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The Discrete Riccati Equation of Optimal Control

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In a previous paper the author has proved a fundamental theorem on Riccati equation solutions in the continuous case. This paper extends the results for the discrete case and summarizes the theory of the discrete Riccati algebraic equation.

1. INTRODUCTION

Consider the discrete dynamical system described by

$$x_{k+1} = Ax_k + Gu_k\,, \quad x_0 \text{ given} \tag{1}$$
$$y_k = Hx_k$$

where $x_k \in \mathrm{R}_n$, $u_k \in \mathrm{R}_r$ and $y_k \in \mathrm{R}_p$ are the state, the input, and the output of the system respectively [10]. Here A, G and H are real matrices of appropriate dimensions and $\det A \neq 0$.

Further consider the discrete linear regulator problem. Given system (1), find a control u_k such that the cost

$$\mathscr{J} = \tfrac{1}{2}\sum_{k=0}^{\infty} y_k' Q y_k + u_k' R u_k \tag{2}$$

is minimized for any $x_0 \in \mathrm{R}_n$.

Both Q and R are assumed to be symmetric positive definite matrices of dimensions $p \times p$ and $r \times r$ respectively.

Sometimes the cost functional involves the state of the system rather than its output. Then a nonnegative definite Q is the proper choice. In our case, however, we can consider Q positive definite in complete generality. On the other hand, the assumption that R be positive definite is certainly restrictive for the *discrete* linear regulator problem. However, the cost functional with nonnegative definite R can be converted into that with positive definite R using the inverse system representation. This transformation is shown for single-input single-output systems in [5].

We invoke the discrete minimum principle [8] to solve the problem. We thus form the Hamiltonian given by

$$h_k = \tfrac{1}{2}x_k'H'QHx_k + \tfrac{1}{2}u_k'Ru_k + p_{k+1}x_{k+1},$$

where p_k, the costate, is coupled with x_k via the equations

$$x_{k+1} = \frac{\partial h_k}{\partial p_{k+1}},$$

$$p_k = \frac{\partial h_k}{\partial x_k}.$$

By minimizing the Hamiltonian with respect to u_k we obtain the following two-point boundary-value problem to be solved [8]:

(3)
$$\begin{aligned} x_{k+1} &= Ax_k - GR^{-1}G'p_{k+1}', \\ p_k &= x_k'H'QH + p_{k+1}A. \end{aligned}$$

Guessing a solution of the form

$$p_k = x_k'P,$$

the P matrix must satisfy the equation

(4)
$$P - A'P(I + GR^{-1}G'P)^{-1} A - H'QH = 0.$$

The optimal control u_k^* is given by

$$u_k^* = -R^{-1}G'A'^{-1}(P - H'QH)\, x_k$$

and thus the optimal closed-loop system obeys the equation

$$x_{k+1} = [A - GR^{-1}G'A'^{-1}(P - H'QH)]\, x_k.$$

The minimal value $\mathscr{J}^*$ of (2) is given [8] as

(5)
$$\mathscr{J}^* = \tfrac{1}{2}x_0'Px_0.$$

Equation (4) is sometimes referred to as the discrete Riccati algebraic equation.

To proceed further we find it convenient to set

$$\begin{aligned} GR^{-1}G' &= BB', \\ H'QH &= C'C, \end{aligned}$$

 where B and C are matrices of full rank such that

$$\operatorname{rank} B = \operatorname{rank} GR^{-1}G',$$
$$\operatorname{rank} C = \operatorname{rank} H'QH.$$

The discrete Riccati equation (4) then reads

$$P - A'P(I + BB'P)^{-1} A - C'C = 0 \tag{6a}$$

or, equivalently,

$$P - A'PA + A'PB(I + B'PB)^{-1} B'PA - C'C = 0 \tag{6b}$$

by virtue of the matrix identity [8]

$$(U^{-1} + VZ^{-1}W)^{-1} = U - UV(Z + WUV)^{-1} WU.$$

Also

$$\begin{aligned} x_{k+1} &= [A - BB'A'^{-1}(P - C'C)]\, x_k = \\ &= [A - B'I + B'PB)^{-1} B'PA]\, x_k. \end{aligned} \tag{7}$$

Equation (6) is closely related to the $2n \times 2n$ composite matrix

$$M = \begin{bmatrix} A + BB'A'^{-1}C'C, & -BB'A'^{-1} \\ -A'^{-1}C'C, & A'^{-1} \end{bmatrix}, \tag{8}$$

which couples equations (3) as follows:

$$\begin{bmatrix} x_{k+1} \\ p'_{k+1} \end{bmatrix} = M \begin{bmatrix} x_k \\ p'_k \end{bmatrix}.$$

The discrete Riccati algebraic equation can have more than one solution. In the sections to follow, conditions will be given for a solution to possess certain special properties.

As the central results, the existence and uniqueness of the nonnegative definite solution is established and the lattice structure of all such solutions is discussed.

2. GENERAL FORM OF SOLUTIONS

Throughout the paper we assume that the M matrix has a diagonal Jordan canonical form, i.e., it has $2n$ eigenvectors. This assumption is made for the sake of simplicity and is by no means essential. All results are easy to generalize to the nondiagonal case.

Let

$$Ma_i = \lambda_i a_i, \quad i = 1, 2, \ldots, 2n$$

and write

$$a_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix},$$

where x_i and y_i are elements of C_n.

Then following [7], [6] we can express the solutions of (6) in terms of a_i as follows.

Theorem 1. *Each solution of* (6) *takes the form*

$$P = YX^{-1},$$

where

$$X = [x_1, x_2, \ldots, x_n],$$

$$Y = [y_1, y_2, \ldots, y_n],$$

correspond to such a choice of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of M that X^{-1} exists.

Proof. Let P satisfies (6) and set

$$K = A - BB'A'^{-1}(P - C'C),$$

the closed-loop system matrix.

Then (6b) and (7) imply that

$$PK = A'^{-1}(P - C'C)$$

and hence

(9)
$$M\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} K.$$

Let $J = X^{-1}KX$ be the Jordan canonical form of K and set $PX = Y$. Then (9) yields

(10)
$$M\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} J.$$

Since J is diagonal, $J = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ constitute the eigenvectors of M associated with $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $P = YX^{-1}$ since X^{-1} exists. Q.E.D.

Corollary 1. *Let $P = YX^{-1}$, where $X = [x_1, x_2, \ldots, x_n]$ and $Y = [y_1, y_2, \ldots, y_n]$ correspond to the choice of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of M. Then the closed-loop system matrix K has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ associated with the eigenvectors $x_1, x_2, \ldots, x_n$.*

Proof. The J matrix is the Jordan form of K and X is the respective transformation matrix. Q.E.D.

We note that if an n-tuple of eigenvectors $\begin{bmatrix} X \\ Y \end{bmatrix}$ generates a solution P to (6), then any nonsingular transformation $\begin{bmatrix} X \\ Y \end{bmatrix} T$ of theirs does so since $YT(XT)^{-1} = YX^{-1} = P$. Specifically, the order of the eigenvectors used is immaterial.

3. THE EIGENVALUE PROPERTIES

The eigenvalues of M enjoy a very interesting property of symmetry. To begin with, let us prove

Lemma 1. *Suppose* $\det A \neq 0$. *Then there is no zero eigenvalue of* M.

Proof. By contradiction, suppose that

$$M \begin{bmatrix} x \\ y \end{bmatrix} = 0 ,$$

i.e.,

$$\begin{aligned} Ax + BB'A'^{-1}C'Cx - BB'A'^{-1}y &= 0 , \\ -A'^{-1}C'Cx + \quad A'^{-1}y &= 0 . \end{aligned}$$

Multiplying the latter equation by BB' and adding it to the former one gives us $Ax = 0$, a contradiction. Q.E.D.

In addition to the (right) eigenvectors a_i defined by $Ma_i = \lambda_i a_i$, we introduce the left eigenvectors of M as follows.

$$r_i M = \lambda_i r_i .$$

It is well-known that the eigenvectors can be chosen so that

$$(11) \qquad \begin{aligned} r_i a_j &= 0 , \quad i \neq j , \\ &\neq 0 , \quad i = j . \end{aligned}$$

Theorem 2. *Let* λ_i *be an eigenvalue of* M *and* $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ *be the associated eigenvector. Then* λ_i^{-1} *is an eigenvalue of* M *and* $[-y_i', x_i']$ *is the corresponding left eigenvector.*

Proof. Let

$$M \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \lambda_i \begin{bmatrix} x_i \\ y_i \end{bmatrix} .$$

It is easy to see that

$$M^{-1} = \begin{bmatrix} A^{-1}, & A^{-1}BB' \\ C'CA^{-1}, & A' + C'CA^{-1}BB' \end{bmatrix}$$

and hence

$$[-y_i', x_i'] M^{-1} = \lambda_i[-y_i', x_i'] .$$

It follows that

$$[-y_i', x_i'] M = \lambda_i^{-1}[-y_i', x_i'] .$$

Q.E.D.

Note that M being a real matrix, its eigenvalues occur in quadruples $(\lambda_i, \lambda_i^*, \lambda_i^{-1}, \lambda_i^{*-1})$. Here and below the asterisk represents the conjugate transpose of a matrix vector, or scalar.

4. REAL SOLUTIONS

There can be both real and complex solutions P to equation (6). In the regulator problem, however, only real solutions are of interest.

Theorem 3. *A solution P of (6) is real if either all eigenvectors used to construct it are real, or with any complex eigenvector the complex conjugated one is also used.*

Proof is trivial and can be found in [6].

5. HERMITIAN SOLUTIONS

In the linear regulator problem, only the symmetric solutions to (6) are of interest consistently with (5). We generalize a little here and characterize the Hermitian solutions.

Theorem 4. *Let $M\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} J$, where $J = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then $\lambda_i^* - \lambda_j^{-1} \neq 0$, $i, j = 1, 2, \ldots, n$ implies that X^*Y is Hermitian.*

Proof. The idea of the proof follows closely that of a similar proof in [7], [6].

By inspection,

$$M'T - TM^{-1} = 0 \tag{12}$$

where

$$T = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} .$$

Set

$$N = X^*Y - Y^*X = [X^*, Y^*]\, T \begin{bmatrix} X \\ Y \end{bmatrix} .$$

Then

$$J^*N - NJ^{-1} = J^*[X^*, Y^*]\, T\begin{bmatrix} X \\ Y \end{bmatrix} -$$

$$- [X^*, Y^*]\, T\begin{bmatrix} X \\ Y \end{bmatrix} J^{-1} = [X^*, Y^*]\,(M'T - TM^{-1})\begin{bmatrix} X \\ Y \end{bmatrix} = 0$$

by (12).

It is well known that only $N = 0$ satisfies the equation $J^*N - NJ^{-1} = 0$ whenever $\lambda_i^* - \lambda_j^{-1} \neq 0$, $i, j = 1, 2, \ldots, n$. Then $X^*Y = Y^*X$. Q.E.D.

As a consequence, $P = YX^{-1} = X^{-1*}(X^*Y)\, X^{-1}$ is Hermitian provided X^*Y is so and X^{-1} exists.

6. NONNEGATIVE DEFINITE SOLUTIONS

The assumption on Q and R in (2) implies that $\mathscr{J}^* \geqq 0$ and, therefore, only nonnegative definite solutions P of equation (6) are considered in the regulator problem. We draw the reader's attention to the fact that positive definite matrices are thought a subset of nonnegative definite matrices.

For Hermitian or real symmetric matrices, the notation $P_1 \geqq P_2$ means that $P_1 - P_2$ is nonnegative definite.

Theorem 5. *Let* $M\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} J$, *where* $J = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. *Then* $|\lambda_i| < 1$, $i = 1, 2, \ldots, n$ *implies that* $X^*Y \geqq 0$.

Proof. Like in [7], [6], set $U_k = \begin{bmatrix} X \\ Y \end{bmatrix} J^k$. Then U_k satisfies the following recurrent equation

$$U_{k+1} = MU_k\,, \quad U_0 = \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Defining $T = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, we get $X^*Y = U_0^*TU_0$. Further introduce

$$\begin{aligned} S_k &= - U_k^*TU_k + U_0^*TU_0 = \\ &= - \sum_{i=0}^{k-1} U_{i+1}^*TU_{i+1} - U_i^*TU_i = \\ &= - \sum_{i=0}^{k-1} U_i^*M'TMU_i - U_i^*TU_i = \\ &= - \sum_{i=0}^{k-1} U_i^*(M'TM - T)\, U_i\,. \end{aligned}$$

The matrix $M'TM - T$ is nonpositive definite since the matrix

$$\begin{bmatrix} I, & C'C \\ 0, & I \end{bmatrix} (M'TM - T) \begin{bmatrix} I, & 0 \\ C'C, & I \end{bmatrix} = \begin{bmatrix} -C'C, & 0 \\ 0, & -A^{-1}BB'A'^{-1} \end{bmatrix}$$

is so; hence $S_k \geqq 0$ for all k.

If J is a stable matrix, $\lim_{k\to\infty} U_k = 0$ and $X^*Y = \lim_{k\to\infty} S_k \geqq 0$. Q.E.D.

Again $P = YX^{-1} \geqq 0$ provided $X^*Y \geqq 0$ and X^{-1} exists.

7. CONTROLLABILITY, OBSERVABILITY, STABILIZABILITY, AND DETECTABILITY

In this section we discuss some preliminary results. First of all, λ is said to be an *uncontrollable eigenvalue* [2] of the pair (A, B) if there exists a row vector $w \neq 0$ such that $wA = \lambda w$ and $wB = 0$. Similarly, λ is an *unobservable eigenvalue* of the pair (C, A) if there exists a vector $z \neq 0$ such that $Az = \lambda z$ and $Cz = 0$.

The pair (A, B) is said to be *stabilizable* [9] if a real matrix S exists such that $A + BS$ is stable, i.e. $|\lambda_i| < 1$ for all its eigenvalues. Alternatively, (A, B) is stabilizable if and only if the unstable eigenvalues of (A, B) are controllable.

In a like manner, (C, A) is *detectable* [9] if a real matrix D exists such that $DC + A$ is stable, or, equivalently, if and only if the unstable eigenvalues of (C, A) are observable.

The concepts defined above play the central role in the subsequent development.

Lemma 2.

$$M \begin{bmatrix} 0 \\ w' \end{bmatrix} = \lambda^{-1} \begin{bmatrix} 0 \\ w' \end{bmatrix}$$

if and only if λ is an uncontrollable eigenvalue of (A, B).

Proof.

$$M \begin{bmatrix} 0 \\ w' \end{bmatrix} = \begin{bmatrix} -BB'A'^{-1}w' \\ A'^{-1}w' \end{bmatrix} = \lambda^{-1} \begin{bmatrix} 0 \\ w' \end{bmatrix}$$

is equivalent to $wA = \lambda w$ and $wB = 0$. Q.E.D.

Lemma 3.

$$M \begin{bmatrix} z \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} z \\ 0 \end{bmatrix}$$

if and only if λ is an unobservable eigenvalue of (C, A).

Proof.

$$M\begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{bmatrix} Az + BB'A'^{-1}C'Cz \\ -A'^{-1}C'Cz \end{bmatrix} = \lambda \begin{bmatrix} z \\ 0 \end{bmatrix}$$

is equivalent to $Cz = 0$ and $Az = \lambda z$. Q.E.D.

Lemma 4. *There is an eigenvalue λ of M such that $|\lambda| = 1$ if and only if there is an uncontrolable eigenvalue λ of (A, B) and/or unobservable eigenvalue λ of (C, A) such that $|\lambda| = 1$.*

Proof. $\Leftarrow$ This part is a consequence of Lemma 2 and 3.

$\Rightarrow$ Let $M\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$, $|\lambda| = 1$. Then

$$\begin{aligned} Ax + BB'A'^{-1}C'Cx - BB'A'^{-1}y &= \lambda x \\ -A'^{-1}C'Cx + A'^{-1}y &= \lambda y . \end{aligned} \tag{13}$$

Multiplying the latter of equations (13) by BB' and adding it to the former gives us

$$\begin{aligned} Ax - \lambda BB'y &= \lambda x , \\ -C'Cx + y &= \lambda A'y . \end{aligned}$$

Premultiplying the first equation by $\lambda^* y^*$ and the second one by x^* yields

$$\begin{aligned} \lambda^* y^* Ax &= \lambda^* \lambda y^* BB'y + \lambda^* \lambda y^* x = \\ &= -x^* C'Cx - y^* x . \end{aligned}$$

Thus

$$-\lambda^* \lambda y^* BB'y - x^* C'Cx = (\lambda^* \lambda - 1) y^* x = 0$$

since $\lambda^* \lambda = |\lambda|^2 = 1$.

It follows that

$$\begin{aligned} y'B &= 0 , \\ Cx &= 0 . \end{aligned} \tag{14}$$

Simple algebraic manipulations based on (13) and (14) show that

(i) $Cx = 0$ implies $y'A = \lambda^{-1} y'$ which together with $y'B = 0$ qualifies λ^{-1} as an uncontrollable eigenvalue of (A, B), $|\lambda^{-1}| = 1$;

(ii) $y'B = 0$ implies $Ax = \lambda x$, which together with $Cx = 0$ qualifies λ as an unobservable eigenvalue of (C, A), $|\lambda| = 1$. Q.E.D.

To develop the fundamental theorem for the discrete Riccati algebraic equation we proceed along the lines of [3].

A solution $P \geqq 0$ of equation (6) is said to be an *optimizing solution* if it yields the optimal closed-loop system; a solution $P \geqq 0$ is said to be a *stabilizing solution* if it yields a stable closed-loop system.

Theorem 6. *There exists a stabilizing solution P to equation* (6) *if and only if* (A, B) *is stabilizable and* $|\lambda| \neq 1$ *for all eigenvalues λ of M.*

Proof. $\Rightarrow$ Suppose there exists a stabilizing solution P to (6). Then n stable eigenvalues λ of M must exist and hence, by Theorem 2, $|\lambda| \neq 1$ for all eigenvalues λ of M.

In addition, the matrix $S = -B'A'^{-1}(P - C'C)$ stabilizes $A + BS$, the closed-loop system matrix, i.e. the pair (A, B) is stabilizable.

$\Leftarrow$ Suppose (A, B) is stabilizable, $|\lambda| \neq 1$ for all eigenvalues λ of M, and that the converse of the statement is true – there is no stabilizing solution of (6). It can be due only to the two facts below:

(i) There exist less then n stable eigenvalues of M, a contradiction.

(ii) The X matrix in (10) is singular. If this is the case, write z for any nonzero vector of $\mathcal{N}(X)$, the null space of X. By (8),

$$\begin{aligned} AX + BB'A'^{-1}C'CX - BB'A'^{-1}Y &= XJ\,, \\ -A'^{-1}C'CX + \qquad A'^{-1}Y &= YJ\,. \end{aligned} \tag{15}$$

On premultiplying the last equation by BB' and summing up equations (15) we obtain

$$AX - BB'YJ = XJ\,. \tag{16}$$

Now postmultiply the above equation by z and premultiply it by $z^*J^*Y^*$ to get

$$-z^*J^*Y^*BB'YJz = z^*J^*Y^*XJz\,.$$

The right hand side of this equation is nonnegative since Y^*X is so by Theorems 4 and 5, while the left hand side is nonpositive. It follows that either is zero and hence

$$B'YJz = 0\,. \tag{17}$$

Multiplying (16) by z and substituting from (17) gives us $XJz = 0$.

It means that $\mathcal{N}(X)$ is a J-invariant subspace of R_n. Hence there exists at least one nonzero vector $\hat{z} \in \mathcal{N}(X)$ such that $J\hat{z} = \mu\hat{z}$ where μ coincides with one of the stable eigenvalues of M, that is,

$$|\mu| < 1\,. \tag{18}$$

The second equation (15) postmultiplied by $\hat{z}$ yields

$$\hat{z}'Y'A^{-1} = \mu\hat{z}'Y' . \tag{19}$$

Collecting (17), (18) and (19) we conclude that

$$\hat{z}'Y'A = \mu^{-1}\hat{z}'Y' , \quad |\mu^{-1}| > 1 ,$$
$$\hat{z}'Y'B = 0 ,$$

i.e., (A, B) is not stabilizable, again a contradiction. Q.E.D.

It has been a well-established fact [9] that (A, B) stabilizable and (C, A) detectable is a sufficient condition for a stabilizing solution to exist. In the light of Lemma 4 it is evident why this condition is unnecessary unlike that of Theorem 6: there is no need for observability of the unstable eigenvalues of A save those with $|\lambda| = 1$.

We also point out that equation (6) can have at most one stabilizing solution due to Theorem 2.

Theorem 7. *The stabilizing solution is the only nonnegative definite solution of* (6) *if and only if* (C, A) *is detectable.*

Proof. $\Leftarrow$ Assuming (C, A) detectable we shall demonstrate that any solution P of (6) yields a stable closed-loop system matrix K.

Suppose to the contrary that a λ exists such that $Kz = \lambda z$, $|\lambda| \geqq 1$. Then (9) can be rewritten as

$$A + BB'A'^{-1}C'C - BB'A'^{-1}P = K ,$$
$$-A'^{-1}C'C + \quad A'^{-1}P = PK .$$

We sum up the first equation and the second one multiplied by BB' to obtain

$$K = A - BB'PK ,$$
$$A'PK = -C'C + P .$$

Easy algebraic manipulations with the last equations result in

$$\lambda^*z^*PAz = \lambda^*\lambda z^*PBB'Pz + \lambda^*\lambda z^*Pz ,$$
$$\lambda z^*A'Pz = -z^*C'Cz + z^*Pz .$$

It follows that

$$-\lambda^*\lambda z^*PBB'Pz - z^*C'Cz = (\lambda^*\lambda - 1)\, z^*Pz .$$

Since $\lambda^*\lambda = |\lambda|^2 \geqq 1$, the right hand side of the above equation is nonnegative, while the left hand side is nonpositive. Therefore both are zero and hence

$$B'Pz = 0 ,$$
$$Cz = 0 .$$

As a result, $\lambda z = Kz = Az - \lambda BB'Pz$ implies

$$Az = \lambda z\,, \quad |\lambda| \geqq 1\,,$$
$$Cz = 0\,.$$

Thus (C, A) is not detectable, contradicting our hypothesis. Hence K is stable.

But there is only one way how to choose the stable eigenvalues of M and hence P, the stabilizing solution of (6), is unique.

$\Rightarrow$ We proceed by contradiction, see [6]. Suppose there is an undetectable eigenvalue λ_1 of (C, A). We are going to show that at least two different nonnegative solutions exist to equation (6). One of them is the stabilizing solution P by hypothesis. It is expressed as $P = YX^{-1}$ in terms of

$$X = [x_1, x_2, \ldots, x_n]\,, \quad Y = [y_1, y_2, \ldots, y_n]$$

and assume that $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ is associated with λ_1^{-1}.

To form another solution $P_1 = Y_1 X_1^{-1}$ we substitute the eigenvector $\begin{bmatrix} z_1 \\ 0 \end{bmatrix}$ of M corresponding to λ_1 for the eigenvector $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ with λ_1^{-1}. We thus have

$$X_1 = [z_1, x_2, \ldots, x_n]\,, \quad Y_1 = [0, y_2, \ldots, y_n]$$

and set

$$\hat{X} = [x_2, \ldots, x_n]\,, \quad \hat{Y} = [y_2, \ldots, y_n]\,.$$

Theorem 2 together with equation (11) implies that

$$[z_1^*, 0] \begin{bmatrix} -y_i \\ x_i \end{bmatrix} = 0\,, \quad i = 2, 3, \ldots, n\,.$$

Hence

$$z_1^* \hat{Y} = 0$$

and

$$X_1^* Y_1 = \begin{bmatrix} 0 & z_1^* \hat{Y} \\ 0 & \hat{X}^* \hat{Y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{X}^* \hat{Y} \end{bmatrix} \geqq 0 \tag{20}$$

because $X^*Y \geqq 0$.

To prove that X_1^{-1} indeed exists, suppose to the contrary that X_1 is singular. Then a vector $v \neq 0$ exists such that $z_1 = \hat{X}v$ and, consequently,

$$0 = z_1^* \hat{Y} = v^* \hat{X}^* \hat{Y}\,,$$

i.e., $\det \hat{X}^*\hat{Y} = 0$. Observe that $\det \hat{X}^*\hat{Y}$ is a principal minor of $X^*Y \geqq 0$ and hence

$$[x_1, \hat{X}]^* \hat{Y}v = X^*(\hat{Y}v) = 0\,,$$

a contradiction since X is nonsingular and $\hat{Y}v \neq 0$.

Thus P_1 does exist and is different from P because it corresponds to a different n-tuple of eigenvalues of M. Q.E.D.

Now the fundamental theorem can easily be deduced.

Theorem 8. *Stabilizability of (A, B) and detectability of (C, A) is necessary and sufficient for equation (6) to have a unique nonnegative definite solution which stabilizes the closed-loop system.*

Proof. $\Rightarrow$ This part is now a classical result [9].

$\Leftarrow$ The existence of a stabilizing solution implies stabilizability of (A, B) by Theorem 6. The uniqueness of the solution implies detectability of (C, A) by Theorem 7.

9. THE LATTICE OF NONNEGATIVE SOLUTIONS

This section summarizes some latest results regarding equation (6). The method of attack mimics that in [4]. We note that Theorem 5 supplies just a sufficient condition for a solution of (6) to be nonnegative definite. We are going to fully characterize the nonnegative solutions of (6) below.

Let there be $\varrho \geqq 0$ undetectable eigenvalues of (C, A), say $\lambda_1, \lambda_2, \ldots, \lambda_\varrho$, and let

$$Az_i = \lambda_i z_i\,, \quad i = 1, 2, \ldots, \varrho\,.$$
$$Cz_i = 0\,,$$

Then, by Lemma 3, $\begin{bmatrix} z_i \\ 0 \end{bmatrix}$ is the eigenvector of M associated with λ_i, $i = 1, 2, \ldots, \varrho$, and no other eigenvector of this form exists with the unstable eigenvalues of M.

Define

$$\mathscr{S} = \{\lambda_1, \lambda_2, \ldots, \lambda_\varrho\}$$

and write $\mathscr{S}_\alpha$, $\alpha = 1, 2, \ldots$ for the subsets of $\mathscr{S}$.

Similarly, define

$$\mathscr{R} = \{\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_\varrho^{-1}\}$$

and relate a subset $\mathscr{R}_\alpha$ of $\mathscr{R}$ to any subset $\mathscr{S}_\alpha$ of $\mathscr{S}$ consistently with the definition of $\mathscr{R}$ and $\mathscr{S}$.

Note that all eigenvalues in $\mathscr{R}$ must be used to form the stabilizing solution since $|\lambda_i^{-1}| < 1$, $i = 1, 2, \ldots, \varrho$.

Finally write P_α for the solution of (6) which is generated from the stabilizing solution by replacing the elements of $\mathscr{R}_\alpha$ by those of $\mathscr{S}_\alpha$.

Theorem 9. *Suppose there exists the stabilizing solution of equation* (6). *Then the solutions P_α generated respectively by all subsets $\mathscr{S}_\alpha$ of $\mathscr{S}$ form the class of all nonnegative definite solutions of* (6).

Proof. The existence of all P_α can be proved in an identical manner as the existence of P_1 in the necessity part of the proof of Theorem 7.

It remains to prove that no other nonnegative solution exists: if an eigenvalue λ_i were substituted for λ_j^{-1}, $\lambda_i \neq \lambda_j$, it would give us

$$\hat{X} = [\ldots, u, \ldots, x, \ldots]\,, \quad \hat{Y} = [\ldots, v, \ldots, y, \ldots]$$

where

$$M\begin{bmatrix} u \\ v \end{bmatrix} = \lambda_i \begin{bmatrix} u \\ v \end{bmatrix}, \quad M\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_i^{-1}\begin{bmatrix} x \\ y \end{bmatrix}.$$

But

$$[u^*, v^*]\begin{bmatrix} -y \\ x \end{bmatrix} \neq 0$$

by (11) and Theorem 2. It follows that $u^*y \neq x^*v$ and hence

$$\hat{X}^*\hat{Y} = \begin{bmatrix} \ddots & & & \\ & u^*v & \ldots & u^*y \\ & \vdots & & \vdots \\ & x^*v & \ldots & x^*y \\ & & & & \ddots \end{bmatrix} \neq \hat{Y}^*\hat{X}\,.$$

We conclude that $\hat{P} = \hat{Y}\hat{X}^{-1}$, if it exists at all, cannot be nonnegative definite. Q.E.D.

Before proceeding any further, we define an eigenvalue λ of A to be *cyclic* if any two eigenvectors of A associated with λ are linearly dependent.

Corollary 2. *Suppose equation* (6) *has the stabilizing solution. Further suppose there exist ϱ undetectable eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_\varrho$ of (C, A), $\varrho \geqq 0$. Then*

(i) *if λ_i, $i = 1, 2, \ldots, \varrho$ are cyclic, there are exactly 2^ϱ nonnegative definite solutions of* (6); and

(ii) *if some of the λ_i's are not cyclic, the set of all nonnegative definite solutions of* (6) *is infinite* (*nondenumerable*).

Proof. (i) follows immediately from Theorem 9 since there are 2^ϱ subsets of a set consisting of ϱ elements.

(ii) Some of the λ_i's not being cyclic, there are infinitely (nondenumerably) many ways of choosing the independent eigenvectors and for each particular choice the case (i) applies. Q.E.D.

The above result (i) is limited to a diagonalizable matrix M. A more refined analysis in [4] shows that, in general, there exist $\prod_{i=1}^{\varrho} (q_i + 1)$ nonnegative solutions, where

$$\begin{aligned} Cz_{i+j} &= 0\,, \quad j = 0, 1, \ldots, q_i - 1\,, \\ &\neq 0\,, \quad j = q_i \end{aligned}$$

and

$$\begin{aligned} Az_{i+j} &= \lambda z_{i+j} + z_{i+j-1}\,, \quad j = 1, 2, \ldots, q_i - 1\,, \\ &= \lambda z_i\,, \qquad\qquad\quad j = 0\,. \end{aligned}$$

Now a very interesting property of all nonnegative solutions will be presented [4].

Theorem 10. *Let the class of the nonnegative definite solutions of* (6) *contains the stabilizing solution. Then*

(i) *if the set is finite, it constitutes a distributive lattice with respect to the partial ordering* $\geqq$. *Moreover, the stabilizing and the optimizing solutions are respectively the identity and the zero elements of the lattice.*

(ii) *If the set is nondenumerable, it generates nondenumerable many distributive lattices. The lattices are isomorphic to one another and have the identity and the zero elements in common.*

Proof. (i) First we note that the family of subsets of $\mathscr{S}$ constitutes a distributive lattice with respect to the partial ordering by inclusion [1].

We shall prove the theorem by establishing an isomorphism between the set of all solutions $P \geqq 0$ of (6) and the underlying lattice of the subsets of $\mathscr{S}$.

By definition, there is one-to-one correspondence between $\mathscr{S}_\alpha$ and P_α. In particular, the stabilizing solution corresponds to the empty set Φ and the optimizing solution to $\mathscr{S}$ itself. Since $\mathscr{S}_\alpha \subseteq \mathscr{S}_\beta$ implies $P_\alpha \geqq P_\beta$ by (20), the isomorphism is established.

It follows that the stabilizing (optimizing) solution is the identity (zero) element of the lattice of solutions since Φ ($\mathscr{S}$) is the zero (identity) element of the underlying lattice of the subsets of $\mathscr{S}$.

Finally, distributivity follows directly from distributivity of the underlying lattice.

(ii) In view of corollary 2, (ii), and the above proof, there exist nondenumerable many distributive lattices. However, the same ordering is always preserved, i.e., the lattices are isomorphic to one another. The uniqueness of the identity and the zero elements corresponds to that of the stable and the optimal closed-loop systems. Q.E.D

In case $\varrho = 0$, that is, (C, A) is detectable, the $\mathscr{S}$ set is empty and the lattice collapses to a single element – the optimizing as well as stabilizing solution. See Theorem 7.

However, not all nonnegative solutions of (6) are, in general, real. The real nonnegative solutions constitute a sublattice of the lattice in Theorem 10.

The physical interpretation of different real nonnegative solutions is as follows. Each nonnegative solution is a conditionally optimizing solution of (6), the condition being a certain degree of stability. Specifically, P_α stabilizes the undetectable eigenvalues of (C, A) included in $\mathscr{S}_\alpha$ and no others. The discrete Riccati algebraic equation (6) thus contains the optimal solutions for all degrees of stability [6]. The idea that the more undetectable eigenvalues is stabilized the higher is the cost (2) is made rigorous via the concept of lattice.

Another important consequence of Theorem 10 is that if more than one nonnegative solution of equation (6) exists, then the stabilizing solution is never the optimizing one and vice versa.

10. EXAMPLE

The following simple example is intended to illustrate some fine points involved in the exposition. Consider

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = 0\,.$$

Thus

$$M = \begin{bmatrix} 2 & 0 & -0.5 & -0.5 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

and its eigenvalues are

$$\lambda_1 = 2\,, \quad \lambda_3 = 0.5\,,$$

$$\lambda_2 = 2\,, \quad \lambda_4 = 0.5\,.$$

Observe that M is not cyclic. Therefore its eigenvectors are, in general,

$$a_1 = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} c \\ d \\ 0 \\ 0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} d \\ -c \\ 3d \\ -3c \end{bmatrix}, \quad a_4 = \begin{bmatrix} -b \\ a \\ -3b \\ 3a \end{bmatrix}$$

where a, b, c, and d are reals such that $ad - bc \neq 0$.

 All general solutions of the discrete Riccati algebraic equation (6) are given by Theorem 1:

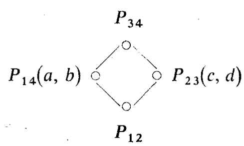
$$P_{12} = 0\,, \qquad P_{13} = \frac{3}{ac + bd}\begin{bmatrix} bd, & -ad \\ -bc, & ac \end{bmatrix},$$

$$P_{14} = \frac{3}{a^2 + b^2}\begin{bmatrix} b^2, & -ab \\ -ab, & a^2 \end{bmatrix}, \qquad P_{23} = \frac{3}{c^2 + d^2}\begin{bmatrix} d^2, & -cd \\ -cd, & c^2 \end{bmatrix},$$

$$P_{24} = \frac{3}{ac + bd}\begin{bmatrix} bd, & -bc \\ -ad, & ac \end{bmatrix}, \qquad P_{34} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Note that all solutions are real and that P_{13} and P_{24} are not symmetric. The pair (A, B) is stabilizable and $|\lambda_i| \neq 1$, $i = 1, 2, 3, 4$. Hence there exists a stabilizing solution and it is not the only nonnegative solution since the pair (C, A) is not detectable.

Theorem 9 yields all nonnegative solutions: P_{12}, P_{14}, P_{23} and P_{34} are their representatives. They generate nondenumerably many lattices, which are isomorphic to one another. The lattices can be visualized as follows [4]



where P_{12} is the optimizing solution, P_{34} is the stabilizing solution, and the coupling represents the partial ordering

$$P_{34} \geqq P_{14} \geqq P_{12}\,,$$

$$P_{34} \geqq P_{23} \geqq P_{12}\,.$$

Note that $P_{14} - P_{23}$ is indefinite, indeed.

11. CONCLUSIONS

The discrete Riccati algebraic equation has been studied. The underlying discrete linear regulator problem has been posed and referred to throughout.

The paper contains a comprehensive theory of the equation studied. Theorems regarding the general form of solutions as well as their special properties have been proved or reproved. The main and highly original result is the fundamental theorem in Section 8 and the lattice theorem in Section 9.

A simple example has been appended to illustrate certain fine points of the theory.

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VÝTAH

Diskrétní Riccatiho rovnice optimálního řízení

VLADIMÍR KUČERA

V článku je podána teorie diskrétní Riccatiho algebraické rovnice, o jejíž řešení se opírá syntéza optimálního řízení dle kvadratických kriterií či syntéza optimálních lineárních filtrů.

Článek vychází z předchozích prací autora o spojité Riccatiho rovnici. Je dokázána věta o obecném tvaru řešení, o reálných, hermitovských a nezáporně definitních řešeních. Nejdůležitější přínos práce je obsažen ve větě o existenci a unicitě nezáporného řešení a ve větě o svazových vlastnostech třídy všech nezáporných řešení.

Článek lze chápat jako matematický základ k praktickému řešení diskrétních optimalizačních problémů.

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