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GEOMETRIC METHODS IN THE THEORY OF SINGULAR 2D LINEAR SYSTEMS

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A geometric approach for systems represented by a singular 2D Fornasini-Marchesini model is developed by introducing suitable notions of invariant subspace and controlled invariant subspace of the state space. The first notion is shown to be usefull in characterizing the set of compatible boundary conditions and in studying the existence and uniqueness of solutions to the state space equation of the considered models. The second notion is proved to be relevant in investigating the solvability of a Disturbance Decoupling Problem and is employed for stating a constructive sufficient condition for the existence of solutions to such problem.

1. INTRODUCTION

Singular linear systems have received an increasing attention during the last years and, recently, also 2D singular systems have been considered by some authors. In particular, a singular general model of 2D linear system has been introduced and studied in [7], and a singular Roesser model has been considered in [11]. Conditions for the existence and uniqueness of solutions to such models have been given in [7] and in [9], [11]. The utility of employing singular 2D models in e.g. image processing, hyperbolic equations, heat equations has been discussed in [11], [12] and [10], which also contains an extensive bibliography.

In this note we consider singular 2D Fornasini-Marchesini models. Our aim is to develop a geometric theory for such models and to explore the potential of the geometric approach both in analysing the properties of singular 2D models and in solving specific synthesis problems.

Geometric methods have been introduced in a 2D framework and used to solve some related control and observation problems, in [2], [3], [4] and, in a slightly different way, in [6]. On the other hand, a geometric approach to singular systems in the 1D framework has been first developed in [1]. In the first part of this note, we combine the ideas of [2] together with those of [1]. This allows us to define suitable geometric objects which are used for characterizing the set of compatible initial conditions and for studying the existence and uniqueness of solutions to the state equation of the models we are dealing with. Then, we consider a Disturbance Decoupling Problem and we provide a constructive sufficient condition for its solvability using geometric tools. From a general point of view, our results show that the geometric approach can be usefully employed in studying the structural properties of singular 2D models, for instance the singular Fornasini-Marchesini model, and in solving related noninteracting control problems, for instance the considered Disturbance Decoupling Problem.

2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Consider the singular 2D Fornasini-Marchesini model $\Sigma = (E, A_1, A_2, B_1, B_2)$ [7], [5] described by

$$Ex(i+1, j+1) = A_1x(i+1, j) + A_2x(i, j+1) + B_1u(i+1, j) + B_2u(i, j+1) \tag{2.1}$$

where i, j are nonnegative integers, $x(i,j) \in X = \boldsymbol{R}^n$ is the local state vector at the point (i,j), $u(i,j) \in U = \boldsymbol{R}^m$ is the input vector and $E \in \boldsymbol{R}^{r\times n}$, $A_k \in \boldsymbol{R}^{r\times n}$ and $B_k \in \boldsymbol{R}^{r\times m}$ for $k = 1, 2$, are real matrices. The special feature of the singular model is that the matrix E is in general not square and if it is square it may be singular.

In our setting, we let Σ evolve over the region $\mathscr{D}$ contained in the plane $\boldsymbol{Z} \times \boldsymbol{Z}$ given by

$$\mathscr{D} = \{(i,j) \in \boldsymbol{Z} \times \boldsymbol{Z}, \text{ with } i \geqq 0 \text{ and } j \geqq 0\}.$$

A boundary condition for Σ is an assignment of the form

$$x(i, 0) = x_{i0} \quad \text{for} \quad i > 0 \quad \text{and} \quad x(0, j) = x_{0j} \quad \text{for} \quad j > 0. \tag{2.2}$$

In the following we will identify the boundary conditions with the elements of $X^{\boldsymbol{N}^+} \times X^{\boldsymbol{N}^+}$.

If W is a linear space, the space of all W-valued doubly-indexed sequences $\mathbf{w}(i,j)$ defined on $\mathscr{D}\backslash\{0, 0\}$ will be denoted by $S(W)$. By a *solution* to Σ over $\mathscr{D}$ with boundary conditions (2.2) for the input $\mathbf{u}(i,j) \in S(U)$, we mean a sequence $\mathbf{x}(i,j) \in S(X)$ which satisfies (2.1) and (2.2). A boundary condition such that there exists a solution for the input $\mathbf{u}(i,j) = 0$ is called a *compatible boundary condition* for Σ.

The space of all the compatible boundary conditions for a given Σ is, in general, an infinite dimensional subspace of $X^{\boldsymbol{N}^+} \times X^{\boldsymbol{N}^+}$ whose description may be very cumbersome. Consider, for instance, the system $\Sigma = (E, A_1, A_2, B_1, B_2)$ where E is the zero $n \times n$ matrix. A boundary condition $(\{x_{i0}, i > 0\}, \{x_{0j}, j > 0\})$ is a compatible one for Σ if and only if

- $A_1x_{10} = -A_2x_{01}$;
- $A_1x_{20} = -A_2x_{11}$ and $A_2x_{02} = -A_1x_{11}$ for some $x_{11} \in X$;

- $A_1x_{30} = -A_2x_{21}$ and $A_2x_{03} = -A_1x_{12}$ for some $x_{12}, x_{21} \in X$ such that $A_1x_{21} = -A_2x_{12}$;
- ..... and so on ......

In setting our framework, we may however restrict our attention to subspaces of the space of all the compatible boundary conditions which enjoy good properties and which have a simpler description. In fact, if $\mathbf{V}$ is the space of boundary conditions we are going to deal with, it is quite reasonable to ask that, for any element of $\mathbf{V}$, $x(i, 0)$, as well as $x(0, j)$, can assume any value in a certain vector space, independently of the values assumed by other points. In order to satisfy these requirements $\mathbf{V}$ must be of the form $\mathbf{V} = V^{\mathbf{N}^+} \times V^{\mathbf{N}^+}$ for some subspace V of X. Then, our first aim is to look for the largest subspace V of X such that any element of $V^{\mathbf{N}^+} \times V^{\mathbf{N}^+}$ is a compatible boundary condition for Σ. This requires to introduce the following definition.

2.1. Definition. A subspace V of the state space X which satisfies the relation

$$A_1V + A_2 \subset EV \tag{2.3}$$

is said to be an invariant subspace of Σ.

The family of invariant subspaces of Σ is not empty, since it contains $\{0\}$, and it is closed under subspace addition. Therefore, it contains a maximum element, namely the *largest* or *maximum invariant subspace* of Σ, which will be denoted by $\tilde{V}$. The computation of such subspace is made possible by the following proposition.

2.2. Proposition. The sequence of subspaces defined recursively by

$$\begin{aligned} V_0 &= X \\ V_{k+1} &= V_k \cap \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^{-1} (EV_k \times EV_k) \end{aligned} \tag{2.4}$$

is decreasing and converges, in a finite number of steps, to $\tilde{V}$, the maximum invariant subspace of Σ.

Proof. The sequence is obviously nonincreasing and, if $V_q = V_{q+1}$, it becomes stationary at the step q. Therefore, since X is finite dimensional, the sequence converges in a finite number of steps, say $q \leqq n$, to V_q. By $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} V_q = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} V_{q+1} \subset$ $\subset EV_q \times EV_q$, we have that V_q is invariant. Moreover, if V is an invariant subspace, it is obviously contained in V_0 and, if it is contained in V_k, it is easily seen to be contained in V_{k+1}. Then, by induction, V is contained in V_q and $V_q = \tilde{V}$. □

We can now state the following results concerning the compatible boundary conditions for Σ.

2.3. Proposition. The maximum invariant subspace $\tilde{V}$ of Σ is the largest subspace of X such that any point in $\tilde{V}^{\mathbf{N}^+} \times \tilde{V}^{\mathbf{N}^+}$ is a compatible boundary condition for Σ.

Proof. Given a point in $\tilde{V}^{N^+} \times \tilde{V}^{N^+}$ it is easy to construct recursively a solution over $\mathscr{D}$ for $\mathbf{u}(i,j) = 0$ using (2.3). Hence, any point in $\tilde{V}^{N^+} \times \tilde{V}^{N^+}$ is a compatible boundary condition for Σ. Conversely, if V is the largest subspace of X such that any point of $V^{N^+} \times V^{N^+}$ is a compatible boundary condition for Σ, V is obviously an invariant subspace of Σ. Hence $V \subset \tilde{V}$ and, since $\tilde{V}^{N^+} \times \tilde{V}^{N^+} \subset V^{N^+} \times V^{N^+}$ by the first part of the proof, $\tilde{V} = V$. □

2.4. Remark. Denoting by V' an $n \times p$ full column rank matrix such that $\operatorname{Im} V' = \tilde{V}$, we can write $A_1 V' = EV'F_1$, $A_2 V' = EV'F_2$ and $x(i,j) = V' z(i,j)$ for $z(i,j) \in \mathbf{R}^p$. Then, given a solution $\mathbf{z}(i,j) \in S(\mathbf{R}^p)$ to the non singular 2D equation

$$z(i+1, j+1) = F_1 z(i+1, j) + F_2 z(i, j+1)$$

with boundary conditions $(\{z_{i0}, i > 0\}, \{z_{0j}, j > 0\})$, $\mathbf{x}(i,j) = V' \mathbf{z}(i,j)$ is a solution to Σ with boundary conditions $(\{x_{i0} = V'z_{i0},\ i > 0\},\ \{x_{0j} = V'z_{0j},\ j > 0\})$. It follows, in particular, that given a boundary condition whose local components belong to $\tilde{V}$ (respectively, to an invariant subspace V) there exists a solution $\mathbf{x}(i,j)$ whose local components $x(i,j)$ belong to $\tilde{V}$ (respectively, to V).

The next result about the existence of solutions corresponding to nonzero inputs will be useful in the sequel.

2.5. Proposition. Given the model $\Sigma = (E, A_1, A_2, B_1, B_2)$, described by (2.1), if

$$\operatorname{Im} B_1 \subset E\tilde{V} \quad \text{and} \quad \operatorname{Im} B_2 \subset E\tilde{V} \tag{2.6}$$

holds, then there exists a solution to Σ over $\mathscr{D}$ for any compatible boundary condition and for any input $\mathbf{u}(i,j) \in S(U)$. Moreover, if V is an invariant subspace and $\operatorname{Im} B_1 \subset EV$ and $\operatorname{Im} B_2 \subset EV$, then, for any compatible boundary condition with local components belonging to V and any input, there exists a solution with local components belonging to V.

Proof. As in 2.3, 2.4. □

After having investigated the existence of solutions to Σ, one is usually concerned with their uniqueness. A sufficient condition for the uniqueness of solutions is stated, in a form that will be useful in the sequel, in the next proposition using the characteristic subspaces of the pairs (E, A_1) and (E, A_2) (cf. [1] Definition 1). Recall that these are the largest subspaces V_1 and V_2 of X such that $A_1 V_1 \subset EV_1$ and $A_2 V_2 \subset EV_2$.

2.6. Proposition. Given the model $\Sigma = (E, A_1, A_2, B_1, B_2)$ described by (2.1), let V_1 and V_2 denote respectively the characteristic subspace of the pair (E, A_1) and of the pair (E, A_2). Assume that $\mathbf{x}(i,j)$, $\mathbf{x}'(i,j) \in S(X)$ be two solutions to Σ corresponding to the same boundary condition $(\{x_{i0},\ i > 0\},\ \{x_{0j}, j > 0\})$ and to the same input $\mathbf{u}(i,j) \in S(U)$. Then, if

$$\text{either} \quad \operatorname{Ker} E \cap V_1 = \{0\} \quad \text{or} \quad \operatorname{Ker} E \cap V_2 = \{0\} \tag{2.7}$$

we have $\mathbf{x}(i,j) = \mathbf{x}'(i,j)$.

Proof. Assume that Ker $E \cap V_1 = \{0\}$ and denote by $\mathbf{x}''(i,j)$ the difference $\mathbf{x}(i,j) - \mathbf{x}'(i,j)$. The sequence $\mathbf{x}''(i,j)$ is a solution to Σ corresponding to the zero boundary condition and to the zero input. In particular, $x''(1,0) = 0$ and $Ex''(1, j+1) = A_1 x''(1,j)$ for all $j > 0$, hence $\{x''(1,j), j \geqq 0\}$ is a solution to the singular 1D linear system (E, A_1, B), for arbitrary B, corresponding to the zero initial condition and to the zero input. Then, by [1] Theorem 2, we get that $x(1,j) = 0$ for all $j \geqq 0$. In turn, this implies $E\, x''(2, j+1) = A_1\, x''(2,j)$. By recalling that $x''(2,0) = 0$ and by applying recursively the same argument as above we get $\mathbf{x}''(i,j) = 0$. The proof is analogous if Ker $E \cap V_2 = \{0\}$. □

2.7. Remark. It is worthwhile to remark that if A_2 and B_2 are zero and, hence, Σ reduces essentially to a singular 1D system, then $\tilde{V}$ reduces to the characteristic subspace of the pair (E, A_1) and (2.7) coincides with the necessary and sufficient condition for the uniqueness of solutions stated in [1] Theorem 2.

3. CONTROLLED INVARIANT SUBSPACES AND DISTURBANCE DECOUPLING

Several control problems concerning 1D have been solved in an elegant and very effective way by employing geometric methods (see for instance [14]). Similar, although generally weaker, results have been obtained in a 2D framework in [2], [3], [4]. A classical problem we can consider from this point of view is that of decoupling a disturbance by means of a static state feedback. A natural formulation of such problem in our context is the following.

3.1. Definition. Let Σ_d be the singular 2D model described by

$$\begin{aligned} E\, x(i+1, j+1) &= A_1\, x(i+1, j) + A_2\, x(i, j+1) + B_1\, u(i+1, j) + \\ &+ B_2\, u(i, j+1) + D_1\, w(i+1, j) + D_2\, w(i, j+1) \\ y(i,j) &= C\, x(i,j) \end{aligned} \tag{3.1}$$

where u is the controlled input and $w \in W = \mathbf{R}^q$ is a disturbance. The *Disturbance Decoupling Problem* (DDP) for Σ_d consists in finding a feedback law $u(h,k) = F\, x(h,k)$ such that the compensated system Σ_{dF} admits a unique solution for all compatible initial conditions and all disturbances and its output is not affected by the disturbance w.

In order to find conditions for the solvability of the DDP, we need to introduce a suitable notion of controlled invariant subspace. So, given a singular 2D Fornasini-Marchesini model $\Sigma = (E, A_1, A_2, B_1, B_2)$ described by (2.1), we begin, following [2], by stating the following Proposition.

3.2. Proposition. Given a subspace V of the state space X, the followings are equivalent:

i) $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} V \subset EV \times EV + \operatorname{Im} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ in the product space $X \times X$;

ii) there exists a feedback $F: X \to U$ such that

$$\left(\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} F \right) V \subset EV \times EV \text{in the product space} \quad X \times X \,. \tag{3.2}$$

Proof. As in [2] 2.1 with the obvious modifications. □

3.3. Definition. i) Any subspace $V \subset X$ for which the equivalent conditions of Proposition 3.2 hold is said to be a *controlled invariant subspace* for Σ or an $(E, A_{1,2}, B_{1,2})$-*invariant subspace.*

ii) Given and $(E, A_{1,2}, B_{1,2})$-invariant subspace V, any feedback $F: X \to U$ for which (3.2) holds is said to be a *friend* of V.

For any subspace $K \subset X$, the family of controlled invariabt subspaces for Σ which are contained in K is not empty, since it contains $\{0\}$, and is closed under subspace addition. It therefore has a maximum element, namely the *maximum controlled invariant subspace* for Σ *contained in* K or the *maximum* $(E, A_{1,2}, B_{1,2})$-*invariant subspace contained in* K, which will be denoted by $V^*(K)$. The computation of such subspace is made possible by the following Proposition.

3.4. Proposition. The sequence of subspaces defined recursively by

$$\begin{aligned} V_0 &= K \\ V_{k+1} &= V_k \cap \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^{-1} \left(EV_k \times EV_k + \operatorname{Im} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) \end{aligned} \tag{3.3}$$

is decreasing and converges, in a finite number of steps, to $V^*(K)$, the maximum $(E, A_{1,2}, B_{1,2})$-invariant subspace for Σ contained in K.

Proofs. As in Proposition 2.2. □

Clearly, if V is an $(E, A_{1,2}, B_{1,2})$-invariant subspace for $\Sigma = (E, A_1, A_2, B_1, B_2)$ and F is one of its friend, V is an invariant subspace for $\Sigma_F = (E, A_1 + B_1F, A_2 + B_2F)$. Conversely, any invariant subspace for Σ_F is an $(E, A_{1,2}, B_{1,2})$-invariant subspace for Σ having F as a friend. In the following, given a model $\Sigma = (E, A_1, A_2, B_1, B_2, C)$ described by

$$\begin{aligned} &E\, x(i+1, j+1) = A_1\, x(i+1, j) + A_2\, x(i, j+1) + B_1\, u(i+1, j) + \\ &+ B_2\, u(i, j+1) \\ &y(i,j) = C\, x(i,j) \end{aligned} \tag{3.4}$$

we will consider in particular the $(E, A_{1,2}, B_{1,2})$-invariant subspace $V^*(\operatorname{Ker} C)$, botained by letting $K = \operatorname{Ker} C$ in (3.3).

3.5. Proposition. For all boundary conditions $(\{x_{i0}, \; i > 0\}, \{x_{0j}, j > 0\}) \in$

$\in (V^*(\mathrm{Ker}\, C))^{\mathbf{N}^+} \times (V^*(\mathrm{Ker}\, C))^{\mathbf{N}^+}$, there exists an input $u(i, j)$ for which the system Σ has a solution over $\mathscr{D}$ belonging to $S(V^*(\mathrm{Ker}\, C))$.

Proof. By direct computation using the $(E, A_{1,2}, B_{1,2})$-invariance. □

The above Proposition says that $V^*(\mathrm{Ker}\, C)$ has an output nulling property analogous to that exhibited by the maximum controlled invariant subspace contained in the Kernel of the output map in the linear 1D case. This implies that $V^*(\mathrm{Ker}\, C)$ can be used, as in the 1D case, to state the conditions for the solvability of the DDP.

3.6. Proposition. Given the system Σ_d described by (3.1), let $V^*(\mathrm{Ker}\, C)$ be the maximum $(E, A_{1,2}, B_{1,2})$-invariant subspace contained in $\mathrm{Ker}\, C$ and let $F: X \to U$ be one of its friends. If we have $\mathrm{Im}\, D_k \subset EV^*(\mathrm{Ker}\, C)$, for $k = 1, 2$, then for any boundary condition belonging to $(V^*(\mathrm{Ker}\, C))^{\mathbf{N}^+} \times (V^*(\mathrm{Ker}\, C))^{\mathbf{N}^+}$ and for any disturbance $\mathbf{w}(i, j)$ there exists a solution of the compensated system Σ_{dF} over $\mathscr{D}$ which gives identically zero output.

Proof. By Proposition 2.5, since $V^*(\mathrm{Ker}\, C)$ is an invariant subspace for Σ_{dF} contained in the Kernel of the output map. □

We can now state a sufficient condition for the solution of the DDP. To this aim, we need to consider the largest (A_1, E, B_1) invariant subspace and the largest (A_2, E, B_2) invariant subspace described in [13]. Recall that these are the largest subspaces V_1^* and V_2^* of X such that $AV_1^* \subset EV_1^* + \mathrm{Im}\, B_1$ and $A_2 V_2^* \subset EV_2^* + \mathrm{Im}\, B_2$.

3.7. Proposition. Given the disturbed system Σ_d described by (3.1), let us assume that $\mathrm{Im}\, D_k \subset EV^*(\mathrm{Ker}\, C)$, for $k = 1, 2$ and that either $\mathrm{Ker}\, E \cap V_1^* = \{0\}$ or $\mathrm{Ker}\, E \cap V_2^* = \{0\}$. Then the DDP is solvable.

Proof. Let F be a friend of $V^*(\mathrm{Ker}\, C)$. By Proposition 3.6, for any disturbance $\mathbf{w}(i, j)$ there exists a solution of the compensated system Σ_{dF} over $\mathscr{D}$ which gives identically zero output. Moreover, since V_1^* and V_2^* contain, respectively, the characteristic subspaces of the pairs $(E, A_1 + B_1 F)$ and $(E, A_2 + B_2 F)$, the solution is unique by Proposition 2.6. □

4. CONCLUSION

Some methods and ideas of the geometric approach have been applied in the context of the singular 2D Fornasini-Marchesini models. This has produced geometric tools which allow us to analyse the existence and uniqueness of solutions to the state equation of the model and to give a sufficient constructive condition for the solvability of a Disturbance Decoupling Problem. Further developments of this approach in connection with observation problems and with noninteracting control problems are under investigation.

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