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Necessary Optimality Conditions for Discrete Systems with State-Dependent Control Region

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Necessary optimality conditions in the form of a discrete maximum principle for general discrete optimal control problems with state-dependent control region are derived using the mathematical programming approach. In a more concrete case of the explicitely given constraining sets as systems of equalities and inequalities, more meaningful optimality conditions were obtained which can be important from a computational point of view. This fact was illustrated by a simple example of discrete optimal control problem with state-dependent control region which was solved applying the obtained optimality conditions.

1. INTRODUCTION

The purpose of this paper is to show that the approach developed in [1; 2], and detailly exposed in [3], to deal with constrained minimization problems in finite dimensional spaces can be modified in such a way that it also applies to discrete optimization problems with state-dependent control region. As far as such optimization problems have practical applications, it is reasonable to study this type of discrete optimal control problems.

Similar problems were studied earlier by Leitman et al., e.g. see [4], using a geometrical approach. Also Boltjanskij in his recent book [5] considered this class of discrete optimal control problems. Problem posed in [5] is a general one, but some assumptions necessary during the proof considerably narrow possible applications. Method used in [5] is, in fact, the same as in [3]. Namely, a general nonlinear mathematical programming problem is studied and necessary optimality conditions for its solution derived. Then a discrete optimal control problem is transcribed to the mathematical programming one and the necessary optimality conditions are decomposed and interpreted for this special case.

The same approach we use also in this contribution. In the next section we summerize some basic results from [3] concerning mathematical programming theory.

The central role here plays a concept of the so called "conical approximation" to the set $\Omega \subset E^n$. The origin of this concept can be found already in the first works devoted to the extremum problems. Let us recall at least the attainability cone introduced by Pontrjagin et al. in [6] or the derived cone as used by Hestenes in [7]. Also cone of forbidden variations originated from Dubovickij and Miljutin [8] has a similar meaning. Finally as an outgrowth of these ideas a very general theory of extremum problems was accomplished by Neustadt [9], Gamkrelidze and Charatišvili [10], and Gamkrelidze [11].

The concept of a conical approximation used in [3] and also here, is nearly the same as the concept of a derived cone of Hestenes [7]. By the way, it can be rather straightforward obtained as a special case of more general concepts, e.g. see [9], interpreted in finite dimensional spaces. Boltajanskij in [5] introduced the so called "tent" to the set. His definition is less general than that used in [3], but in finite dimensional spaces no difficulties arise. Using the concept of a conical approximation it is possible to derive necessary optimality conditions for a general mathematical programming problem. We state these conditions without proofs which can be found in [3] – see Section 2.

A general discrete optimal control problem is precisely formulated in Section 3. The admissible control region is allowed to be state-dependent. Also state constraints may be present. This fundamental formulation does not specify various constraining sets. To obtain more familiar and practical formulation some special cases are pointed out. Then we state and discuss certain assumptions which are needed to obtain necessary optimality conditions for a discrete optimal control problem.

The most important assumption concernes the so called locally smooth selection of a multivalued mapping, which enables us to treat problems with state-dependent control region. This assumption is not too much restrictive and it is satisfied in a large class of discrete optimal control problems with explicitely given constraints.

The derivation of necessary optimality conditions is subject of study in Section 4, which is partially based on the author's thesis [12]. Applying the concept of a locally smooth selection we are able to modify a general scheme from [3] to obtain a set of necessary optimality conditions for our formulation of a discrete optimal control problem. By an analogy with a continuous case one can also speak about a discrete maximum principle. For explicitely given discrete optimal control problems we state necessary optimality conditions separately in Section 5.

In this way we have obtained certain results of Boltjanskij [5] using more simple and straightforward construction and, moreover, it was possible to replace the convexity assumption in [5] by a weaker assumption of the so called directional convexity. Also the explicite continuity and compactness assumptions on an admissible control region were not necessary in our construction. Especially in the case of explicitely given constraints the derived necessary optimality conditions have a simple form in comparison with those of Boltjanskij in [5]. This fact is important from the both practical and computational point of view.

To illustrate practical importance of the obtained results a simple example with state-dependent control region is included which was solved using our necessary optimality conditions.

2. PRELIMINARY RESULTS

Our main tool in the next section will consist of some results concerning mathematical programming theory in finite dimensional spaces. For convenience, some basic concepts and results of this theory are summerized in this section, which is based on the monograph of Canon et al. [3], where the interested reader can find a more detailed exposition and proofs. Similar questions were also discussed by Boltjanskij [5] and the author [12].

A general mathematical programming problem can be stated in the following way (E^n denotes n-dimensional Euclidean space and, if not otherwise stated, all vectors are supposed to be column-vectors with except of the gradients of functions).

Definition 1. Given a real-valued function $f : E^n \to E^1$ and a subset $\Omega \subset E^n$. Find a vector $\hat{z} \in \Omega$ such that $f(\hat{z}) \leqq f(z)$ for all $z \in \Omega$. This problem is denoted as a general mathematical programming problem.

The constraining set Ω can posses a quite general structure. In general, the more concrete structure of Ω is assumed, the more meaningfull and stronger results can be achieved. For the later used construction from [3] it is convenient to treat some explicitely given equality constraints separately, although some other possibilities also exist – see [5]. Thus we have the next modification of Definition 1.

Definition 2. Let $f : E^n \to E^1$, $r : E^n \to E^m$ be continuously differentiable in E^n and let $\Omega \subset E^n$. The problem of finding a vector $\hat{z} \in E^n$ satisfying

$$\hat{z} \in \Omega\,, \quad r(\hat{z}) = 0 \tag{2.1}$$

such that $f(\hat{z}) \leqq f(z)$ for all $z \in E^n$ satisfying (2.1) we denote as a basic problem. Vector $\hat{z}$ is then said to be an optimal solution for the basic problem and vector z satisfying (2.1) is said to be a feasible solution for the basic problem.

If our aim is to derive meaningful necessary optimality conditions for the basic problem we must further characterize the set Ω. This will be done by the so called conical approximation to the set Ω at the point $\hat{z} \in \Omega$. Some historical aspects of this concept were briefly discussed in Introduction. Let us denote by co Γ the convex hull of the set Γ and by $\|\cdot\|$ the usual Euclidean norm in E^n. As a cone we shall understand a cone with vertex in the origin.

Definition 3. A convex cone $C(\hat{z}, \Omega) \subset E^n$ will be called a conical approximation to the set $\Omega \subset E^n$ at the point $\hat{z} \in \Omega$ if for any collection $\{\delta z_1, \ldots, \delta z_N\}$ of linearly independent vectors in $C(\hat{z}, \Omega)$ there exist an $\varepsilon > 0$, possibly depending on $\hat{z}, \delta z_1, \ldots$ $\ldots, \delta z_N$, and a continuous map ζ from $\mathrm{co}\{\hat{z}, \hat{z} + \varepsilon\, \delta z_1, \ldots, \hat{z} + \varepsilon\, \delta z_N\}$ into Ω such that $\zeta(\hat{z} + \delta z) = \hat{z} + \delta z + o(\delta z)$, where for function $o(\delta z)$ holds $\lim_{\|\delta z\| \to 0} \|o(\delta z)\| / \|\delta z\| = 0$.

An important special case of Definition 3 we obtain if the map ζ is the identity map, i.e. $\mathrm{co}\{\hat{z}, \hat{z} + \varepsilon\delta z_1, \ldots, \hat{z} + \varepsilon\delta z_N\} \subset \Omega$. This special case is then said to be a conical approximation of the first kind and is denoted by $C_1(\hat{z}, \Omega)$.

Now we are able to formulate necessary optimality conditions for the basic problem. Proof of this fundamental theorem is based on the Brouwer fixed-point theorem and can be found in [1; 2; 3]. In the formulation of this theorem we denote by $\bar{\Gamma}$ the closure of the set $\Gamma \subset E^n$ in E^n.

Theorem 1. If $\hat{z}$ is an optimal solution to the basic problem and $C(\hat{z}, \Omega)$ is a conical approximation to Ω at $\hat{z}$, then there exist a scalar $\mu \leqq 0$ and a row-vector $\psi \in E^m$ such that the following conditions are satisfied:

(1) If $\mu = 0$, then ψ is nonzero.

(2) For all $\delta z \in \bar{C}(\hat{z}, \Omega)$ holds the relation

$$\left(\mu \frac{\partial f(\hat{z})}{\partial z} + \psi \frac{\partial r(\hat{z})}{\partial z}\right) \delta z \leqq 0\,.$$

It is clear that if we admit $C(\hat{z}, \Omega) = \{0\}$, then the statement of Theorem 1 becomes trivial. Therefore we shall be primarily interested in cases with $C(\hat{z}, \Omega) \neq \{0\}$. Later we shall study discrete optimal control problems for which the "relaxed" extension of Theorem 1 will play a central role. Proof of this extension can be directly obtained from Theorem 1.

Theorem 2. Let $\Omega^* \subset E^n$ be any set with the property that for every $z^* \in \Omega^*$ there exists $z \in \Omega$ satisfying $r(z) = r(z^*)$ and $f(z) \leqq f(z^*)$. If $\hat{z}$ is an optimal solution to the basic problem, if $\hat{z} \in \Omega^*$, and if $C(\hat{z}, \Omega^*)$ is a conical approximation to the set Ω^* at $\hat{z}$, then there exist a scalar $\mu \leqq 0$ and a row-vector $\psi \in E^m$ such that the following conditions are satisfied:

(1) If $\mu = 0$, then ψ is nonzero.

(2) For all $\delta z \in \bar{C}(\hat{z}, \Omega^*)$ holds the relation

$$\left(\mu \frac{\partial f(\hat{z})}{\partial z} + \psi \frac{\partial r(\hat{z})}{\partial z}\right) \delta z \leqq 0\,.$$

Condition (2) in the both theorems can be written alternatively using a concept of the dual cone.

Definition 4. Let $C \subset E^n$ be a convex cone. The closed convex cone

$$DC[C] = \{z \in E^n \mid \langle c, z \rangle \leqq 0 , c \in C\}$$

is denoted as a dual cone of the cone C.

Here by $\langle \cdot, \cdot \rangle$ we denoted the scalar product. It is easy to see that, e.g. condition (2) of Theorem 1 can be equivalently written

$$\mu \frac{\partial f(\hat{z})}{\partial z} + \psi \frac{\partial r(\hat{z})}{\partial z} = \beta ,$$

where $\beta \in DC[C(\hat{z}, \Omega)]$, and similarly for Theorem 2.

Finally let us discuss two important cases of the constraining set Ω. In the both cases the corresponding conical approximation is of the first kind, which fact is useful in our later construction. First, let us study a more special case of the basic problem. Namely, we shall assume that the set Ω is given explicitely as a system of inequalities, i.e.

$$\Omega = \{z \in E^n \mid g(z) \leqq 0\} , \tag{2.2}$$

where function $g : E^n \to E^s$ is continuously differentiable and $g(z) \leqq 0 \Leftrightarrow g^i(z) \leqq 0$, $i = 1, \ldots, s$. This notation will be used for vectors throughout the paper. If we want to find conical approximation to the set Ω given by (2.2), we divide the components of g, i.e. functions g^i, $i = 1, \ldots, s$, into two sets as indicated in the following definition.

Definition 5. Let Ω be given by (2.2). For any $\hat{z} \in \Omega$ the active index set $I[g(\hat{z})]$ of the function g at the point $\hat{z}$ is defined by

$$I[g(\hat{z})] = \{i \mid g^i(\hat{z}) = 0 , \ i \in \{1, \ldots, s\}\} .$$

The complement of $I[g(\hat{z})]$ in $\{1, \ldots, s\}$ we shall denote by $I^c[g(\hat{z})]$.

Definition 6. Let Ω be given as in (2.2). For any $\hat{z} \in \Omega$ the internal cone to Ω at $\hat{z}$, denoted by $IC(\hat{z}, \Omega)$, is defined by

$$IC(\hat{z}, \Omega) = \left\{\delta z \in E^n \mid \frac{\partial g^i(\hat{z})}{\partial z} \delta z < 0, \ i \in I[g(\hat{z})]\right\} \cup \{0\} .$$

For this special case of Ω we have the following three propositions – see [3].

Proposition 1. Let Ω be given as in (2.2). Then the internal cone $IC(\hat{z}, \Omega)$ is a conical approximation of the first kind. Further, if $IC(\hat{z}, \Omega)$ is not the origin, then

$$\overline{IC}(\hat{z}, \Omega) = \left\{\delta z \in E^n \mid \frac{\partial g^i(\hat{z})}{\partial z}\, \delta z \leqq 0,\ i \in I[g(\hat{z})]\right\}$$

and

$$DC[IC(\hat{z}, \Omega)] = \left\{y \in E^n \mid y = \sum_i \alpha_i \frac{\partial g^i(\hat{z})}{\partial z},\ \alpha_i \geqq 0,\ i \in I[g(\hat{z})]\right\}.$$

Proposition 2. A sufficient condition for $IC(\hat{z}, \Omega)$ not to be only the origin is that the vectors $\partial g^i(\hat{z})/\partial z$, $i \in I[g(\hat{z})]$ are linearly independent. On the other hand, if $IC(\hat{z}, \Omega) = \{0\}$, then there exists a nonzero row-vector $v \in E^s$ such that

$$v \leqq 0\,, \quad v\,\frac{\partial g(\hat{z})}{\partial z} = 0\,, \quad vg(\hat{z}) = 0\,.$$

Proposition 3. If $\hat{z}$ is an optimal solution to the basic problem with Ω given by (2.2), then there exist a scalar $\mu \leqq 0$ and row-vectors $\psi \in E^m$, $v \in E^s$ such that the following conditions are satisfied:

(1) If $\mu = 0$, then not the both ψ and v are zero.

(2) $\mu\,\dfrac{\partial f(\hat{z})}{\partial z} + \psi\,\dfrac{\partial r(\hat{z})}{\partial z} + v\,\dfrac{\partial g(\hat{z})}{\partial z} = 0\,.$

(3) $v \leqq 0\,, \quad vg(\hat{z}) = 0\,.$

The last statement in Proposition 1 is, in fact, the well-known Farkas' lemma, which is very often used in [3]. From the practical point of view only such necessary optimality conditions are meaningful, where $\mu \neq 0$, i.e. without any loss of generality we may assume $\mu = -1$. Examining carefully the last proposition we obtain

Proposition 4. If in Proposition 3 the vectors

$$\frac{\partial r^i(\hat{z})}{\partial z}\,,\ i = 1, \ldots, m\,, \quad \frac{\partial g^j(\hat{z})}{\partial z}\,, \quad j \in I[g(\hat{z})]$$

are additionally linearly independent, then we may put $\mu = -1$.

Second important case arises when the constraining set Ω is convex.

Definition 7. Suppose that Ω is a convex subset of E^n. The support cone to Ω at $\hat{z} \in \Omega$, denoted by $SC(\hat{z}, \Omega)$, is a convex cone generated by the set $\Omega - \{\hat{z}\}$, i.e.

$$SC(\hat{z}, \Omega) = \{\delta z \in E^n \mid \delta z = \lambda(z - \hat{z}),\ z \in \Omega,\ \lambda \geqq 0\} .$$

Proposition 5. The support cone $SC(\hat{z}, \Omega)$ to the convex set Ω at the point $\hat{z} \in \Omega$ is a conical approximation of the first kind.

The proof of this proposition follows immediately from the definition of a support cone $SC(\hat{z}, \Omega)$. Let us also note that the concept of a support cone coincides with the concept of a radial cone to the convex set used in [3].

3. DISCRETE OPTIMAL CONTROL PROBLEM

Now we shall formulate the so called discrete optimal control problem with prescribed number of stages which will be studied from the point of view of necessary optimality conditions. The standard notation of [3] will be mainly used. First, let us consider a general case.

Suppose that the dynamical behaviour of our system can be fully described by a vector difference equation

$$x_{k+1} = f_k(x_k, u_k), \quad k = 0, 1, \ldots, K - 1, \tag{3.1}$$

where a positive integer K is given number of stages, $x_k \in E^n$ denotes state of the system at the stage k, $u_k \in E^m$ is control at the stage k and $f_k : E^n \times E^m \to E^n$. As usual, x_k and u_k are assumed to be column-vectors.

Our aim is to choose a control sequence $\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{K-1})$ and a corresponding trajectory $\hat{x} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K)$, determined by (3.1), which minimize the sum (cost functional)

$$J = \sum_{k=0}^{K-1} h_k(x_k, u_k), \tag{3.2}$$

where $h_k : E^n \times E^m \to E^1$, $k = 0, 1, \ldots, K - 1$. This minimization is subject to the following system of constraints.

(a) The control constraints are assumed to be state-dependent, namely,

$$u_k \in U_k(x_k), \quad k = 0, 1, \ldots, K - 1, \tag{3.3}$$

where U_k is a multivalued mapping from E^n into the set of the all nonempty subsets of E^m, which is sometimes denoted by $\mathscr{P}(E^m)$, i.e. $U_k : E^n \to \mathscr{P}(E^m)$, $k = 0, 1, \ldots, K - 1$.

(b) State constraints are simply given as nonempty admissible sets in E^n, i.e.

$$x_k \in A_k, \quad k = 0, 1, \ldots, K. \tag{3.4}$$

The pair of sequence $(\hat{x}, \hat{u})$ is then denoted as an optimal process in the system (3.1) to (3.4) and the pair (x, u) satisfying constraints (3.1), (3.3) and (3.4) is called an admissible process in the given system. The question of the existence of optimal processes in general discrete systems was studied in the previous paper of the author [13] and is, therefore, omitted here.

Our formulation of a discrete optimal control problem is fairly general and to obtain meaningful necessary optimality conditions we have to impose certain assumptions on the above defined problem. Let us also note that, in principle, it would be possible to suppose some structural properties of constraining sets in (3.3) and (3.4) as suggested by Boltjanskij in [5], but the all discussion and notation is then more complicated with only a little substantial gain. Moreover, the results of a practical interest can be obtained without this modification in the problem statement.

Before we proceed further let us give a more concrete formulation, the so called explicitely given case, of the constraints (3.3) and (3.4) supposing that the each constraining set can be given explicitely by a system of equalities and inequalities. So we assume

$$(3.5)\qquad U_k(x) = \{u \in E^m \mid Q_k(x, u) = 0,\ q_k(x, u) \leqq 0\}\,, \quad k = 0, 1, \ldots, K - 1\,,$$

$$(3.6)\qquad A_k = \{x \in E^n \mid S_k(x) = 0\,,\ s_k(x) \leqq 0\}\,, \quad k = 0, 1, \ldots, K\,.$$

Here $Q_k : E^n \times E^m \to E^{\gamma_k}$, $q_k : E^n \times E^m \to E^{\varkappa_k}$, $S_k : E^n \to E^{\varrho_k}$ and $s_k : E^n \to E^{\pi_k}$, i.e. these mappings are finite dimensional. The inequality sign for vectors has the same meaning as in previous section – see (2.2). Let us also remark that to our formulation of a discrete optimal control problem a large class of discrete optimization problems can be brought in an obvious way, e.g. problems with terminal cost functional, problems with delays, periodic problems etc.

Assumption 1. The all functions appearing in the relations (3.1) and (3.2), and in the case of explicitely given constraints also in (3.5) and (3.6), are continuously differentiable in their domains of definition.

This assumption is a quite natural one if we use mathematical programming approach. In a special case of a constant admissible control region $U_k(x) = U_k \subset E^m$, $k = 0, 1, \ldots, K - 1$, we can Assumption 1 somewhat release as far as the continuity of f_k and h_k with respect to u is concerned. This will be more clear from our later construction.

The next assumption concerns the so called "directional convexity" of a discrete optimal control problem. This concept is due to Holtzman – see [14; 15; 16]. It is the author's opinion that this assumption is a natural one if the summation type of the cost functional (3.2) is considered. In the Halkin's paper [17], which can be viewed as a principal one in the field of discrete optimal control theory, the terminal cost functional was assumed and the convexity of a discrete optimal control problem required. Otherwise the obtained results were not valid. Halkin himself speaks about the directional convexity to be a generalization of his results – for details see [14 to 17].

However, this "generalization" must be carefully interpreted. In fact, if we transform our cost functional (3.2) to the terminal one using the additional state variable, we must due to [17] assume the convexity of this problem with a very special cost functional given by $(n + 1)$-th component of the terminal state. Especially to this type of discrete optimal control problems the relaxed assumption of the directional convexity applies and the use of the convexity assumption would be to much restrictive, e.g. also simple linear discrete optimal control problems with summation quadratic cost functional violate the convexity assumption. On the other hand, for problems with a general nonlinear terminal cost functional the former convexity assumption is necessary.

These ideas are also confirmed by results of Propoj summerized in his book [18]. Thus we can conclude that a certain convexity assumption is always necessary to obtain a discrete maximum principle. This situation is then very similar to the existence theory for continuous system, where the convexity of the so called epigraph is assumed, e.g. see Olech [19]. The analogical parallelism exists also between the maximum principle for continuous systems and the existence theory for discrete systems, where no convexity assumptions are needed – see [6; 13].

By the all above discussion we have tried to explain in a certain sense some statements in the literature which overestimate the role of directional convexity. As we showed, the main difference lies in various problem formulations.

Definition 8. Let e be any vector in E^n. A set $\Omega \subset E^n$ is said to be e-directionally convex if for every vector z^* in the convex hull of Ω there exists a vector $z \in \Omega$ such that

$$z = z^* + \lambda e , \quad \lambda \geqq 0 .$$

Now define functions $F_k : E^n \times E^m \to E^{n+1}$, $k = 0, 1, \ldots, K - 1$, by the relation

$$F_k(x, u) = \begin{pmatrix} h_k(x, u) \\ f_k(x, u) \end{pmatrix}, \quad k = 0, 1, \ldots, K - 1 . \tag{3.7}$$

Further consider in E^{n+1} sets

$$V_k(x) = F_k(x, U_k(x)) = \{v \in E^{n+1} \mid v = F_k(x, u),\ u \in U_k(x)\} , \tag{3.8}$$
$$k = 0, 1, \ldots, K - 1 ,$$

and a vector $e_0 = (-1, 0, 0, \ldots, 0)$, $e_0 \in E^{n+1}$.

Assumption 2. For $k = 0, 1, \ldots, K - 1$ and every $x \in E^n$ the sets $V_k(x)$ are e_0-directionally convex.

More detailly, Assumption 2 says that for every $v' = F_k(x, u')$, $v'' = F_k(x, u'')$ in $V_k(x)$, i.e. $u', u'' \in U_k(x)$, and every α, $0 \leqq \alpha \leqq 1$, there exists a $v(\alpha)$ in $V_k(x)$, i.e. $u(\alpha) \in U_k(x)$, $v(\alpha) = F_k(x, u(\alpha))$, such that

$$\begin{aligned} h_k(x, u(\alpha)) &\leqq \alpha\, h_k(x, u') + (1 - \alpha)\, h_k(x, u'') , \\ f_k(x, u(\alpha)) &= \alpha\, f_k(x, u') + (1 - \alpha)\, f_k(x, u'') , \end{aligned} \qquad k = 0, 1, \ldots, K - 1 . \tag{3.9}$$

Up to now the sets A_k, $k = 0, 1, \ldots, K$ in (3.4) were not further specified. For the later construction we shall need the following property of these sets.

Assumption 3. For $k = 0, 1, \ldots, K$ there exists a conical approximation of the first kind $C_1(\hat{x}_k, A_k)$ to the set A_k at every point $\hat{x}_k \in A_k$.

This assumption is clearly satisfied (see Section 2) if, either A_k, $k = 0, 1, \ldots, K$ are convex (the support cone is then a conical approximation of the first kind) or if the conical approximations of inequality constraints in (3.6) are given by internal cones.

Finally, the last assumption arises in the connection with a state-dependent control region. As we know, $U_k(x)$, $k = 0, 1, \ldots, K - 1$, are multivalued mappings on E^n, and we shall need a concept of the so called selection of a multivalued mapping.

Definition 9. Let $U : E^n \to \mathscr{P}(E^m)$ be a multivalued mapping. A function $\omega : E^n \to E^m$ such that $\omega(x) \in U(x)$ for all $x \in E^n$ is said to be a selection of the multivalued mapping U.

Assumption 4. For $k = 0, 1, \ldots, K - 1$ and every $\bar{x} \in E^n$, $\bar{u} \in E^m$ such that $\bar{u} \in U_k(\bar{x})$ there exists a selection $\omega^k_{\bar{x},\bar{u}} : E^n \to E^m$ such that

(a) there exists a neighbourhood $O_{\bar{x}}$ of the point $\bar{x}$ on which $\omega^k_{\bar{x},\bar{u}}$ is continuously differentiable function;

(b) $\omega^k_{\bar{x},\bar{u}}(\bar{x}) = \bar{u}$;

(c) $\omega^k_{\bar{x},\bar{u}}(x) \in U_k(x)$ for all $x \in O_{\bar{x}}$.

From the evident reasons this assumption is also denoted as a "locally smooth selection" one and is, in fact, a discrete analogy of a "program" introduced by Hestenes [7, p. 305] for continuous time control systems with explicitely given admissible control region of the type (3.5). It has shown that this concept is also useful to deal with discrete systems and, moreover, not only for the constraining sets given by (3.5). We also preferred the denotation as a "selection" which is more appropriate in a general case and coincides with a terminology used in the theory of multivalued mappings. Let us note that the so called "local section" assumption used by Boltjanskij in [5; 20] is evidently the same as a "program" assumption of Hestenes introduced earlier.

Assumption 4 gives an implicite description of feasible types of an admissible control region. We are, therefore, interested in which cases this assumption will be fulfilled. Two most important cases are given by Propositions 6 and 7.

Proposition 6. Let $U_k(x) = U_k \subset E^m$, $k = 0, 1, \ldots, K - 1$, i.e. a constant region of admissible controls. Then Assumption 4 is satisfied.

Proof. It is trivial to see that for each $\bar{x} \in E^n$, $\bar{u} \in U_k$, $k = 0, 1, \ldots, K - 1$, a function $\omega^k_{\bar{x},\bar{u}}(x) = \bar{u}$ is the desired selection, constant in the whole E^n, i.e. a globally smooth selection. □

Proposition 7. Let $U_k(x)$, $k = 0, 1, \ldots, K - 1$, be given by (3.5) and Q_k and q_k, $k = 0, 1, \ldots, K - 1$ continuously differentiable on $E^n \times E^m$. Suppose that for $k = 0, 1, \ldots, K - 1$ and every $x \in E^n$ $u \in U_k(x)$ the vectors

$$\frac{\partial}{\partial u} Q^i_k(x, u),\ i = 1, \ldots, \gamma_k\,, \qquad \frac{\partial}{\partial u} q^j_k(x, u),\ j \in I[q_k(x, u)]$$

are linearly independent. Then the Assumption 4 is satisfied.

Proof. We can prove this proposition either applying the result of Hestenes [7] or we can prove it directly using the implicite function theorem. Let us choose the first possibility which is a shorter one. Fix $k \in \{0, 1, \ldots, K - 1\}$ and define a new variable $v \in E^{\varkappa_k}$. Consider the following set of equations

$$\begin{aligned} &Q^i_k(x, u) = 0\,, && i = 1, \ldots, \gamma_k\,, \\ &q^j_k(x, u) + (v^j)^2 = 0\,, && j = 1, \ldots, \varkappa_k\,. \end{aligned} \tag{3.10}$$

It is easy to see that for each $k = 0, 1, \ldots, K - 1$ a point $(x, u) \in E^n \times E^m$ satisfies the constraints (3.5), iff we can find v such that (x, u, v) satisfy equation (3.10). Denote by $\mathscr{R}_k \subset E^n \times E^m \times E^{\varkappa_k}$ a set of the all points (x, u, v) satisfying (3.10). Then there exist functions – see [7, p. 206]

$$\alpha_k : E^n \times E^m \times E^{\varkappa_k} \to E^m\,,$$
$$\beta_k : E^n \times E^m \times E^{\varkappa_k} \to E^m\,,$$

continuously differentiable in some open set $\mathscr{R}^*_k$ which contains $\mathscr{R}_k$, such that if $(x, u, v) \subset \mathscr{R}^*_k$, we have

$$\begin{aligned} &Q^i_k(x, \alpha_k(x, u, v)) = 0\,, && i = 1, \ldots, \gamma_k\,, \\ &q^j_k(x, \alpha_k(x, u, v)) + (\beta_k(x, u, v))^2 = 0\,, && j = 1, \ldots, \varkappa_k\,. \end{aligned}$$

 Further, if $(x, u, v) \subset \mathscr{R}_k$, then

$$\alpha_k(x, u, v) = u\,, \quad \beta_k(x, u, v) = v\,.$$

Now consider $(\bar{x}, \bar{u})$ such that $\bar{u} \in U_k(\bar{x})$ and define

$$\bar{v}_j = [-q_k^j(\bar{x}, \bar{u})]^{1/2}\,, \quad j = 1, \ldots, \varkappa_k\,,$$

i.e. $(\bar{x}, \bar{u}, \bar{v}) \subset \mathscr{R}_k$. Clearly we can choose a neighbourhood $O_{\bar{x}}$ of $\bar{x}$ in E^n such that $x \in O_{\bar{x}}$ implies $(x, \bar{u}, \bar{v}) \in \mathscr{R}_k^*$. Then it is easy to verify that the function

$$\tilde{\alpha}_k(x) = \begin{cases} \alpha_k(x, \bar{u}, \bar{v}) & \text{for } x \in O_{\bar{x}}\,, \\ \text{any } u \in U_k(x) & \text{otherwise,} \end{cases}$$

defines a locally smooth selection with respect to $(\bar{x}, \bar{u})$, i.e. $\omega^k_{\bar{x},\bar{u}}(x) = \tilde{\alpha}_k(x)$, and this is clearly possible for the all $k = 0, 1, \ldots, K - 1$. □

In [5] the additional assumption of continuity and compactness of admissible control regions $U_k(x)$, $k = 0, 1, \ldots, K - 1$ was used. For the definition of the continuous multivalued mapping see e.g. [5; 21; 22]. This assumption was in [5] the principal one to construct a conical approximation to the discrete optimal control problem. In our construction described in the next section is this assumption not necessary. Also the mentioned compactness assumption in [5] seems not to be appropriate when dealing with necessary optimality conditions. However, the compactness assumption plays a crutial role if existence conditions for a discrete optimal control problem should be obtained – see [13].

4. MAXIMUM PRINCIPLE FOR GENERAL CONTROL PROBLEMS

In this section we obtain necessary optimality conditions for a general discrete optimal control problem (3.1)–(3.4). First, let us perform some preliminary steps, which will considerably simplify the final proof. In our construction we follow, to a great extent, the technique of [3], which we update and modify to deal with discrete optimal control problems with state-dependent control region. In this sense this section can be considered as a generalization of the corresponding parts of [3].

Problem transcription. The aim is to bring a discrete optimal control problem to the form of a mathematical programming one, as given in Section 2. For this purpose let us introduce the following substitutions. For $k = 0, 1, \ldots, K - 1$ let $v_k = (a_k, w_k) \in E^{n+1}$, where $a_k \in E^1$, $w_k \in E^n$. Equation (3.1) is then equivalent to

$$x_{k+1} = w_k\,, \quad k = 0, 1, \ldots, K - 1 \tag{4.1}$$

with $w_k \in f_k(x_k, U_k(x_k))$, $k = 0, 1, \ldots, K - 1$. Further define $z = (x_0, x_1, \ldots \ldots, x_K, v_0, v_1, \ldots, v_{K-1})$ to be a point of $[(K + 1) n + K(n + 1)]$-dimensional space $\tilde{E}$. Then we can define functions f and r from Section 2 and the constraining set Ω as follows.

$$f(z) = \sum_{k=0}^{K-1} z_{n(K+1)+k(n+1)+1} = \sum_{k=0}^{K-1} a_k \,, \tag{4.2}$$

$$r(z) = \begin{pmatrix} -x_1 + w_0 \\ -x_2 + w_1 \\ \vdots \\ -x_K + w_{K-1} \end{pmatrix}, \tag{4.3}$$

$$\Omega = \{z \in \tilde{E} \mid x_k \in A_k,\ k = 0, 1, \ldots, K\,,\quad v_k \in V_k(x_k),\ k = 0, 1, \ldots, K - 1\}\,. \tag{4.4}$$

For the definition of $V_k(x)$ see (3.8). It is easy to see that the original discrete optimal control problem (3.1)–(3.4) is equivalent to the mathematical programming problem (4.1)–(4.4) as far as the values of functionals (3.2) and (4.2) are concerned.

In general, we are not able to handle the set Ω given by (4.4) directly, because we do not know how to construct a suitable conical approximation to this set. However, it is possible to use Theorem 2, where the set Ω^* is defined by the following relation.

$$\Omega^* = \{z \in \tilde{E} \mid x_k \in A_k,\ k = 0, 1, \ldots, K,\ v_k \in \operatorname{co} V_k(x_k),\ k = 0, 1, \ldots, K - 1\}\,. \tag{4.5}$$

To the set Ω^* we shall be able to construct a conical approximation $C(\hat{z}, \Omega^*)$ at the optimal point $\hat{z}$.

Now let us show that the sets Ω and Ω^* satisfy the conditions of Theorem 2. Let $z^* \in \Omega^*$, $z^* = (x_0^*, x_1^*. \ldots, x_K^*, v_0^*, v_1^*, \ldots, v_{K-1}^*)$. Since the sets $V_k(x_k^*)$ are e_0-directionally convex, there exist points $\tilde{v}_k = (\tilde{a}_k, \tilde{w}_k) \in V_k(x_k^*)$, $k = 0, 1, \ldots, K - 1$ such that $\tilde{a}_k \leqq a_k^*$ $\tilde{w}_k = w_k^*$, $k = 0, 1, \ldots, K - 1$. Hence, for the point $\tilde{z} = (x_0^*, x_1^*, \ldots \ldots, x_K^*, \tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_{K-1})$, which evidently lies in Ω, we have

$$r(\tilde{z}) = r(z^*)\,, \quad f(\tilde{z}) \leqq f(z^*)\,.$$

Thus the set Ω^* given by (4.5) satisfies the hypothesis of Theorem 2 with respect to the set Ω given by (4.4).

Conical approximation to the set Ω^*. As before, we denote by $\hat{z} = (\hat{x}_0, \hat{x}_1, \ldots \ldots, \hat{x}_K, \hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{K-1})$ an optimal solution of the transcribed problem (4.1)–(4.4) corresponding to $(\hat{x}, \hat{u})$. Clearly then $\hat{z} \in \Omega^*$. For each $k = 0, 1, \ldots, K - 1$ choose any locally smooth selection $\hat{\omega}_k(x)$ corresponding to $\hat{x}_k$ and $\hat{u}_k$, $\hat{u}_k \in U_k(\hat{x}_k)$. The existence of such selection is guaranteed by Assumption 4. Let $\tilde{F}_k(x) = F_k(x, \hat{\omega}_k(x))$, $k = 0, 1, \ldots, K - 1$, where F_k is given by (3.7). Since $\hat{\omega}_k(\hat{x}_k) = \hat{u}_k$, we may write

$$\frac{\partial \tilde{F}_k(\hat{x}_k)}{\partial x}\, \delta x_k = \left(\frac{\partial}{\partial x} F_k(\hat{x}_k, \hat{u}_k) + \frac{\partial}{\partial u} F_k(\hat{x}_k, \hat{u}_k)\, \frac{\partial \hat{\omega}_k(\hat{x}_k)}{\partial x}\right) \delta x_k\,, \quad k = 0, 1, \ldots, K - 1.$$

 Now consider in $\tilde{E}$ a set

$$(4.6) \qquad C(\hat{z}, \Omega^*) = \{\delta z = (\delta x_0, \delta x_1, \ldots, \delta x_K, \delta v_0, \delta v_1, \ldots, \delta v_{K-1}) \mid \delta x_k \in C_1(\hat{x}_k, A_k), \quad k = 0, 1, \ldots, K,$$
$$\left(\delta v_k - \frac{\partial \tilde{F}_k(\hat{x}_k)}{\partial x} \delta x_k\right) \in SC(\hat{v}_k, \text{co}\, V_k(\hat{x}_k)), \ k = 0, 1, \ldots, K-1\},$$

where $C_1(\hat{x}_k, A_k)$ is the assumed conical approximation of the first kind to A_k at $\hat{x}_k$ and $SC(\hat{v}_k, \text{co}\, V_k(\hat{x}_k))$ is a support cone to $\text{co}\, V_k(\hat{x}_k)$ at $\hat{v}_k$. We claim that the set (4.6) is a conical approximation to the set Ω^* at $\hat{z}$. The stated form of $C(\hat{z}, \Omega^*)$ can be in a certain sense guessed, if we realize that in (4.6) only such δv_k are considered, which eliminate the "first order" changes of $SC(\hat{v}_k, \text{co}\, V_k(\hat{x}_k))$ due to the admissible δx_k, $k = 0, 1, \ldots, K-1$.

It is easy to see that the set $C(\hat{z}, \Omega^*)$ is a convex cone. Now let us assume any finite collection $\delta z_1, \ldots, \delta z_N$ of independent vectors in $C(\hat{z}, \Omega^*)$, i.e. $\delta z_i = (\delta x_{0i}, \delta x_{1i}, \ldots, \delta x_{Ki}, \delta v_{0i}, \delta v_{1i}, \ldots, \delta v_{(K-1)i}$, $i = 1, \ldots, N$. By (4.6) we have that

$$\delta x_{ki} \in C_1(\hat{x}_k, A_k), \quad k = 0, 1, \ldots, K, \quad i = 1, \ldots, N,$$
$$\delta v_{ki} = \frac{\partial \tilde{F}(\hat{x}_k)}{\partial x} \delta x_{ki} + (v_{ki} - \hat{v}_k), \quad k = 0, 1, \ldots, K-1, \quad i = 1, \ldots, N,$$

where $(v_{ki} - \hat{v}_k) \in SC(\hat{v}_k, \text{co}\, V_k(\hat{x}_k))$.

Since by Proposition 5 a support cone is a conical approximation of the first kind, and since the same is true for the cones $C_1(\hat{x}_k, A_k)$, $k = 0, 1, \ldots, K$, we can find an $\varepsilon > 0$ such that for any scalars $\mu_1, \ldots, \mu_N$ satisfying $\mu_i \geqq 0$, $i = 1, \ldots, N$, and $\sum_{i=1}^{N} \mu_i \leqq 1$, we have

$$\left(\hat{x}_k + \varepsilon \sum_{i=1}^{N} \mu_i \delta x_{ki}\right) \in A_k, \qquad k = 0, 1, \ldots, K,$$
$$\left(\hat{v}_k + \varepsilon \sum_{i=1}^{N} \mu_i (v_{ki} - \hat{v}_k)\right) \in \text{co}\, V_k(\hat{x}_k), \quad k = 0, 1, \ldots, K-1.$$

Denote by $\Delta = \text{co}\,\{\hat{z}, \hat{z} + \delta z_1, \ldots, \hat{z} + \delta z_N\}$. Then for any $z \in \Delta$ we may write

$$(4.7) \qquad \delta z = z - \hat{z} = \varepsilon \sum_{i=1}^{N} \mu_i(z)\, \delta z_i, \quad \mu_i(z) \geqq 0, \quad \sum_{i=1}^{N} \mu_i(z) \leqq 1.$$

For any $z \in \Delta$ the vector $\mu(z) = (\mu_1(z), \ldots, \mu_N(z))^{\mathrm{T}}$ is uniquely determined by the expression $\mu(z) = Y \delta z$, where Y is a matrix, whose rows y_i, $i = 1, \ldots, N$ satisfy $\langle y_i, \varepsilon \delta z_j \rangle = \delta_{ij}$, the Kronecker delta, for $i, j = 1, \ldots, N$.

Since $(\hat{v}_k + \varepsilon(v_{ki} - \hat{v})) \in \mathrm{co}\, V_k(\hat{x}_k)$ for $k = 0, 1, \ldots, K - 1$, $i - 1, \ldots, N$, there exists a finite set of controls (not necessarily unique) $u^\alpha_{ki} \in U_k(\hat{x}_k)$, $\alpha = 1, \ldots, p_{ki}$ such that

$$(4.8) \qquad \hat{v}_k + \varepsilon(v_{ki} - \hat{v}_k) = \sum_{\alpha=1}^{p_{ki}} \lambda^\alpha_{ki} F_k(\hat{x}_k, u^\alpha_{ki}), \quad k = 0, 1, \ldots, K - 1, \quad i = 1, \ldots, N,$$

where $\lambda^\alpha_{ki} \geqq 0$ and $\sum_{\alpha=1}^{p_{ki}} \lambda^\alpha_{ki} = 1$. Thus, for any $\delta z = (\delta x_0, \delta x_1, \ldots, \delta x_k, \delta v_0, \delta v_1, \ldots \ldots, \delta v^V_{K-1})$ with $\hat{z} + \delta z = z \in \Delta$, we obtain from (4.6)–(4.8)

$$(4.9) \qquad \delta x_k = \varepsilon \sum_{i=1}^{N} \mu_i(z)\, \delta x_{ki}, \quad k = 0, 1, \ldots, K,$$

and

$$(4.10) \qquad \delta v_k = \varepsilon \sum_{i=1}^{N} \mu_i(z) \left[\frac{\partial \tilde{F}_k(\hat{x}_k)}{\partial x} \delta x_{ki} + (v_{ki} - \hat{v}_k) \right] =$$

$$= \frac{\partial \tilde{F}_k(x_k)}{\partial x} \delta x_k + \sum_{i=1}^{N} \mu_i(z) \left(\sum_{\alpha=1}^{p_{ki}} \lambda^\alpha_{ki} F_k(\hat{x}_k, u^\alpha_{ki}) - \hat{v}_k \right),$$

The last two expressions give the desired representation for vectors in Δ in the terms of vectors in A_k and $V_k(\hat{x}_k)$.

Define the map $\zeta : \Delta \to \Omega^*$ as follows. Consider $\delta z = z - \hat{z}$ with $z \in \Delta$. Then

$$\zeta(z) = (\varphi_0(z), \varphi_1(z), \ldots, \varphi_K(z), \sigma_0(z), \sigma_1(z), \ldots, \sigma_{K-1}(z)).$$

where

$$(4.11) \qquad \varphi_k(z) = x_k, \quad k = 0, 1, \ldots, K$$

and

$$(4.12) \qquad \sigma_k(z) = \tilde{F}_k(x_k) + \sum_{i=1}^{N} \mu_i(z) \left[\sum_{\alpha=1}^{p_{ki}} \lambda^\alpha_{ki} F_k(x_k, \omega^\alpha_{ki}(x_k)) - \tilde{F}_k(x_k) \right],$$
$$k = 0, 1, \ldots, K - 1.$$

Here ω^α_{ki} is a locally smooth selection corresponding $\hat{x}_k$ and u^α_{ki}, i.e. $u^\alpha_{ki} = \omega^\alpha_{ki}(\hat{x}_k)$, k, i, α range as indicated in (4.12). These selections exist, again, due to Assumption 4. Values of $\mu_i(z)$ are uniquely given by (4.7). By our construction, for every $z \in \Delta$ the corresponding $x_k \in A_k$, $k = 0, 1, \ldots, K$, which shows that $\varphi_k(z)$ is also in A_k. Further, since $u^\alpha_{ki} \in U_k(\hat{x}_k)$, from (4.7) we immediately obtain that also $\sigma_k(z) \in \mathrm{co}\, V_k(\hat{x}_k)$. Thus our map ζ is, indeed, from Δ into Ω^*.

It remains to show that this map can be written in the form $\zeta(\hat{z} + \delta z) = \hat{z} + \delta z + o(\delta z)$ as required in Definition 3. The all locally smooth selections needed in (4.12) pertain to the same point $\hat{x}_k$. Hence, for each $k = 0, 1, \ldots, K - 1$ there exists a neighbourhood $O_{\hat{x}_k}$ of the point $\hat{x}_k$ such that $\tilde{F}_k$ is continuously differentiable there. Then, reducing ε, if necessary, we see that $\tilde{F}_k$, $k = 0, 1, \ldots, K - 1$, is continuously differentiable in Δ.

From the previous considerations we know that $\mu(z) = Y(z - \hat{z}) = Y\delta z$ for any $z \in \Delta$. Denote by $Z_k(z)$ a matrix whose i-th column is ($i = 1, \ldots, N$, $k = 0, 1, \ldots, \ldots, K - 1$)

$$\sum_{\alpha=1}^{p_{ki}} \lambda_{ki}^{\alpha} F_k(x_k, \omega_{ki}^{\alpha}(x_k)) - \tilde{F}_k(x_k) .$$

From (4.12) we have that

$$\sigma_k(\hat{z} + \delta z) = \tilde{F}_k(\hat{x}_k + \delta x_k) + Z_k(\hat{z} + \delta z)\, Y\delta z\,, \quad k = 0, 1, \ldots, K - 1 .$$

Further, from previous discussion we may write

$$\tilde{F}_k(\hat{x}_k + \delta x_k) = F_k(\hat{x}_k + \delta x_k, \hat{\omega}_k(\hat{x}_k + \delta x_k)) = \tilde{F}_k(\hat{x}_k) + \frac{\partial \tilde{F}_k(\hat{x}_k)}{\partial x}\delta x_k + o(\delta x_k) =$$

$$= F_x(\hat{x}_k, \hat{u}_k) + \left[\frac{\partial}{\partial x} F_k(\hat{x}_k, \hat{u}_k) + \frac{\partial}{\partial u} F_k(\hat{x}_k, \hat{u}_k)\right]\delta x_k + o(\delta x)\,,$$

$$k = 0, 1, \ldots, K - 1$$

and

$$Z_k(\hat{z} + \delta z)\, Y\delta z = Z_k(\hat{z})\, Y\delta z + o(\delta z)\,,$$

where function $o(\delta z)$ is obviously continuous and satisfies $\|o(\delta z)\|/\|\delta z\| \to 0$ as $\|\delta z\| \to 0$.

Thus we have obtained the expansion of (4.12) about $\hat{z}$, namely,

$$(4.13) \qquad \sigma_k(\hat{z} + \delta z) = \tilde{F}_k(\hat{x}_k) + \frac{\partial \tilde{F}_k(\hat{x}_k)}{\partial x}\delta x_k + \sum_{i=1}^{N} \mu_i(z) \,.$$

$$.\,[\sum_{\alpha=1}^{p_{ki}} \lambda_{ki}^{\alpha} F_k(\hat{x}_k, \omega_{ki}^{\alpha}(\hat{x}_k)) - \tilde{F}_k(\hat{x}_k)] + o(\delta z)\,, \quad k = 0, 1, \ldots, K - 1\,,$$

where $o(\delta z)$ is continuous and $\|o(\delta z)\|/\|\delta z\| \to 0$ as $\|\delta z\| \to 0$. If we compare (4.13) with (4.10) and also take into the account (4.11) we conclude that for any $z = \hat{z} + \delta z \in \Delta$, $\zeta(\hat{z} + \delta z) = \hat{z} + \delta z + o(\delta z)$, where $o(\delta z)$ is continuous and $\|o(\delta z)\|/\|\delta z\| \to 0$ as $\|\delta z\| \to 0$. Hence, the set $C(\hat{z}, \Omega^*)$ given by (4.6) is, indeed, a conical approximation to the set Ω^* at $\hat{z}$.

Remark 1. If U_k, $k = 0, 1, \ldots, K - 1$ do not depend on x, then by Proposition 6 $\hat{\omega}_k(x) = \hat{u}_k$, $\omega^\alpha_{ki}(x) = u^\alpha_{ki}$, $\alpha = 1, \ldots, p_{ki}$, $i = 1, \ldots, N$, $k = 0, 1, \ldots, K - 1$, and we obtain the same result as in [3].

Remark 2. As follows from the just described construction of $C(\hat{z}, \Omega^*)$, it is, in general, not necessary that the sets co $V_k(\hat{x}_k)$, $k = 0, 1, \ldots, K - 1$, have a full dimension in E^{n+1}, i.e. have an interior point in E^{n+1}. However, this fact was needed in the construction described in [5].

Necessary optimality conditions. We have shown that the all assumption of Theorem 2 are satisfied with Ω and Ω^* given by (4.4) and (4.5), respectively, and f and r given by (4.2) and (4.3), respectively. Finally, the set (4.6) defines a conical approximation $C(\hat{z}, \Omega^*)$ to the set Ω^* at $\hat{z}$. From Theorem 2 it follows that there exist a scalar $\mu \leqq 0$ and a row-vector $\chi = (\lambda_1, \ldots, \lambda_K)$ with $\lambda_k \in E^n$, $k = 1, \ldots, K$ such that

(1) if $\mu = 0$, then χ is nonzero;

(2) for all $\delta z \in \bar{C}(\hat{z}, \Omega^*)$ holds the relation

$$\left(\mu \frac{\partial f(\hat{z})}{\partial z} + \chi \frac{\partial r(\hat{z})}{\partial z}\right) \delta z \leqq 0 .$$

If we substitute for f and r from (4.2) and (4.3), respectively, into condition (2), we get that

$$\mu \sum_{k=0}^{K-1} \delta a_k + \sum_{k=0}^{K-1} \lambda_{k+1}(-\delta x_{k+1} + \delta w_k) \leqq 0 \tag{4.14}$$

for all $\delta z \in \bar{C}(\hat{z}, \Omega)$.

This result can be converted to a more familiar form as stated in the next theorem. For this purpose we introduce the Hamiltonian

$$\begin{aligned} H_{k+1}(x, u) &= \mu h_k(x, u) + \lambda_{k+1} f_k(x, u) , \quad k = 0, 1, \ldots, K - 1 , \\ \tilde{H}_{k+1}(x) &= H_{k+1}(x, \hat{\omega}_k(x)) , \quad k = 0, 1, \ldots, K - 1 , \end{aligned} \tag{4.15}$$

with μ and λ_{k+1} introduced above and $\hat{\omega}_k$, $k = 0, 1, \ldots, K - 1$ is again, a locally smooth selection corresponding to $\hat{x}_k$, $\hat{u}_k$. For the sake of the notational simplicity only variables x and u in (4.15) with respect to which the Hamiltonian will be later differentiated, are written explicitely. Let us also note that from (4.15) we formally have

$$x_{k+1} = \frac{\partial}{\partial \lambda_{k+1}} H_{k+1}(x_k, u_k) , \quad k = 0, 1, \ldots, K - 1 .$$

Now we are able to prove a general theorem which will give us a set of necessary optimality conditions for the discrete optimal control problem (3.1)–(3.4). Sometimes is such theorem denoted also as a discrete maximum principle. In fact, necessary optimality conditions for the mathematical programming problem (4.1)–(4.6) will be decomposed with respect to the number of stages.

Theorem 3. Consider discrete optimal control problem (3.1)–(3.4) and suppose that the Assumptions 1–4 are satisfied. If $(\hat{x}, \hat{u})$ is an optimal control process, then there exist a scalar $\mu \leqq 0$, adjoint (costate) row-vectors $\lambda_k \in E^n$, $k = 1, \ldots, K$ and vectors β_k, $k = 0, 1, \ldots, K$, $\beta_k \in DC[C_1(\hat{x}_k, A_k)]$, the dual cone to the convex cone $C_1(\hat{x}_k, A_k)$ in E^n, such that the following conditions (a)–(c) are satisfied:

(a) If $\mu = 0$, then at least one of the row-vectors λ_k, $k = 1, \ldots, K$ is nonzero.

(b) The row-vectors λ_k, $k = 1, \ldots, K$ satisfy the adjoint equation

$$\lambda_k = \frac{\partial}{\partial x} \tilde{H}_{k+1}(\hat{x}_k) - \beta_k\,, \quad k = 0, 1, \ldots, K\,,$$

where we define $\lambda_0 = 0$, $\tilde{H}_{k+1} \equiv 0$.

(c) The maximum condition

$$H_{k+1}(\hat{x}_k, \hat{u}_k) = \max_{u \in U_k(\hat{x}_k)} H_{k+1}(\hat{x}_k, u)\,, \quad k = 0, 1, \ldots, K - 1\,,$$

is satisfied along the optimal process $(\hat{x}, \hat{u})$.

Proof. Condition (a) is, in fact, the condition (2) established earlier. Now let $\delta z = (0, \ldots, 0, \delta x_k, 0, \ldots, 0, \delta v_k, 0, \ldots, 0)$ be in $C(\hat{z}, \Omega^*)$, $k \in \{0, 1, \ldots, K - 1\}$ with $\delta v_k = (\partial \bar{F}_k(\hat{x}_k)/\partial x)\, \delta x_k$. From (4.14) we get

$$\left[\mu \frac{\partial}{\partial x} h_k(\hat{x}_k, \hat{\omega}_k(\hat{x}_k)) + \lambda_{k+1} \frac{\partial}{\partial x} f_k(\hat{x}_k, \hat{\omega}_k(\hat{x}_k)) - \lambda_k\right] \delta x_k \leqq 0\,, \quad k = 0,1, \ldots, K - 1\,,$$

for all $\delta x_k \in \bar{C}_1(\hat{x}_k, A_k)$. Taking into the account (4.15) and Definition 4 we may write

$$\lambda_k = \frac{\partial}{\partial x} \tilde{H}_{k+1}(\hat{x}_k) - \beta_k\,, \quad \beta_k \in DC[C_1(\hat{x}_k, A_k)]\,, \quad k = 0, 1, \ldots, K\,,$$

if we formally define $\lambda_0 = 0$ and $\tilde{H}_{k+1} \equiv 0$. This proves condition (b).

Finally, suppose $\delta z = (0, \ldots, 0, \delta v_k, 0, \ldots, 0)$, $k \in \{0, 1, \ldots, K - 1\}$, is in $C(\hat{z}, \Omega^*)$. From (4.14)

$$\mu\, \delta a_k + \lambda_{k+1}\, \delta w_k \leqq 0$$

for all $\delta v_k \in \overline{SC}(\hat{v}_k, \text{co } V_k(\hat{x}_k))$, $k = 0, 1, \ldots, K - 1$. However, $V_k(\hat{x}_k) \subset \text{co } V_k(\hat{x}_k) \subset \overline{SC}(\hat{v}_k, \text{co } V_k(x_k))$, and, thus, we obtain that the function $\mu a_k + \lambda_{k+1} w_k$ considered on the set $V_k(\hat{x}_k)$ attains its maximum at $\hat{v}_k$, i.e.

$$\mu \hat{a}_k + \lambda_{k+1} \hat{w}_k = \max_{(a,w) \in V_k(\hat{x}_k)} (\mu a + \lambda_{k+1} w), \quad k = 0, 1, \ldots, K - 1,$$

which is nothing else than the condition (c). □

The obtained form of a discrete maximum principle is a general one, but this generality was paid by the less practical importance of these necessary optimality conditions. Therefore, it seems to the author that further generalizations in this direction, i.e. assuming more complicated structure of constraining sets, are interesting only from the theoretical point of view. Such attempts were made in [5] using rather complicated constructions. From these reasons we shall study an explicitely given discrete optimal control problem in the next section, for which fairly deep results can be obtained.

Sometimes it is also important to know if the scalar multiplier $\mu \neq 0$ in Theorem 3. Then we may put $\mu = -1$ without any loss of generality. Only such problems are, in fact, interesting from an engineering or economic point of view. In classical calculus of variations such problems (with $\mu = -1$) are denoted as normal ones, see [7]. In our case we easily obtain the next result.

Corollary 1. Suppose that the state constraints (3.4) except possibly of a given initial point $\hat{x}_0$, are absent. Then the discrete optimal control problem (3.1)–(3.3) is normal, i.e. $\mu = -1$.

Proof. If there are no state constraints, i.e. $A_k = E^n$, $k = 0, 1, \ldots, K$, then $C_1(\hat{x}_k, E^n) = E^n$ and $DC[E^n] = 0$. Hence, $\beta_k = 0$, $k = 0, 1, \ldots, K$. Assuming now $\mu = 0$, we obtain from condition (b) of Theorem 3 that $\lambda_k = 0$, $k = 1, \ldots, K$, which is a contradiction with condition (a) of the same theorem. Now if the initial point $\hat{x}_0$ is given, the same reasoning shows that $\beta_k = 0$, $k = 1, \ldots, K$ and $\beta_0 = \lambda_0 = 0$ by definition, i.e. $\beta_0 \in DC[C_1(\hat{x}_0, \hat{x}_0)] = E^n$. □

Remark 3. Suppose now $U_k(x) = U_k$, $k = 0, 1, \ldots, K - 1$ constant. In this special case we see that it is sufficient to assume h_k and f_k to be continuously differentiable in x for every $u \in U_k$, $k = 0, 1, \ldots, K - 1$, – cf. Assumption 1.

5. MAXIMUM PRINCIPLE FOR EXPLICITE CONTROL PROBLEMS

In this section we shall consider the explicitely given discrete optimal control problem from Section 3, i.e. we shall assume that the constraints are given by (3.5) and (3.6). This concretization will result in more detailed necessary optimality conditions, which are of a practical interest. As mentioned in Section 2, the equality constraints are treated separately in this approach. In the explicite case it means that the state equality constraints $S_k(x) = 0$, $k = 0, 1, \ldots, K - 1$ in (3.6) are added to the equations (4.3). This leads to the following expression for $r(z)$.

$$r(z) = \begin{bmatrix} -x_1 + w_0 \\ -x_2 + w_1 \\ \vdots \\ -x_K + w_{K-1} \\ S_0(x_0) \\ S_1(x_1) \\ \vdots \\ S_K(x_K) \end{bmatrix}. \tag{5.1}$$

Denote $A'_k = \{x \in E^n \mid s_k(x) \leqq 0\}$, $k = 0, 1, \ldots, K$. From Proposition 1 we know that as far as function s_k is continuously differentiable in E^n, the corresponding internal cone $IC(\hat{x}_k, A'_k)$ will be a conical approximation of the first kind to the set A'_k at $\hat{x}_k$, $k = 0, 1, \ldots, K$. Thus Assumption 3 is a priori satisfied.

Through the construction presented in the last section the Assumption 4 was necessary. Taking into the account Proposition 7 we see that this assumption will be also satisfied, provided that the hypothesis of Proposition 7 is fulfilled.

If we now look through the construction of $C(\hat{z}, \Omega^*)$ in Section 4 we can conclude the following. There exist a scalar $\mu \leqq 0$ and a row-vector $\chi = (\lambda_1, \ldots, \lambda_K, \psi_0, \psi_1, \ldots \ldots, \psi_K)$ with $\lambda_k \in E^n$, $k = 1, \ldots, K$, $\psi_k \in E^{\varrho_k}$, $k = 0, 1, \ldots, K$ such that

(1) if $\mu = 0$, then χ is nonzero;

(2) the relation

$$\left(\mu \frac{\partial f(\hat{z})}{\partial z} + \chi \frac{\partial r(\hat{z})}{\partial z}\right) \delta z \leqq 0 \tag{5.2}$$

holds for all $\delta z \in \bar{C}(\hat{z}, \Omega^*)$. Here $C(\hat{z}, \Omega^*)$ is again given by (4.6) with the only change that $C_1(\hat{x}_k, A_k) \equiv IC(\hat{x}_k, A'_k)$, $k = 0, 1, \ldots, K$, as discussed above. Substituting for f and r from (4.2) and (5.1), respectively, into (5.2) we get

$$\mu \sum_{k=0}^{K-1} \delta a_k + \sum_{k=0}^{K-1} \lambda_{k+1}(-\delta x_{k+1} + \delta w_k) + \sum_{k=0}^{K} \psi_k \frac{\partial S_k(\hat{x}_k)}{\partial x} \delta x_k \leqq 0 \tag{5.3}$$

for all $\delta z \in \bar{C}(\hat{z}, \Omega^*)$. Now we can state an analogy of Theorem 3 to the explicitely given discrete optimal control problem. Again, the Hamiltonian notation (4.15) will be used.

Theorem 4. Consider the explicite discrete optimal control problem (3.1), (3.2), (3.5) and (3.6), and suppose that the Assumptions 1 and 2 are satisfied. Further assume that for $k = 0, 1, \ldots, K - 1$ and every $x \in E^n$, $u \in U_k(x)$ the vectors

$$(5.4) \qquad \frac{\partial}{\partial u} Q_k^i(x, u),\ i = 1, \ldots, \gamma_k\,, \quad \frac{\partial}{\partial u} q_k^j(x, u),\ j \in I[q_k(x, u)]\,,$$

where $I[q_k(x, u)]$ is the active index set of the inequality constraints in (3.6), are linearly independent.

If $(\hat{x}, \hat{u})$ is an optimal process, then there exist a scalar $\mu \leqq 0$, adjoint row-vectors $\lambda_k \in E^n$, $k = 1, \ldots, K$, row-vector multipliers $\psi_k \in E^{\varrho_k}$, $v_k \in E^{\pi_k}$, $k = 0, 1, \ldots, K$ and matrices W_k, $k = 0, 1, \ldots, K - 1$ of dimension $(m \times n)$ such that the following conditions (a)–(f) are satisfied:

(a) If $\mu = 0$, then at least one of the row-vectors λ_k, $k = 1, \ldots, K$, ψ_k, v_k, $k = 0, 1, \ldots, K$ is nonzero.

(b) The row-vectors λ_k, $k = 1, \ldots, K$ satisfy the adjoint equation

$$\lambda_k = \frac{\partial}{\partial x} H_{k+1}(\hat{x}_k, \hat{u}_k) + \left(\frac{\partial}{\partial u} H_{k+1}(\hat{x}_k, \hat{u}_k)\right) W_k + \\ + \psi_k \frac{\partial S_k(\hat{x}_k)}{\partial x} + v_k \frac{\partial s_k(\hat{x}_k)}{\partial x}\,, \quad k = 0, 1, \ldots, K\,.$$

where we define $\lambda_0 = 0$ and $H_{K+1} \equiv 0$.

(c) The maximum condition

$$H_{k+1}(\hat{x}_k, \hat{u}_k) = \max_{u \in U_k(\hat{x}_k)} H_{k+1}(\hat{x}_k, u)\,, \quad k = 0, 1, \ldots, K - 1$$

is satisfied along the optimal process $(\hat{x}, \hat{u})$.

(d) $v_k \leqq 0\,, \quad v_k s_k(\hat{x}_k) = 0\,, \quad k = 0, 1, \ldots, K\,.$

(e) $\dfrac{\partial}{\partial x} Q_k(\hat{x}_k, \hat{u}_k) + \left(\dfrac{\partial}{\partial u} Q_k(\hat{x}_k, \hat{u}_k)\right) W_k = 0\,, \quad k = 0, 1, \ldots, K - 1\,.$

(f) $\dfrac{\partial}{\partial x} q_k^j(\hat{x}_k, \hat{u}_k) + \left(\dfrac{\partial}{\partial u} q_k^j(\hat{x}_k, \hat{u}_k)\right) W_k = 0\,, \quad j \in I[q_k(\hat{x}_k, \hat{u}_k)]\,,$
$k = 0, 1, \ldots, K - 1.$

Proof. Evidently the all assumptions of Theorem 3 are satisfied. Condition (b) of Theorem 3 yields

$$\lambda_k = \frac{\partial}{\partial x} H_{k+1}(\hat{x}_k, \hat{u}_k) + \left(\frac{\partial}{\partial u} H_{k+1}(\hat{x}_k, \hat{u}_k)\right) W_k - \beta_k\,, \quad k = 0, 1, \ldots, K\,,$$

where $\lambda_0 = 0$, $H_{K+1} \equiv 0$, $\beta_k \in DC[IC(\hat{x}_k, A'_k)]$ and we denoted by W_k a matrix with elements $w_k^{ij} = \partial\hat{\omega}_k^i(\hat{x}_k)/\partial x^j$, $i = 1, \ldots, m, j = 1, \ldots, n$. The existence of $\hat{\omega}_k$, $k = 0, 1, \ldots, K - 1$ follows from Proposition 7. Suppose that $IC(\hat{x}_k, A'_k)$, $k = 0, 1, \ldots, K$ is not only the origin. From Proposition 1 we obtain that $\beta_k = -\nu_k(\partial s_k(\hat{x}_k)/\partial x)$ with $\nu_k \leqq 0$, $\nu_k^i = 0$, $i \in I^c[s_k(\hat{x}_k)]$. Then also a stronger version of condition (a) holds, namely, we can delete ν_k, $k = 0, 1, \ldots, K$ in the statement of condition (a), which is thus a direct analogy of the corresponding condition in Theorem 3. If we admit also the case $IC(\hat{x}_k, A'_k) = \{0\}$, we see from Propositions 2 that the all conditions of Theorem 4 are trivially fulfilled. This proves conditions (a), (b) and (d).

By definition, $\hat{\omega}_k(x) \in U_k(x)$ for $x \in E^n$, $k = 0, 1, \ldots, K - 1$. So we can write that $Q_k(x, \hat{\omega}_k(x)) \equiv 0$, $k = 0, 1, \ldots, K - 1$ for $x \in E^n$. Differentiating this expression with respect to x at the point $\hat{x}_k$ gives condition (e). Similarly, as $\hat{\omega}_k(x) \in U_k(x)$ we have that $q_k(x, \hat{\omega}_k(x)) \leqq 0$ for $x \in E^n$, $k = 0, 1, \ldots, K - 1$. Moreover,

$$q_k^j(\hat{x}_k, \hat{\omega}_k(\hat{x}_k)) = q_k^j(\hat{x}_k, \hat{u}_n) = 0\,, \quad j \in I[q_k(\hat{x}_k, \hat{u}_k)]\,, \quad k = 0, 1, \ldots, K - 1\,,$$

which shows that functions $q_k^j(x, \hat{\omega}_k(x))$ attain maximum at the point $\hat{x}_k$. Hence, condition (f) is then only a necessary condition for this fact. □

Similar conditions were obtained also by Boltjanskij in [5], but he assumed that the discrete optimal control problem in question is convex, which is more restrictive. Further, conditions stated in Theorem 4 cannot be easily applied, because a number of introduced unknown coefficients, in general, does not coincide with a number of equations, which can be used to determine them – see conditions (e) and (f) of Theorem 4. Therefore, we are interested, whether it would be possible to bring these two conditions to some more convenient form.

For this purpose let us state the maximum condition (c) in a more detailed way. Namely, condition (c) together with (3.6) and (5.4) give – see Proposition 3.

$$\frac{\partial}{\partial u} H_{k+1}(\hat{x}_k, \hat{u}_k) + \zeta_k \frac{\partial}{\partial u} Q_k(\hat{x}_k, \hat{u}_k) + \xi_k \frac{\partial}{\partial u} q_k(\hat{x}_k, \hat{u}_k) = 0\,, \tag{5.5}$$

where row-vectors $\zeta_k \in E^{\gamma_k}$, $\xi_k \in E^{\varkappa_k}$, and

$$\xi_k \leqq 0\,, \quad \xi_k\, q_k(\hat{x}_k, \hat{u}_k) = 0\,, \quad k = 0, 1, \ldots, K - 1\,. \tag{5.6}$$

From (5.5) and conditions (e) and (f) of Theorem 4 we obtain that

$$\left(\frac{\partial}{\partial u} H_{k+1}(\hat{x}_k, \hat{u}_k)\right) W_k = \zeta_k \frac{\partial}{\partial x} Q_k(\hat{x}_k, \hat{u}_k) + \xi_k \frac{\partial}{\partial x} q_k(\hat{x}_k, \hat{u}_k)\,, \quad k = 0, 1, \ldots, K - 1\,. \tag{5.7}$$

Using (5.5)–(5.7), the following alternative formulation of Theorem 4 can be stated, where it is also guaranteed that $IC(\hat{x}_k, A'_k) \neq \{0\}$, $k = 0, 1, \ldots, K$, and that this theorem cannot be trivially satisfied due to state constaints (3.6).

Theorem 5. Consider again the explicitely given discrete optimal control problem as in Theorem 4 and let the all assumptions of Theorem 4 be satisfied. Additionally, for $k = 0, 1, \ldots, K$ let each of the following two systems of vectors

$$\frac{\partial}{\partial x} S_k^i(\hat{x}_k)\,, \quad i = 1, \ldots, \varrho_k\,; \quad \frac{\partial}{\partial x} s_k^j(\hat{x}_k)\,, \quad j \in I[s_k(\hat{x}_k)]$$

be linearly independent. If $(\hat{x}_k, \hat{u}_k)$ is an optimal process, then there exist a scalar $\mu \leqq 0$, adjoint row-vectors $\lambda_k \in E^n$, $k = 1, \ldots, K$, and row-vector multipliers $\psi_k \in \in E^{\varrho_k}$, $v_k \in E^{\pi_k}$, $k = 0, 1, \ldots, K$, $\zeta_k \in E^{\gamma_k}$, $\xi_k \in E^{\varkappa_k}$, $k = 0, 1, \ldots. K - 1$, such that the following conditions (a)–(e) are satisfied:

(a) If $\mu = 0$, then at least one the row-vectors λ_k, $k = 1, \ldots, K$, ψ_k, $k = 0, 1, \ldots, K$, is nonzero.

(b) The row-vectors λ_k, $k = 1, \ldots, K$, satisfy the adjoint equation

$$\lambda_k = \frac{\partial}{\partial x} H_{k+1}(\hat{x}_k, \hat{u}_k) + \zeta_k \frac{\partial}{\partial x} Q_k(\hat{x}_k, \hat{u}_k) + \xi_k \frac{\partial}{\partial x} q_k(\hat{x}_k, \hat{u}_k) +$$

$$+ \psi_k \frac{\partial}{\partial x} S_k(\hat{x}_k) + v_k \frac{\partial}{\partial x} s_k(\hat{x}_k)\,, \quad k = 0, 1, \ldots, K - 1\,,$$

where $\lambda_0 = 0$ and

$$\lambda_K = \psi_K \frac{\partial}{\partial x} S_K(\hat{x}_K) + v_K \frac{\partial}{\partial x} s_K(\hat{x}_K).$$

(c) $$\frac{\partial}{\partial u} H_{k+1}(\hat{x}_k, \hat{u}_k) + \zeta_k \frac{\partial}{\partial u} Q_k(\hat{x}_k, \hat{u}_k) + \xi_k \frac{\partial}{\partial u} q_k(\hat{x}_k, \hat{u}_k) = 0\,, \quad k = 0, 1, \ldots, K - 1\,.$$

(d) $v_k \leqq 0\,, \quad v_k\, s_k(\hat{x}_k) = 0\,, \quad k = 0, 1, \ldots, K\,.$

(e) $\xi_k \leqq 0\,, \quad \xi_k\, q_k(\hat{x}_k, \hat{u}_k) = 0\,, \quad k = 0, 1, \ldots, K - 1\,.$

This form of a discrete maximum principle enables us to compute an optimal control process using the conditions (a)–(e). From the same reasons like in Section 4 we cannot simply put $\mu = -1$, as it is desirable from practical and computational aspects. However, if the state constraints (3.5) are not present, the Corollary 1 applies also to this explicite case and we can then assume $\mu = -1$, i.e. the normality of a discrete optimal control problem.

Such version of the discrete maximum principle without state constraints was also used by the author to derive necessary optimality conditions for various solutions of the so called multistage games – see [23; 24].

6. EXAMPLE

The practical importance of Theorem 5, together with Corollary 1, is demonstrated by the following well-known example of Bellman [25] which can be converted to a discrete optimal control problem with state-dependent control region.

Find numbers $\alpha_1, \ldots, \alpha_K$ such that

(1) $\sum_{k=1}^{K} \alpha_k \leqq a\,, \quad a > 0\,;$

(2) $\alpha_k \geqq 0\,, \quad k = 1, \ldots, K\,;$

(3) $\prod_{k=1}^{K} \alpha_k = \max.$

It is not very hard to see that this maximization problem is equivalent to the following discrete optimal control problem. Minimize the cost functional

$$J = -x_{K-1}^{(2)} u_{K-1}$$

subject to

(a) $x_{k+1}^{(1)} = x_k^{(1)} + u_k\,, \quad k = 0, 1, \ldots, K-1\,,$

$\quad x_{k+1}^{(2)} = x_k^{(2)} u_k\,,$

(b) $x_0^{(1)} = 0\,, \quad x_0^{(2)} = 1\,,$

(c) $0 \leqq u_k \leqq a - x_k^{(1)} \Leftrightarrow u_k(u_k + x_k^{(1)} - a) \leqq 0\,, \quad k = 0, 1, \ldots, K-1\,.$

In this very simple case the all assumption of Theorem 5 are evidently satisfied, provided that $x_k^{(1)} \neq a$, $k = 0, 1, \ldots, K-1$. From Corollary 1 we have $\mu = -1$. Then the Hamiltonian (4.15) is written as

$$H_{k+1} = \lambda_{k+1}^{(1)}(\hat{x}_k^{(1)} + \hat{u}_k) + \lambda_{k+1}^{(2)} \hat{x}_k^{(2)} \hat{u}_k\,, \quad k = 0, 1, \ldots, K-2\,,$$

$$H_K \;\; = \hat{x}_{K-1}^{(2)} \hat{u}_{K-1} + \lambda_K^{(1)}(\hat{x}_{K-1}^{(1)} + \hat{u}_{K-1}) + \lambda^{(2)} \hat{x}_{K-1}^{(2)} \hat{u}_{K-1}\,.$$

(b') $\lambda_k^{(1)} = \lambda_{k+1}^{(1)} + \xi_k \hat{u}_k\,, \quad k = 1, \ldots, K - 1\,,$

$\lambda_K^{(1)} = 0\,,$

$\lambda_k^{(2)} = \lambda_{k+1}^{(2)} \hat{u}_k\,, \quad k = 1, \ldots, K - 2\,,$

$\lambda_{K-1}^{(2)} = (\lambda_K^{(2)} + 1)\, \hat{u}_{K-1}\,,$

$\lambda_K^{(2)} = 0\,,$

(c') $\lambda_{k+1}^{(1)} + \lambda_{k+1}^{(2)} \hat{x}_k^{(2)} + \xi_k(2\hat{u}_k + \hat{x}_k^{(1)} - a) = 0\,, \quad k = 0, 1, \ldots, K - 2\,,$

$\lambda_K^{(1)} + (\lambda_K^{(2)} + 1)\, \hat{x}_{K-1}^{(2)} + \xi_{K-1}(2\hat{u}_{K-1} + \hat{x}_{K-1}^{(1)} - a) = 0\,,$

(f') $\xi_k \leqq 0\,, \quad \xi_k\, \hat{u}_k(\hat{u}_k + \hat{x}_k^{(1)} - a) = 0\,, \quad k = 0, 1, \ldots, K - 1\,.$

It is clear that $J \leqq 0$ for all admissible processes (x, u). The case $\hat{u}_k = 0$ for some, at least one, $k = 0, 1, \ldots, K - 1$ is not interesting. because then necessarily $J = 0$ and such processes cannot be optimal (minimizing) ones. Thus we, without any loss of generality, further assume $\hat{u}_k > 0$, $k = 0, 1, \ldots, K - 1$. If now $\hat{u}_{k_0} = a - \hat{x}_{k_0}^{(1)}$ for certain $k_0 = 0, 1, \ldots, K - 2$, i.e. $x_{k_0+1}^{(1)} = a$, we see that $\hat{u}_k = 0$, $k > k_0$ and the above discussion applies. So we also suppose $\hat{u}_k < a - \hat{x}_k^{(1)}$, $k = 0, 1, \ldots, K - 2$, and this means that constraints (c) are not active for $k = 0, 1, \ldots, K - 2$, i.e. $\xi_k = 0$, $k = 0, 1, \ldots, K - 2$.

Let also $\xi_{K-1} = 0$. Then (b') and (c') determine $\hat{x}_{K-1}^{(2)} = 0$. Then $J = 0$, and this is a contradiction with $\hat{u}_k > 0$, $k = 0, 1, \ldots, K - 1$. Hence, $\xi_{K-1} < 0$ and constraint (c) is active at the stage $K - 1$, which, in turn, means that

$$\hat{x}_{K-1}^{(1)} = a - \hat{u}_{K-1}\,, \quad \text{i.e.} \quad \hat{x}_K^{(1)} = a\,. \tag{6.1}$$

From (b') we obtain

$$\lambda_k^{(1)} = \lambda_{K-1}^{(1)} = \xi_{K-1} \hat{u}_{K-1}\,, \quad k = 1, \ldots, K - 1\,, \quad \lambda_K^{(1)} = 0\,,$$

$$\lambda_k^{(2)} = \prod_{i=k}^{K-1} \hat{u}_i\,, \qquad k = 1, \ldots, K - 1\,, \quad \lambda_K^{(2)} = 0\,.$$

From (c') it follows that

$$\xi_{K-1} \hat{u}_{K-1} + \hat{x}_{K-1}^{(2)} \prod_{i=k+1}^{K-1} \hat{u}_i = 0\,, \quad k = 0, 1, \ldots, K - 2\,, \tag{6.2}$$

$$\hat{x}_{K-1}^{(2)} + \xi_{K-1} \hat{u}_{K-1} = 0\,. \tag{6.3}$$

Multiplying (6.2) by $\hat{u}_k > 0$, $k = 0, 1, \ldots, K - 2$, (6.3) by $\hat{u}_{K-1} > 0$ and realizing system equations (a), we have that

$$\xi_{K-1} \hat{u}_{K-1} \hat{u}_k = \xi_{K-1} (\hat{u}_{K-1})^2\,,$$

and because of $\xi_{K-1}\hat{u}_{K-1} \neq 0$, we get

$$\hat{u}_k = \hat{u}_{K-1}, \quad k = 0, 1, \ldots, K-2. \tag{6.4}$$

If we combine (6.1) and (6.4), we finally obtain

$$\hat{x}_K^{(1)} = \sum_{k=0}^{K-1} \hat{u}_k = a,$$

which implies

$$\hat{u}_k = \frac{a}{K}, \quad k = 0, 1, \ldots, K-1. \tag{6.5}$$

Thus, using derived necessary optimality conditions we have found a unique „candidate" of optimality, which is given by (6.5). From the author's paper [13] we know that an optimal solution in this example exists. This proves that the process $(\hat{x}, \hat{u})$ resulting from (6.5) is necessarily the unique optimal one.

The just solved illustrative example was chosen for its computational simplicity. However, let us at least recall some areas, where discrete optimal control problems with state-dependent control region can arise. Primarily such problems are met when optimizing the flight of airplanes or missiles, because various dynamical constraints depend on the position, velocity, etc. of the system, i.e. on the state variables – e.g. see the book of Leitmann [26].

More transparent application of the developed theory is the so called multiproduct inventory problem, see Drew [27]. In this case the limits on total production or total inventory can be directly expressed by (3.5). For other possible approach to discrete optimal control problems with various constraints the reader should consult papers of Ritch [28; 29], where also some computational aspects are given and which are illustrated by an interesting example from the sugar industry [29].

7. CONCLUSIONS

Necessary optimality conditions a discrete maximum principle for general discrete optimal control problems with state-dependent control region were obtained using mathematical programming approach. In fact, the construction presented in [3] was generalized to include also problems with state-dependent control region. This generalization was possible due to the concept of a locally smooth selection.

Similar results were recently obtained also by Boltjanskij in [5]. However, the approach described here is a simpler one and only weaker assumptions than those in [5] were imposed on a discrete optimal control problem. This primarily concerns the convexity of these problems, and the continuity and the full dimensionality of an admissible control region.

Moreover, in the so called explicite case we were able to bring general necessary optimality conditions to a form attractive from computational point of view. This fact was illustrated by a simple example with state-dependent control region, which was solved in the detail. Also an application of sufficient existence conditions was discussed in this connection.

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