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Karel Zvára

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CONSISTENCY OF AN ESTIMATE IN LINEAR REGRESSION WITH NON-NEGATIVE ERRORS

KAREL ZVÁRA

For a linear regression model with non-negative errors the method of regression coefficients estimation, that origins in Anděl's procedure for AR(2), is described. The strong consistency of the estimate is proved.

1. INTRODUCTION

In some applications statisticians should look for a "boundary line" that can be approached to only from one side. For example in [4] the authors search for the dependence of marginal possible grain yield on the size of a chosen growth factor. They fit a boundary line that confines the data, thus separating real from nonreal situations. The boundary line they estimate by least squares method on extremal observations. In our paper we propose a different method.

A linear model is given by

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + e_i, \; i = 1, \dots, n, \tag{1}$$

where $\boldsymbol{\beta} \in \mathsf{R}^k$ is an unknown vector of parameters, $\mathbf{x}_i \in \mathsf{R}^k$ are known vectors and $e_i \geq 0$ are independent identically distributed random errors. Since we cannot assume that $\mathsf{E}\, e_i = 0$, we will try to find some alternative to the method of least squares for estimating $\boldsymbol{\beta}$. To this end we will suppose firstly that the random errors e_i are exponentially distributed with expectation θ. The density of $y_1, \dots, y_n$ is

$$f(y_1, \dots, y_n; \boldsymbol{\beta}, \theta) = \begin{cases} \theta^{-n} \exp\Big(-\theta^{-1} \sum_{i=1}^{n} (y_i - \mathbf{x}_i'\boldsymbol{\beta})\Big) & \text{if } y_i \geq \mathbf{x}_i'\boldsymbol{\beta} \text{ for all } i = 1, \dots, n; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the maximum likelihood estimate $\mathbf{b}_n$ of $\boldsymbol{\beta}$ can be computed by maximization of $\bar{\mathbf{x}}_n'\mathbf{b}$ on a set M_n defined by

$$M_n = \left\{\mathbf{b} \in \mathsf{R}^k : \mathbf{x}_i'\mathbf{b} \le y_i,\ i = 1,\dots,n\right\},$$

where

$$\bar{\mathbf{x}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i.$$

Maximum likelihood estimate of θ is given by

$$t_n = \bar{y}_n - \bar{\mathbf{x}}_n'\mathbf{b}_n,$$

where

$$\bar{y}_n = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Similar problem for autoregressive process AR(2) was solved by Anděl [1, 2]. He proposed an estimate which is similar to $\mathbf{b}_n$.

To show some properties of the estimator $\mathbf{b}_n$ we will give an example. Let $\boldsymbol{\beta} = (\alpha,\beta)'$, $\mathbf{x}_i = (1, x_i)'$, where

$$x_i = \begin{cases} x_L,\ i \in I_L = \{m\nu + 1, \dots,\ m\nu + k\}, \\ x_U,\ i \in I_U = \{m\nu + k + 1, \dots,\ m\nu + m\}, \end{cases} \qquad \nu = 0, 1, \dots, r,$$

$x_L < x_U$ are given reals and $k,\ m,\ r$ are given integers. Let us have $n = m(r+1) > k$ observations $y_1, \dots, y_n$. An estimate $(a,b)'$ of $(\alpha,\beta)'$ is in M_n only if

$$a + bx_L \le y_L, \text{ where } y_L = \min_{i \in I_L} y_i$$

and

$$a + bx_U \le y_U, \text{ where } y_U = \min_{i \in I_U} y_i.$$

The maximization of $a + bc$ over M_n for every $c \in (x_L, x_U)$, especially for $c = \bar{x}_n$, implies that the last two inequalities must be fulfilled as equalities. It follows that the points (x_L, y_L), (x_U, y_U) must lie on the fitted line $a_n + b_n x$. Then

$$b_n = \frac{y_U - y_L}{x_U - x_L}, \qquad a_n = \frac{x_U y_L - x_L y_U}{x_U - x_L}.$$

Let us define

$$e_L = \min_{i \in I_L} e_i = y_L - \alpha - \beta x_L,$$
$$e_U = \min_{i \in I_U} e_i = y_U - \alpha - \beta x_U.$$

From the well known properties of exponential distribution it follows that the random variables e_L and e_U are independent and exponentially distributed with expectations

$$\frac{\theta}{k(r+1)} \quad \text{and} \quad \frac{\theta}{(m-k)(r+1)},$$

respectively. Considering

$$a_n = \alpha + \frac{x_U e_L - x_L e_U}{x_U - x_L},$$

$$b_n = \beta + \frac{e_U - e_L}{x_U - x_L},$$

it follows that

$$\mathsf{E}\, b_n = \beta + \frac{\theta}{x_U - x_L} \frac{1}{r+1} \left(\frac{1}{m-k} - \frac{1}{k} \right),$$

$$\mathsf{Var}\, b_n = \left(\frac{\theta}{x_U - x_L} \frac{1}{r+1} \right)^2 \left(\left(\frac{1}{m-k} \right)^2 + \left(\frac{1}{k} \right)^2 \right),$$

$$\mathsf{MSE}\, b_n = 2 \left(\frac{\theta}{x_U - x_L} \frac{1}{r+1} \right)^2 \left(\left(\frac{1}{m-k} \right)^2 + \left(\frac{1}{k} \right)^2 - \frac{1}{k(n-k)} \right),$$

$$\mathsf{E}\, a_n = \alpha + \frac{\theta}{x_U - x_L} \frac{1}{r+1} \left(\frac{x_U}{k} - \frac{x_L}{m-k} \right),$$

$$\mathsf{Var}\, a_n = \left(\frac{\theta}{x_U - x_L} \frac{1}{r+1} \right)^2 \left(\left(\frac{x_L}{m-k} \right)^2 + \left(\frac{x_U}{k} \right)^2 \right),$$

$$\mathsf{MSE}\, a_n = 2 \left(\frac{\theta}{x_U - x_L} \frac{1}{r+1} \right)^2 \left(\left(\frac{x_L}{m-k} \right)^2 + \left(\frac{x_U}{k} \right)^2 - \frac{x_L x_U}{k(n-k)} \right).$$

The estimate b_n is unbiased only for $m = 2k$. Because of $x_L < x_U$, the estimate a_n is biased in this case. Therefore, at least one of the estimates a_n, b_n is biased. But both of them are asymptotically ($r \to \infty$) unbiased and their variances tend to zero. Therefore, the estimates a_n, b_n are consistent estimates of α, β. All these properties are valid for the estimates a_{nc}, b_{nc} defined by maximization of $a + bc$ on the set M_n, where $c \in (x_L, x_U)$. The assumption on the exponential distribution of e_i enabled us to find an explicit expression of the characteristics of estimates a_n, b_n. In the following part of the paper we will assume more general assumptions on distribution of e_i, therefore we will prove only a strong consistency of the proposed estimates.

2. ASSUMPTIONS ON THE RANDOM ERRORS

Let us suppose that the following assumptions are satisfied:

(A) Let $e_1, \ldots, e_n$ be independent identically distributed random variables.

(B) Let $\mathsf{P}[e_1 \geq 0] = 1$.

(C) Let $\mathsf{P}[0 \leq e_1 < \varepsilon] > 0$ for all $\varepsilon > 0$.

Lemma 1. The constant $c = 0$ is a limit point of $\{e_i\}_{i=1}^{\infty}$ with probability 1.

Proof. Let $\varepsilon > 0$ be a given number. Let us consider the events

$$G_i = \{\omega : 0 \le e_i < \varepsilon\}, \ i = 1, \dots .$$

From the assumptions (A) – (C) it follows that $\mathsf{P}[G_i] = \delta > 0$ for all i. The events $G_1, G_2, \dots$ are independent and $\sum_{i=1}^{\infty} \mathsf{P}[G_i] = \infty$. Borel-Cantelli lemma yields that infinitely many G_i occur with probability 1. □

3. ASSUMPTIONS ON THE REGRESSORS

Investigating asymptotic properties of $\mathbf{b}_n$ we will use a modified concept of the limit point of $\{\mathbf{x}_i\}_{i=1}^{\infty}$. A point $\mathbf{z} \in \mathsf{R}^k$ is called a Q^+-limit point of the sequence $\{\mathbf{x}_i\}_{i=1}^{\infty}$ if and only if there exists an infinite subsequence $\{\mathbf{x}_{i_j}\}_{j=1}^{\infty}$ and real numbers $0 < \lambda_j \le 1$ such that

$$\lim_{j \to \infty} \lambda_j \mathbf{x}_{i_j} = \mathbf{z}.$$

Our definition differs slightly from the definition by Wu [5] because we accept only positive numbers λ_j. The concept of Q^+-limit point allows us to deal with unbounded sequences. For example let $\mathbf{x}_i = (1, (-1)^i i)'$. The vectors $(0,1)'$ or $(0, -1/10)'$ are examples of Q^+-limit points of this sequence.

The set of all Q^+-limit points of $\{\mathbf{x}_i\}_{i=1}^{\infty}$ will be denoted by $\mathcal{Z}$. Let us consider

$$M_\beta = \bigcap_{\mathbf{z} \in \mathcal{Z}} \{\mathbf{b} \in \mathsf{R}^k : \mathbf{z}'\mathbf{b} \le \mathbf{z}'\boldsymbol{\beta}\} = \boldsymbol{\beta} + M,$$

where

$$M = \bigcap_{\mathbf{z} \in \mathcal{Z}} \{\mathbf{a} \in \mathsf{R}^k : \mathbf{z}'\mathbf{a} \le 0\}.$$

The set M is an intersection of closed half-spaces, therefore M is a closed convex cone.

The set of all vectors $\mathbf{c} \in \mathsf{R}^k$ with property that the function $f(\mathbf{x}) = \mathbf{c}'\mathbf{x}$ is minimized on the convex set M for $\mathbf{x} = \mathbf{0}$ is equal to

$$M^* = \{\mathbf{c} \in \mathsf{R}^k : \mathbf{c}'\mathbf{a} \ge 0 \text{ for every } \mathbf{a} \in M\}.$$

M^* is called a dual convex cone of the convex cone M.

Let us denote

$$K = \left\{\mathbf{c} \in \mathsf{R}^k : \mathbf{c} = \sum_{i=1}^{r} \lambda_i \mathbf{z}_i, \text{ for some } \lambda_1, \dots, \lambda_r > 0, \mathbf{z}_1, \dots, \mathbf{z}_r \in \mathcal{Z}, r \in \mathsf{N}\right\}.$$

The set K is a convex cone. Therefore it follows

$$\begin{aligned}
(-K)^* &= \{\mathbf{a} \in \mathsf{R}^k : \mathbf{a}'(-\mathbf{c}) \ge 0 \text{ for all } \mathbf{c} \in K\} \\
&= \left\{\mathbf{a} \in \mathsf{R}^k : \mathbf{a}'\left(\sum_{i=1}^{r} \lambda_i \mathbf{z}_i\right) \le 0 \text{ for all } \lambda_i > 0, \ \mathbf{z}_i \in \mathcal{Z}, \ r \in \mathsf{N}\right\} \\
&= \{\mathbf{a} \in \mathsf{R}^k : \mathbf{a}'\mathbf{z} \le 0 \text{ for all } \mathbf{z} \in \mathcal{Z}\} \\
&= M.
\end{aligned}$$

Let us denote the closure of a set C by $\bar{C}$, the interior of a C by C° and the affine subspace of R^k defined by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ by $\mathcal{M}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. For every convex cone C it is true that $C \subset C^{**} = \bar{C}$, therefore

$$(-M)^* = (-(-K)^*)^* = K^{**} = \bar{K}.$$

We are interested in situations where $\mathbf{a} = \mathbf{0}$ is the only one point of maxima of function $\mathbf{c}'\mathbf{a}$ on M. This property reflects the shape of the cone. A convex cone C is said to be pointed if C does not contain $\mathbf{x}$ and $-\mathbf{x}$ at the same time for every nonzero $\mathbf{x}$.

Lemma 2. Let C be a convex cone in R^k. Then we have
(i) If C^* has an interior point, then C is pointed.
(ii) If C is closed and pointed, then C^* has an interior point.
(iii) If C is closed and pointed, then there is some $\mathbf{p} \in \mathsf{R}^k$, such that $\mathbf{p}'\mathbf{x} > 0$ for all nonzero $\mathbf{x} \in C$.

For the proof see [3, Thm. 3.13]. □

Lemma 3. If the cone $\bar{K}$ is pointed, then there is some $\mathbf{q} \in \mathsf{R}^k$, such that $\mathbf{q}'\mathbf{c} < 0$ for all nonzero $\mathbf{c} \in \bar{K}$.

Proof. The assertion of the lemma follows from Lemma 2 (iii), if we take $C = -\bar{K}$. □

Now we can state basic assumptions on the sequence $\{\mathbf{x}_i\}_{i=1}^{\infty}$ for consistency of some estimates of $\boldsymbol{\beta}$.

(D) There exist k linearly independent Q^+-limit points of sequence $\{\mathbf{x}_i\}_{i=1}^{\infty}$.

(E) The convex set $\bar{K}$ is pointed.

Lemma 4. Let assumptions (D), (E) be satisfied. A point $\mathbf{c} \in \mathsf{R}^k$ is an interior point of K if and only if there exist some $\lambda_1 > 0, \ldots, \lambda_r > 0$, $\mathbf{z}_1 \in \mathcal{Z}, \ldots, \mathbf{z}_r \in \mathcal{Z}$, $r \in \mathsf{N}$ such that

$$\dim \mathcal{M}(\mathbf{z}_1, \ldots, \mathbf{z}_r) = k \tag{2}$$

and

$$\mathbf{c} = \sum_{i=1}^{r} \lambda_i \mathbf{z}_i. \tag{3}$$

Proof. Let the assumptions of the lemma be satisfied. Then it follows from (3) that $\mathbf{c} \in K$. Let vectors $\mathbf{z}_1, \ldots, \mathbf{z}_k$ be linearly independent. Let us denote $\mathbf{Z} = (\mathbf{z}_1, \ldots, \mathbf{z}_k)$. The matrix $\mathbf{Z}$ is regular, therefore

$$\max_{\|\mathbf{s}\|=1} \|\mathbf{Z}^{-1}\mathbf{s}\|^2 = \max_{\|\mathbf{s}\|=1} \mathbf{s}'(\mathbf{Z}\mathbf{Z}')^{-1}\mathbf{s} = \delta^2 > 0.$$

Let $0 < \varepsilon \leq \min_{1\leq j\leq k} \lambda_j/\delta$ and let us denote $\boldsymbol{\lambda}_I = (\lambda_1, \ldots, \lambda_k)'$. The vector $\boldsymbol{\lambda}_I + \varepsilon\, \mathbf{Z}^{-1}\mathbf{s}$ has only non-negative coordinates, hence

$$\mathbf{c} + \varepsilon\mathbf{s} = \mathbf{Z}(\boldsymbol{\lambda}_I + \varepsilon\mathbf{Z}^{-1}\mathbf{s}) + \sum_{j=k+1}^{r} \lambda_j \mathbf{z}_j$$

is a non-negative linear combination of $\mathbf{z}_1, \ldots, \mathbf{z}_r$ and $\mathbf{c} + \varepsilon\mathbf{s} \in K$. The point $\mathbf{c}$ is an interior point of K.

Now, let the existence condition of the lemma be not satisfied. It means that

$$\dim \mathcal{M}(\mathbf{z}_1, \ldots, \mathbf{z}_r) = d < k \tag{4}$$

for every positive combination (3). From (D) it follows that there exists some $\mathbf{z}_{r+1} \in \mathcal{Z}$, such that $\dim \mathcal{M}(\mathbf{z}_1, \ldots, \mathbf{z}_{r+1}) = d + 1$. If the point $\mathbf{c}$ were an interior point of K, then for sufficiently small $\varepsilon > 0$ it would follow

$$\mathbf{c}_1 = \mathbf{c} + \varepsilon\mathbf{z}_{r+1} \in K,$$

$$\mathbf{c}_2 = \mathbf{c} - \varepsilon\mathbf{z}_{r+1} \in K.$$

Let $\lambda_{r+1} = \varepsilon$, then $\mathbf{c}_1 = \sum_{i=1}^{r+1} \lambda_i \mathbf{z}_i$. From the definition of cone K we can write

$$\mathbf{c}_2 = \sum_{j=1}^{q} (\lambda_j^{\triangle}/2)\mathbf{z}_j^{\triangle}, \ \lambda_j^{\triangle} > 0, \ \mathbf{z}_j^{\triangle} \in \mathcal{Z}$$

for some $q > 0$. (It can be proved that $q \leq k$.) But the vector $\mathbf{c}$ can be also written in the form

$$\mathbf{c} = \frac{1}{2}(\mathbf{c}_1 + \mathbf{c}_2) = \sum_{i=1}^{r+1} (\lambda_i/2)\mathbf{z}_i + \sum_{j=1}^{q} (\lambda_j^{\triangle}/2)\mathbf{z}_j^{\triangle}.$$

Equation (4) holds if for some of $\mathbf{z}_t$, $t = 1, \ldots, r+1$, there exist $\alpha_i \geq 0, i \neq t$, $i = 1, \ldots, r+1$, and $\alpha_j^{\triangle} \geq 0$, $j = 1, \ldots, q$, $\sum_{i=1}^{r+1} \alpha_i + \sum_{j=1}^{q} \alpha_j^{\triangle} > 0$ so that

$$\mathbf{z}_t + \sum_{i\neq t,\ i=1}^{r+1} \alpha_i \mathbf{z}_i + \sum_{j=1}^{q} \alpha_j^{\triangle} \mathbf{z}_j^{\triangle} = \mathbf{0}.$$

But from Lemma 3 there exists some $\mathbf{q} \in \mathrm{R}^k$ such that $\mathbf{q}'\mathbf{z} < 0$ for all non-null $\mathbf{z} \in \mathcal{Z}$. Therefore we found the contradiction

$$0 = \mathbf{q}'\mathbf{z}_t + \sum_{i\neq t,\ i=1}^{r+1} \alpha_i \mathbf{q}'\mathbf{z}_i + \sum_{j=1}^{q} \alpha_j^{\triangle} \mathbf{q}'\mathbf{z}_j^{\triangle} < 0$$

and the point $\mathbf{c}$ cannot lie in the interior of K. $\square$

4. CONSISTENCY OF ESTIMATES

We will introduce a modified estimate. Let $\mathbf{b}_{nc}$ be a vector maximizing $\mathbf{c}'\mathbf{b}$ on M_n and let us denote

$$N_c = \left\{\mathbf{b} \in \mathsf{R}^k : \mathbf{c}'\mathbf{b} \geq \mathbf{c}'\boldsymbol{\beta}\right\}.$$

The set M_n is an intersection of n closed half-spaces, therefore M_n is a closed convex set. The same holds for N_c. For every $\mathbf{b} \in M_n, n = 1, \ldots$, we have (cf. assumption (B))

$$\mathbf{x}_i'\boldsymbol{\beta} \leq \mathbf{x}_i'\boldsymbol{\beta} + e_i = y_i$$

therefore $\boldsymbol{\beta} \in M_n \cap N_c$.

Let us define intersection of all sets M_n, i. e.

$$M_0 = \bigcap_{n=1}^{\infty} M_n = \lim_{n\to\infty} M_n.$$

Lemma 5. If assumptions (A) - (C) are satisfied then $M_0 \subset M_\beta$ almost surely.

Proof. If $\mathbf{b} \in M_0$, then

$$\mathbf{x}_i'\mathbf{b} \leq y_i = \mathbf{x}_i'\boldsymbol{\beta} + e_i, \; i = 1, \ldots .$$

For every $\mathbf{z} \in \mathcal{Z}$ there exist $\{i_j\}_{j=1}^{\infty}, 0 < \lambda_j \leq 1$, such that $\lim_{j\to\infty} \lambda_j \mathbf{x}_{i_j} = \mathbf{z}$. From the sequence of inequalities

$$\lambda_j \mathbf{x}_{i_j}'\mathbf{b} \leq \lambda_j \mathbf{x}_{i_j}'\boldsymbol{\beta} + \lambda_j e_{i_j}, \; j = 1, \ldots$$

we can select with probability 1 a subsequence (cf. Lemma 1) such that

$$\lim_{r\to\infty} e_{i_{j_r}} = 0.$$

Then, in the limit, we get $\mathbf{z}'\mathbf{b} \leq \mathbf{z}'\boldsymbol{\beta}$ and therefore $\mathbf{b} \in M_\beta$ almost surely. □

Theorem 1. Let the assumptions (A) - (E) be satisfied and let $\mathbf{c} \in K^\circ$. Then $\mathbf{b}_{nc}$ is a consistent estimate of $\boldsymbol{\beta}$.

Proof. We will use Lemma 2 with $C = M$. We know that $M^* = -\bar{K}$ has an interior point (e. g. $\mathbf{c}$), and hence the only maximum of $\mathbf{c}'\mathbf{a}$ on M is $\mathbf{a} = \mathbf{0}$, and hence the only maximum of $\mathbf{c}'\mathbf{b}$ on M_β is $\mathbf{b} = \boldsymbol{\beta}$. It follows that $M_\beta \cap N_c = \{\boldsymbol{\beta}\}$. But we know that $\beta \in M_o \cap N_c$ and $M_o \subset M_\beta$ a.s.. Then, the only limit point of $\{\mathbf{b}_{nc}\}$ is $\boldsymbol{\beta}$ almost surely, therefore $\mathbf{b}_{nc}$ is a consistent estimator of $\boldsymbol{\beta}$. □

We have proved a consistency of $\mathbf{b}_{nc}$ for every $\mathbf{c} \in K^\circ$. However, according to the motivation introduced at the beginning of our paper, we expect that the choice $\mathbf{c} = \bar{\mathbf{x}}_n$ is appropriate. Results of a simulation experiment (see Chapter 6) confirm well this choice.

We are able to return to the estimate $\mathbf{b}_n$. The last assumption for its consistency is

(F) Let $\lim_{n\to\infty} \bar{\mathbf{x}}_n = \boldsymbol{\xi} \in K^\circ$.

Theorem 2. Under the assumptions (A) – (F) $\mathbf{b}_n$ is a consistent estimate of $\boldsymbol{\beta}$.

Proof. For sufficiently large n the vector $\bar{\mathbf{x}}_n$ lies in K° and the assumptions of Theorem 1 are satisfied. □

5. REGRESSION LINE

Let us consider a model of regression line. In this special case $k = 2$, $\mathbf{x}_i = (1, x_i)'$, $\boldsymbol{\beta} = (\beta_1, \beta_2)'$, therefore the conditions (D), (E) and (F) can be simplified.

Let us denote

$$x_L = \liminf x_i, \quad x_U = \limsup x_i.$$

If the sequence $\{x_i\}_{i=1}^\infty$ were not bounded from above (from below) then we define $x_U = \infty$ ($x_L = -\infty$). Let x_0 be a finite limit point of $\{x_i\}_{i=1}^\infty$. Then $\lambda(1, x_0)'$ for $\lambda \geq 0$ are Q^+-limit points of $\{\mathbf{x}_i\}_{i=1}^\infty$. If $x_U = \infty$ or $x_L = -\infty$ then the corresponding Q^+-limit points of $\{\mathbf{x}_i\}_{i=1}^\infty$ are $\lambda(0,1)'$ or $\lambda(0,-1)'$ for $\lambda \geq 0$. Assumption (D) is not satisfied only if every Q^+-limit point of $\{\mathbf{x}_i\}_{i=1}^\infty$ is a multiple of some fixed vector. This can happen only in two cases: 1) if $x_L = x_U = x_0$ and x_0 is finite; 2) if $x_L = -\infty$, $x_U = \infty$ and there does not exist any finite limit point of $\{x_i\}_{i=1}^\infty$. Therefore, for regression line the condition (D) can be replaced by the condition

(D′) There exists a finite limit point of the sequence $\{x_i\}_{i=1}^\infty$ and simultaneously $x_L < x_U$ holds.

In this case the cone K is generated by the Q^+-limit points expressed in the following way:

$$\begin{pmatrix} 1 \\ x_0 \end{pmatrix} \quad \text{for a finite limit point } x_0 \text{ of sequence } \{x_i\}_{i=1}^\infty,$$

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{for } x_L = -\infty,$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } x_U = \infty.$$

Evidently, the cone K (and then $\bar{K}$, because K can always be generated by two Q^+-limit points only) is in the half-plane $[0,\infty) \times \mathrm{R}^1$. This cone is pointed except of the case when $x_L = -\infty$ and $x_U = \infty$ simultaneously. Therefore condition (E) can be replaced by the new condition

(E′) At least one of the values x_L or x_U is finite.

Theorem 3. Let conditions (A) – (C), (D′) and (E′) be satisfied. Then for any $c \in (x_L, x_U)$ maximization of the function $a + bc$ over M_n yields to a consistent estimator of parameters α, β.

Condition (F) can be replaced by the condition

(F′) Let $\lim_{n\to\infty} \bar{x}_n = \bar{x}_0 \in (x_L, x_U)$.

It is easy to verify that for a regression line the conditions (F) and (F′) are equivalent.

Theorem 4. Under the assumptions (A) – (C), (D′), (E′) and (F′) $\mathbf{b}_n$ is a consistent estimate of parameters of a regression line.

6. SIMULATION EXPERIMENT

To verify properties of the proposed estimate a simulation experiment with the regression line

$$y_i = \beta_0 + \beta_1 x_i + e_i, \qquad i = 1, \ldots, n,$$

for $\beta_0 = 0$, $\beta_1 = 1$ and $n = 2m$, was made. To compare behavior of the estimate of slope β_1 for different designs $x_1, \ldots, x_n$, we choose the following designs:

A: $$x_i = \begin{cases} 1, & i = 1, \ldots, m, \\ -1, & i = m+1, \ldots, n; \end{cases}$$

B: $$x_i = \begin{cases} (4i-2)/n, & i = 1, \ldots, m, \\ 4 - (4i-2)/n, & i = m+1, \ldots, n; \end{cases}$$

C: $$x_i = \begin{cases} (4i-2)/n, & i = 1, \ldots, m, \\ -1, & i = m+1, \ldots, n. \end{cases}$$

For each of the proposed designs it holds

$$\sum_{i=1}^{n} x_i/n = 0, \ \sum_{i=1}^{n} |x_i|/n = 1.$$

The first ideas about the dependence of behavior of the estimate on the distribution of error term can be obtained from the three following choices of distributions:

E: exponential with expectation 1;

U: uniform on $(0, 2\sqrt{3})$;

AN: absolute value of $N(0, \pi/(\pi-2))$.

In all three cases $\mathsf{Var}\, e_i = 1$ and assumptions (A), (B) and (C) are fulfilled.

The sensitivity of the estimate of regression line slope on the choice of the parameter c (see the definition of $\mathbf{b}_{nc}$ estimate) is shown by computation of estimates for $c = 0$ $(= \bar{x})$, $c = -0.5$ and $c = 0.5$. For each combination of design, distribution of error term and the constant c, 1000 simulations were computed for $n = 10$, 20, 50 and 100 and only 100 simulations were computed for $n = 200$.

In Table 1 standard deviations of the slope estimated by the proposed method are compared with standard deviations of the slope estimated by the least squares methods that can be computed directly. It is obvious that for medium values of n the new estimate has smaller variability.

Table 1: Standard deviations of the slope.

		Proposed estimate									LS
		$c = -0.5$			$c = 0$			$c = 0.5$			
	n	E	U	AN	E	U	AN	E	U	AN	
A	10	0.139	0.339	0.219	0.142	0.322	0.223	0.138	0.345	0.226	0.316
	20	0.070	0.197	0.127	0.071	0.195	0.126	0.074	0.200	0.137	0.224
	50	0.028	0.093	0.055	0.030	0.091	0.054	0.027	0.086	0.057	0.141
	100	0.014	0.046	0.030	0.014	0.046	0.029	0.014	0.044	0.030	0.100
	200	0.008	0.028	0.012	0.006	0.021	0.016	0.009	0.020	0.017	0.071
B	10	0.195	0.437	0.320	0.161	0.395	0.295	0.216	0.434	0.304	0.275
	20	0.102	0.278	0.188	0.084	0.251	0.148	0.113	0.270	0.184	0.194
	50	0.044	0.133	0.087	0.036	0.117	0.068	0.042	0.128	0.083	0.122
	100	0.021	0.069	0.044	0.017	0.059	0.033	0.022	0.076	0.046	0.087
	200	0.011	0.035	0.021	0.009	0.031	0.015	0.010	0.038	0.020	0.061
C	10	0.158	0.366	0.251	0.159	0.389	0.256	0.177	0.392	0.274	0.294
	20	0.078	0.230	0.141	0.080	0.231	0.138	0.102	0.266	0.169	0.207
	50	0.032	0.104	0.062	0.031	0.096	0.062	0.044	0.111	0.081	0.131
	100	0.017	0.054	0.030	0.015	0.053	0.032	0.021	0.067	0.042	0.093
	200	0.009	0.023	0.015	0.009	0.023	0.017	0.012	0.033	0.016	0.065

Table 2: Biases of the slope.

		$c = -0.5$			$c = 0$			$c = 0.5$		
	n	E	U	AN	E	U	AN	E	U	AN
A	10	0.000	0.002	0.005	0.007	0.018	0.013	-0.002	-0.003	-0.001
	20	-0.001	0.004	0.004	-0.001	0.003	-0.003	0.001	0.000	0.001
	50	0.001	-0.003	-0.001	-0.002	-0.004	0.000	-0.000	-0.002	-0.003
	100	0.001	-0.001	0.000	-0.000	0.000	-0.002	-0.001	0.001	-0.001
	200	0.000	0.001	-0.002	0.000	0.001	0.002	-0.000	-0.002	-0.001
B	10	-0.051	-0.137	-0.094	-0.003	0.010	0.008	0.070	0.136	0.076
	20	-0.041	-0.122	-0.069	-0.004	0.007	0.003	0.046	0.117	0.076
	50	-0.016	-0.043	-0.032	-0.002	-0.000	-0.000	0.015	0.050	0.032
	100	-0.007	-0.022	-0.016	0.000	-0.000	-0.002	0.009	0.029	0.019
	200	-0.006	-0.013	-0.009	-0.001	-0.001	-0.000	0.003	0.013	0.005
C	10	-0.011	-0.007	-0.008	-0.001	-0.033	-0.015	0.057	0.121	0.080
	20	-0.003	0.001	-0.003	-0.003	-0.012	-0.007	0.036	0.113	0.067
	50	-0.001	-0.008	-0.003	-0.001	0.001	-0.004	0.015	0.037	0.031
	100	-0.001	-0.003	-0.001	-0.001	-0.003	-0.000	0.008	0.023	0.016
	200	-0.002	-0.001	-0.001	0.000	-0.002	0.000	0.006	0.012	0.006

In Table 2 biases of the slope estimated by the proposed method are given. It is obvious that the choice of c different from $c = \bar{x}$ can cause a bias of the proposed estimate. In our experiment the estimate is unbiased under an inappropriate choice of c only in the case that the estimate is not dependent on the concrete value of the constant from a given interval (i.e. design A, design C for $c \in (-1, 2/n)$).

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RNDr. Karel Zvára, CSc., matematicko-fyzikální fakulta University Karlovy (Faculty of Mathematics and Physics – Charles University), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.