_n]$ which is the ring of real polynomials in n indeterminates $z_1, z_2, \ldots, z_n$.

When facing a discrete n-D system (such as an n-D digital filter [1] or the discrete model of a complex plant described by partial differential equations [2]), the z_i's are usually interpreted as delay operators working in various directions. In the case of a delay-differential system [7], they stand for $\exp(-h_i s)$ where s is a complex variable in the Laplace transform and h_i's are various (possibly noncommesurate) delays durations.

Let us now recall some basic concepts for left fractions such as (1). They are mostly cited from [6]. Similar facts for (2) are dual.

Definition 1 (Factor Coprimeness). The matrices D_L and N_L are *left factor coprime* iff they have only unimodular left common factors (i.e. those with nonzero but real determinants).

Definition 2 (Zero Coprimeness). The matrices D_L and N_L are *left zero coprime* iff the composite matrix $[D_L N_L]$ has a full row rank for every n-tuple $(z_1, z_2, \ldots, z_n)$ of complex numbers.

These two different types of coprimeness project into the following two different concepts of equivalence.

Definition 3 (Intertwining Equivalence). The matrices D_L and D_R from $\mathscr{R}[z_1, z_2, \ldots, z_n]$ are *equivalent* (or *intertwined*) iff there are two matrices N_L and N_R from $\mathscr{R}[z_1, z_2, \ldots, z_n]$ such that

$$D_L^{-1} N_L = N_R D_R^{-1} \tag{3}$$

where the both fractions are factor coprime.

Definition 4 (Strict Equivalence). The matrices D_L and D_R from $\mathscr{R}[z_1, z_2, \ldots, z_n]$ are *strictly equivalent* iff there are two matrices N_L and N_R from $\mathscr{R}[z_1, z_2, \ldots, z_n]$ such that (3) holds with the both fractions zero coprime.

It can be shown [6] that the matrices D_L and D_R of the same sizes are strictly equivalent iff there are two unimodular matrices U and V in $\mathscr{R}[z_1, z_2, \ldots, z_n]$ such that

$$D_L = U D_R V \tag{4}$$

Needless to say that the strict equivalence implies the intertwining one. The vice versa, however, is not true and two intertwined matrices need not be related by (4) in general.

Invariant polynomials of n-D matrices can be formally defined in the same way as in 1-D:

Definition 5 (Invariant Polynomials). Let, for an $m \times m$ full rank matrix D, the greatest common divisors of its $k \times k$ minors are denoted by d_k, $1 \leqq k \leqq m$, and $d_0 = 1$. Then the *invariant polynomials* of D are defined by

$$i_1 = d_1/d_0\,, \quad i_2 = d_2/d_1, \ldots, i_m = d_m/d_{m-1} \tag{5}$$

The *Smith form* of D (in $\mathscr{R}[z_1, z_2, \ldots, z_n]$) is the matrix diag $(i_1, i_2, \ldots, i_m)$.

As expected, two n-D polynomial matrices D_L and D_R are intertwined iff they have the same (nonunit) invariant polynomials [6]. Consequently, every n-D polynomial matrix is intertwined with its Smith form but, in contrast to 1-D, is *not* strictly equivalent to it in general. That is why the unimodular operations (4) are not sufficient to calculate Smith forms in n-D. Fortunately, just the intertwining equivalence is what we often need to solve control problems.

3. FEEDBACK SYSTEMS

Consider an n-D linear l-input m-output *system* given by its transfer matrix

$$A_L^{-1}B_L \tag{6}$$

where A_L and B_L are left factor coprime matrices in $\mathscr{R}[z_1, z_2, \ldots, z_n]$, A_L is $m \times m$ and invertible while B_L is $m \times l$.

Further consider a linear output feedback *controller* described by the transfer matrix

$$Q_R P_R^{-1} \tag{7}$$

where P_R, Q_R are, respectively, $m \times m$ and $l \times m$ matrices in $\mathscr{R}[z_1, z_2, \ldots, z_n]$ and P_R is invertible.

As usually, we assume that both the system and controller have their characteristic polynomials equal to the determinants of their transfer matrices which means that they are free of hidden modes.

Let them be connected in the standard feedback structure in Figure 1. Similarly as in [4] for 1-D systems, we can derive that the matrix

$$A_L P_R + B_L Q_R \tag{8}$$

is interwined with the matrix

$$P_L A_R + Q_L B_R \tag{9}$$

where A_R, B_R and P_L, Q_L are defined via the following factor coprime matrix fractions

$$B_R A_R^{-1} = A_L^{-1} B_L \quad \text{and} \quad P_L^{-1} Q_L = Q_R P_R^{-1} \tag{10}$$

In addition, both the matrices (8) and (9) are intertwined with the denominators of any transfer matrix in the feedback loop in Figure 1 (provided that no cancellation nor extension has been made in these transfer matrices). Consequently, (8) and (9) have the same nonunit invariant polynomials and these polynomials equal (up to real multiples) nonunit invariant polynomials of every realization of the feedback system (provided that no hidden modes are added in this realization).

So to place desirably the poles (invariant polynomials) of the resultant feedback

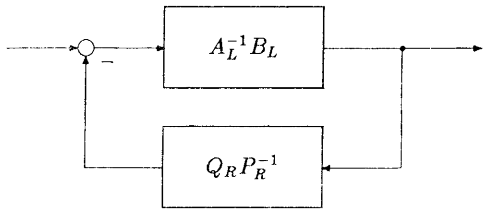

Fig. 1. Standard feedback structure.

system, one must first choose an $m \times m$ matrix C_L or an $l \times l$ matrix C_R (both in $\mathscr{R}[z_1, z_2, \ldots, z_n])$ having these desired invariant polynomials. Then the problem reads as follows:

Formulation 1. For the A_L, B_L and C_L find P_R invertible and Q_R such that

$$A_L P_R + B_L Q_R = C_L \tag{11}$$

or, equivalently, for the A_R, B_R and C_R find P_L invertible and Q_L such that

$$P_L A_R + Q_L B_R = C_R \,. \tag{12}$$

4. SOLUTION

We have now transformed the problem of pole placement (invariant polynomials assignment) into the solution of one from the n-D matrix polynomial equations (11) or (12). The method of solution for such equations can be found in [8] and need not be repeated here. However, it may be interesting to analyze the results of simple examples. We focus our attention on (11) in what follows. The equation (12) is dual.

Example 1. Let be given a 4-D system with

$$A_L = \begin{bmatrix} 1 + z_1 z_2 & 0 \\ z_3 + z_4 & 1 \end{bmatrix}$$

$$B_L = \begin{bmatrix} z_1 \\ z_3 + z_4 \end{bmatrix}$$

and

$$C_L = \begin{bmatrix} 2 + z_1 & 0 \\ 0 & 2 + z_2 \end{bmatrix}$$

When using the algorithm from [8], we result in the controller with

$$P_R = \begin{bmatrix} 2 & 0 \\ 2z_2 z_3 - 3z_3 + 2z_2 z_4 - 3z_4 & 2 + z_2 \end{bmatrix}$$

$$Q_R = [1 - 2z_2 \;\; 0]$$

It is well known that matrix polynomial equations may possess infinitely many solutions. In fact, here any matrices

$$P_R + B_R T \quad \text{and} \quad Q_R - A_R T \tag{13}$$

where T is an arbitrary compatible matrix with entries in $\mathscr{R}[z_1, z_2, \ldots, z_n]$, where

$$A_R = [1 + z_1 z_2]$$

$$B_R = \begin{bmatrix} z_1 \\ z_3 + z_4 - z_1 z_3 - z_1 z_4 + z_1 z_2 z_3 + z_1 z_2 z_4 \end{bmatrix},$$

give rise to the same invariant polynomials ($i_1 = 1$, $i_2 = (2 + z_1)(2 + z_2)$).

Whenever the A_L and B_L are zero coprime (as in Example 1) then one can assign any invariant polynomials and choose any C_L. In particular, all the invariant polynomials can bė taken units by setting $C_L = I$. This choice results in the so called *deadbeat controller* [3, 9] which for the given data reads

$$P_R = \begin{bmatrix} 1 & 0 \\ z_2 z_3 + z_2 z_4 - z_3 - z_4 & 1 \end{bmatrix}$$

$$Q_R = [-z_2 \;\; 0]$$

In fact, this is rather typical situation for A_L and B_L are generically left zero coprime whenever $l \geqq n$ (see [5]).

When, on the contrary, A_L and B_L have some (left) zeros in common then these zeros must occur in every choice of the resulting invariant polynomials with the right multiplicities. This is now well understood in scalar 2-D systems [7] where such common zeros are usually called the fixed poles of the system.

Example 2. Consider now

$$A_L = \begin{bmatrix} 1 & 1 + z_1 \\ -1 - z_1 & 0 \end{bmatrix}, \quad B_L = \begin{bmatrix} 0 \\ 2 + z_2 \end{bmatrix} \tag{14}$$

and, at the moment undeterminate, a right side matrix

$$C_L = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

In such a case, the algorithm from [8] yields directly a parametrization of all acceptable right hand sides. In fact, here the equation (11) is solvable iff $(2 + z_2)$ divides both $c_1(1 + z_1) + c_3$ and $c_2(1 + z_1) + c_4$ at the same time. So for

$$C_1 = \begin{bmatrix} 1 & 0 \\ -1 - z_1 & 2 + z_2 \end{bmatrix}$$

the solution exists being, e.g.,

$$P_R = \begin{bmatrix} 1 & -(1 + z_1)(2 + z_2) \\ 0 & 2 + z_2 \end{bmatrix}$$

and

$$Q_R = [0 \;\; -2z_1 - z_1^2]$$

but

$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 + z_2 \end{bmatrix}$$

makes (11) unsolvable.

This is rather surprising for C_1 and C_2 have not only the same invariant polynomials but they are even strictly equivalent:

$$C_1 = UC_2$$

where

$$U = \begin{bmatrix} 1 & 0 \\ -1 - z_1 & 1 \end{bmatrix}$$

Further studies will be sure useful to explain (system theoretically) why (11) possesses no solution for $C_L = C_2$ even if its invariant polynomials ($i_1 = 1, i_2 = 2 + z_2$) can be easily assigned to the given system (6), either by employing $C_L = C_1$ or by using, from the very beginning, another matrix fraction representation for the plant, say $\bar{A}_L = U^{-1}A_L$ and $\bar{B}_L = U^{-1}B_L$ (here clearly $\bar{A}_L^{-1}\bar{B}_L = A_L^{-1}B_L$).

Let us note that the same situation may happen in 1-D if A_L and B_L are not coprime and also there it is not yet understood. However, in 1-D one can always pre-cancel the fraction (6) to result in coprime left side of (11) before a C_L is chosen which is not the case in n-D (for $n \geqq 2$).

Fortunately, when using the method [8] to solve (11), one can calculate a parametrization of all acceptable right sides before choosing C_L (as in the Example 2). Such a way, all the problems above are overcome.

Example 3. As another example, consider an unstable delay-differential plant with the transfer matrix

$$A_L^{-1}B_L = \begin{bmatrix} s & 1 + s \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} \exp(-hs) \\ 1 \end{bmatrix} \tag{15}$$

Using again the method of [7], we first solve (11) for $C_L = I$ to get (substituting for brevity $d = \exp(-hs)$)

$$\begin{bmatrix} s & 1 + s \\ 0 & s \end{bmatrix} \begin{bmatrix} d - 1 & d - d^2 \\ 1 & -d \end{bmatrix} + \begin{bmatrix} d \\ 1 \end{bmatrix} \begin{bmatrix} -s & 1 + ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

and the right side fraction for (15) as a by-product

$$B_R A_R^{-1} = \begin{bmatrix} -1 - s + sd \\ s \end{bmatrix} [s^2]^{-1} \tag{17}$$

It is easy to see from (16) that the fraction (15) is zero coprime. As a consequence, any stable invariant polynomials can be assigned to stabilize the plant (15).

However, the solution appearing in (16) can not be used directly as $\det P = 0$. In such a case, one must apply (13) to get a suitable solution.

In practical cases one usually wishes to design a proper controller. Clearly, the controller made directly from (16) is an improper one but this is just a preparatory step. It is well known that, to get a proper controller, the degrees of C_L must be sufficiently high. Applying the result of [10], one must take the first-row-degree of C_L to be the first-row-degree of the matrix $[A_L \; B_L]$ plus the maximum degree of $\begin{bmatrix} A_R \\ B_R \end{bmatrix}$ minus one which here equals 2. Similarly, the second-row-degree of C_L must be, at least, 2 as well. For example,

$$C_L = \begin{bmatrix} (1+s)^2 & 0 \\ 0 & (1+s)^2 \end{bmatrix} \tag{18}$$

will do the job. Using now the algorithm [7], we obtain the controller (7) with

$$P_R = \begin{bmatrix} 1 + s + \exp(-hs) & 1 + s - \exp(-hs) - \exp(-2hs) \\ 1 & \exp(-hs) - s \end{bmatrix}$$

$$Q_R = [-s \quad 1 + s(2 + \exp(-hs))]$$

This proper retarded controller stabilizes the plant (15) by assigning (18), i.e. the finite number of poles characterized by the invariant polynomials $i_1 = i_2 = (1 + s^2)$.

Finally, recall that if the system (6) is strictly causal (i.e., $A_L(0)$ is invertible and $B_L(0) = 0$) then every solution of (11) is causal ($P_R(0)$ is invertible) for a causal C_L. Otherwise, noncausal solutions also exist and if a causal one is desired then (13) can be used if necessary.

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